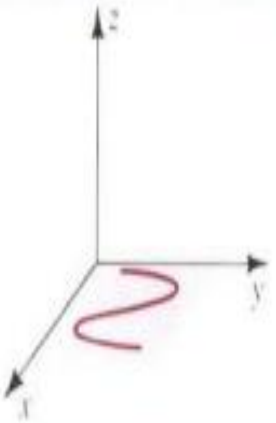
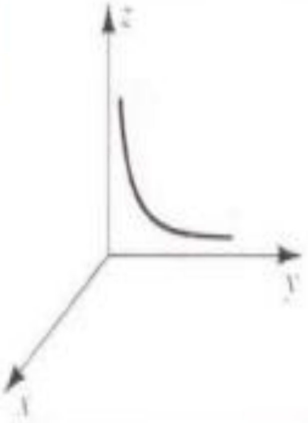
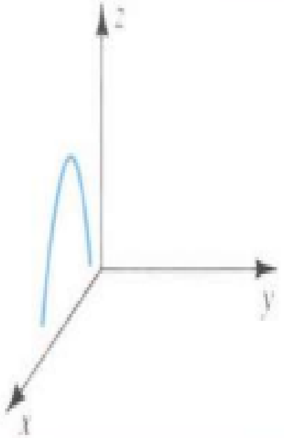
(14.6) SURFACES

TRACE

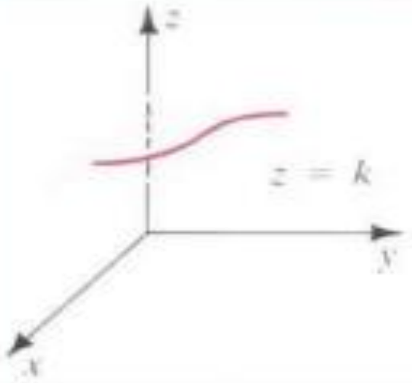
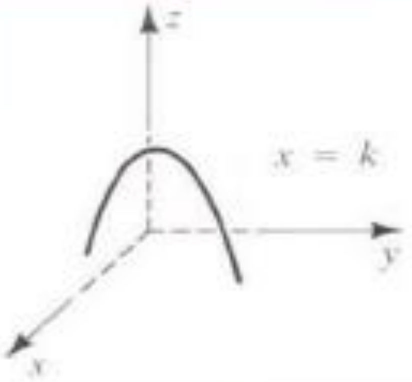
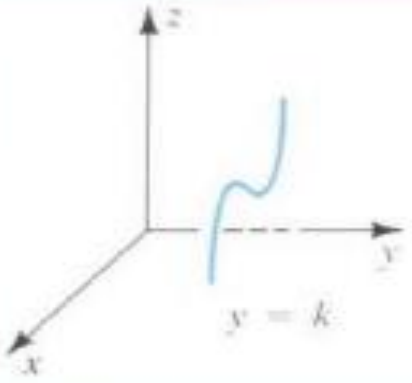
THE FOLLOWING CHARTS SHOW HOW TO FIND EQUATIONS OF TRACES FROM AN EQUATION OF A SURFACE .

Trace	To find equation of trace	Sketch of trace
xy-trace	Let $z = 0$	

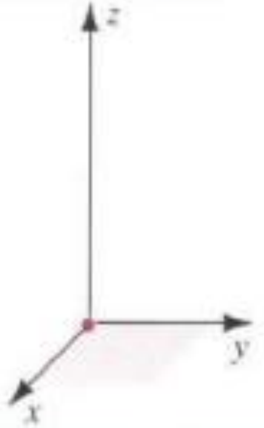
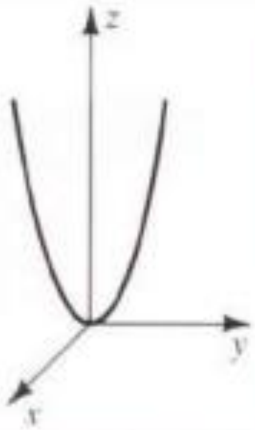
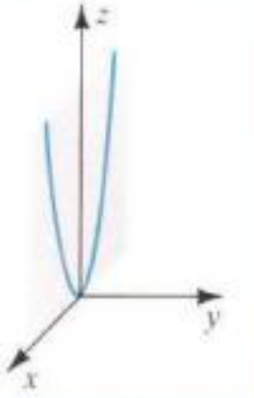
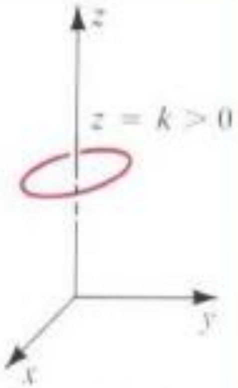
Trace	To find equation of trace	Sketch of trace
yz-trace	Let $x = 0$	

Trace	To find equation of trace	Sketch of trace
xz-trace	Let $y = 0$	

IT MAY ALSO BE CONVENIENT TO FIND TRACES IN PLANES THAT ARE PARALLEL TO THE COORDINATE PLANES.

On $z = k$	Let $z = k$	
On $x = k$	Let $x = k$	
On $y = k$	Let $y = k$	

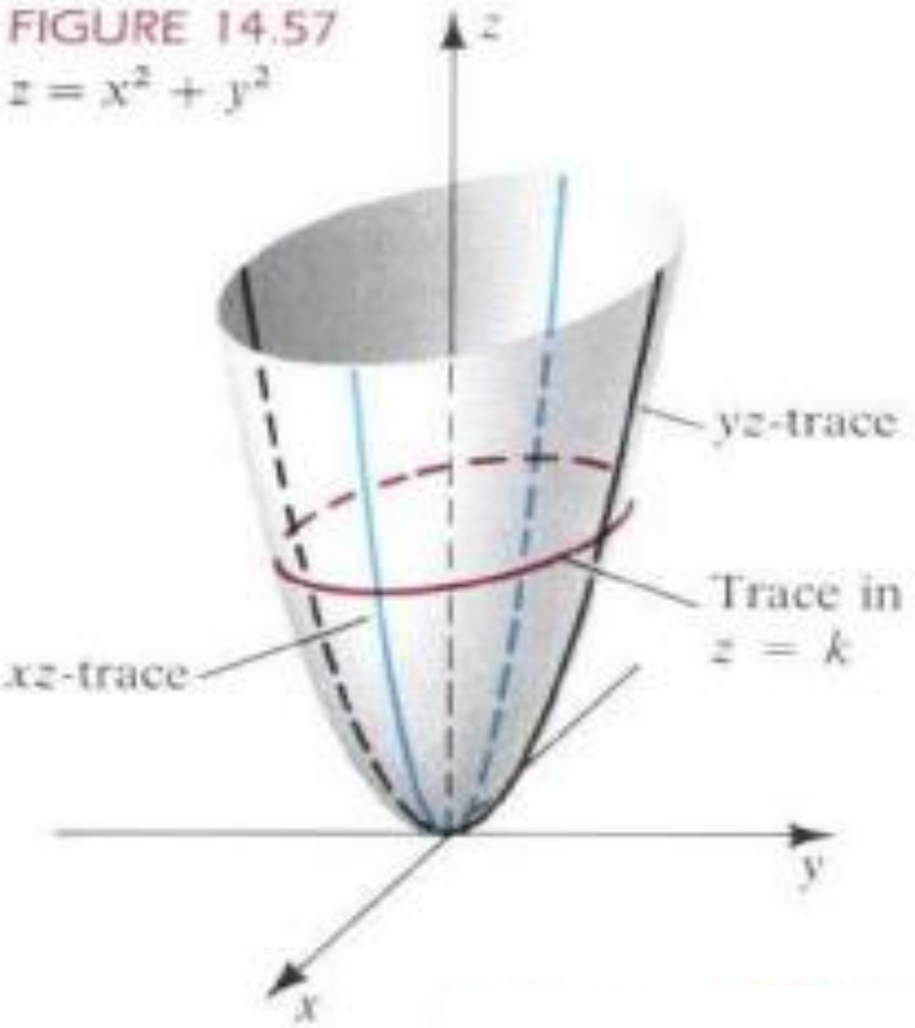
EXAMPLE 1: SKETCH THE GRAPH OF $z = x^2 + y^2$.

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$0 = x^2 + y^2$	Origin	
yz-trace	$z = y^2$	Parabola	
xz-trace	$z = x^2$	Parabola	
On $z = k$	$k = x^2 + y^2$, or $x^2 + y^2 = k$	Circle, point, or no graph	 No graph if $k < 0$

This surface is called a circular paraboloid , or paraboloid of revolution .

FIGURE 14.57

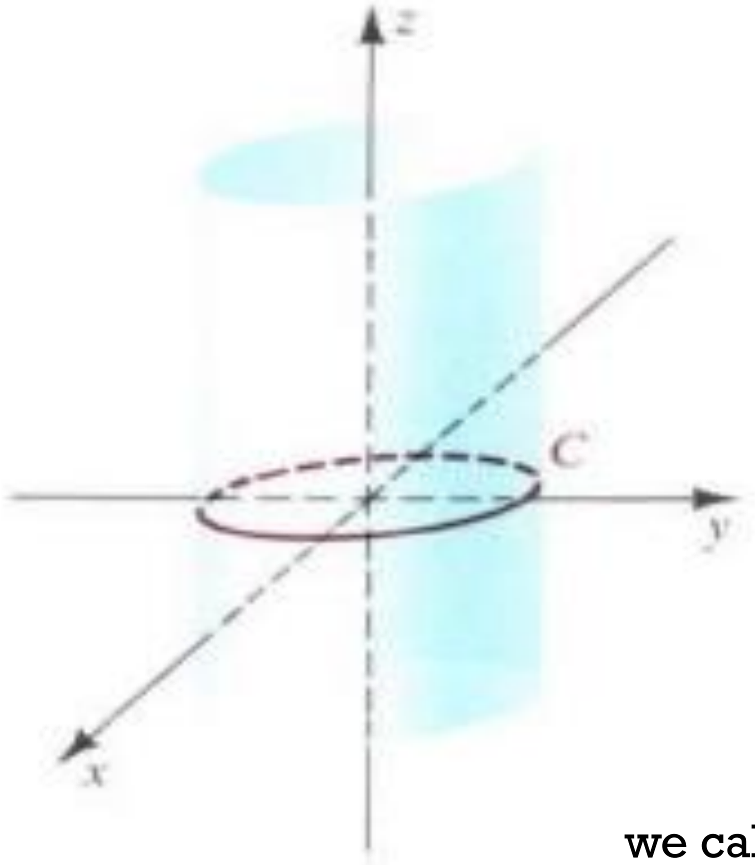
$z = x^2 + y^2$



EXAMPLE 2: SKETCH THE GRAPH OF $\frac{x^2}{4}+\frac{y^2}{9}=1$ IN THREE DIMENSIONS.

FIGURE 14.60

$\frac{x^2}{4}+\frac{y^2}{9}=1$



we call this surface an elliptic cylinder.

ELLIPSOID (14.41) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

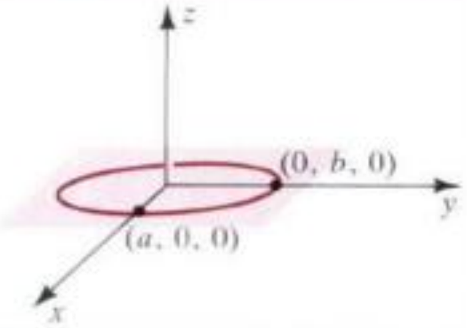
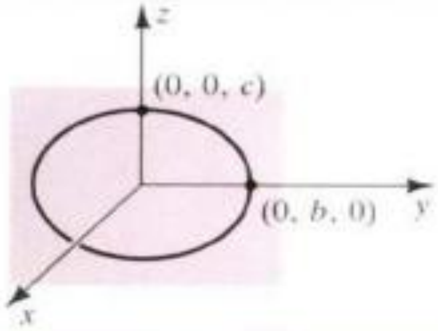
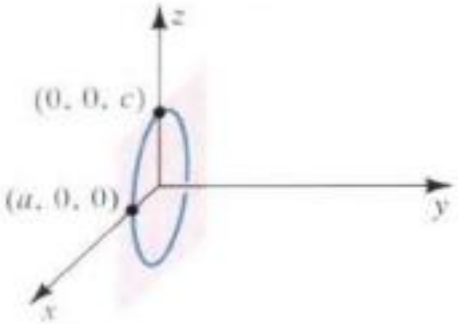
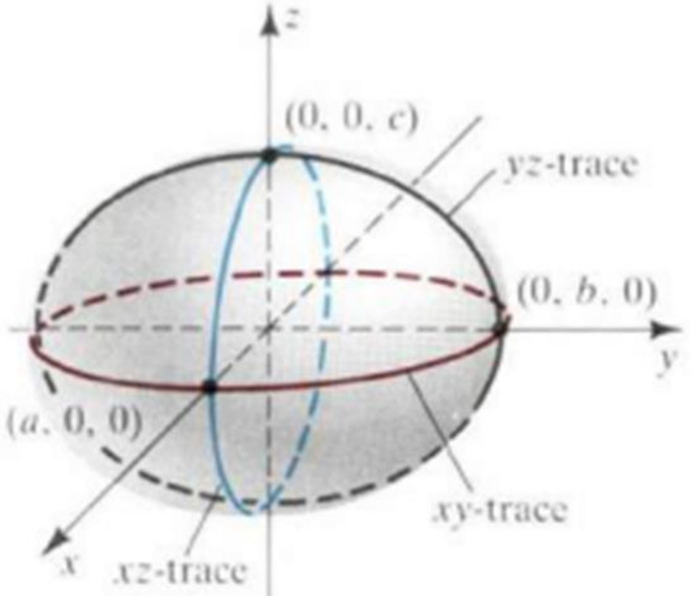
Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Ellipse	
yz-trace	$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipse	
xz-trace	$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$	Ellipse	

FIGURE 14.63
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



To find the trace on a plane z = k parallel to the xy-plane, we let z = k in the equation of the ellipsoid, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2},$$

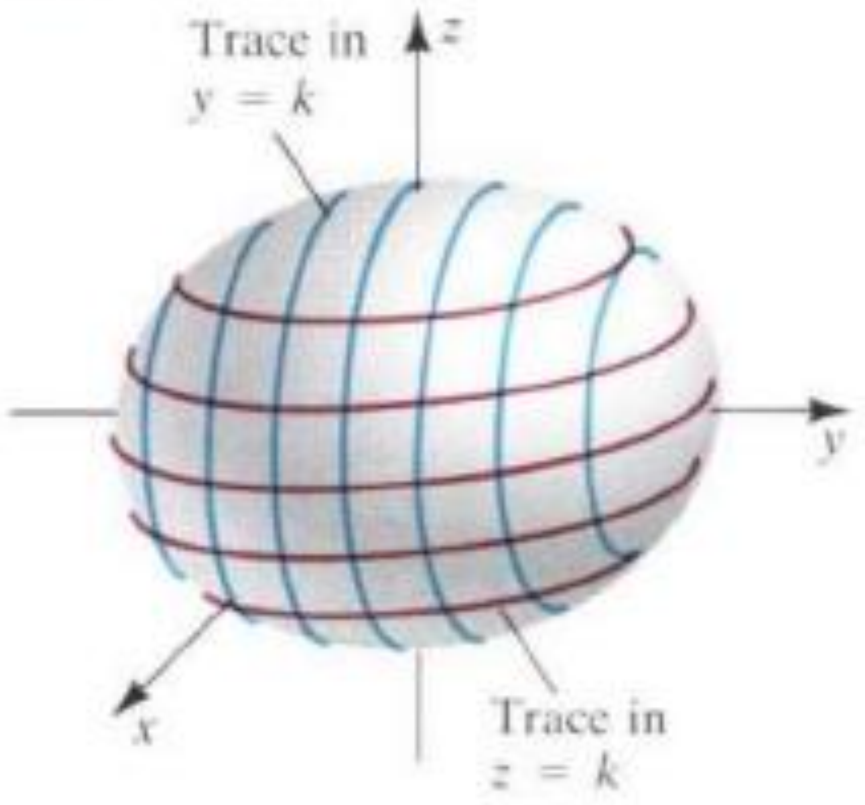
- If $|k| > c$, then $1 - \left(\frac{k^2}{c^2}\right) < 0$

and there is no graph, Thus, the graph of (14.41) lies between the planes
z = - k and z = k.

- If $|k| < c$, then $1 - \left(\frac{k^2}{c^2}\right) > 0$

and hence the trace in the plane z = k is an ellipse, as illustrated in Figure 14.64

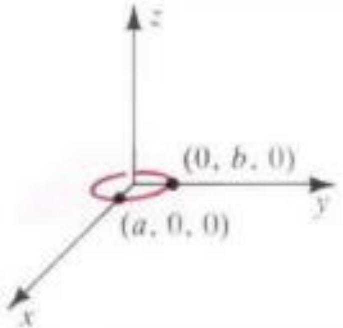
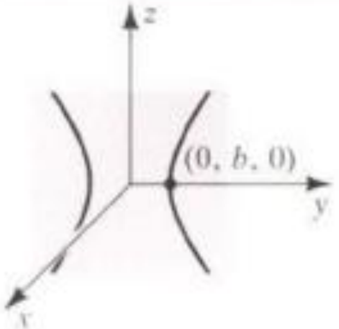
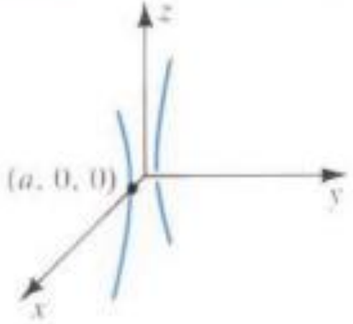
FIGURE 14.64



Note that if a = b = c, then the graph of (14.41) is a sphere of radius (a) with center at the origin.

HYPERBOLOID OF ONE SHEET (14.42)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Ellipse	
yz-trace	$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperbola	
xz-trace	$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$	Hyperbola	

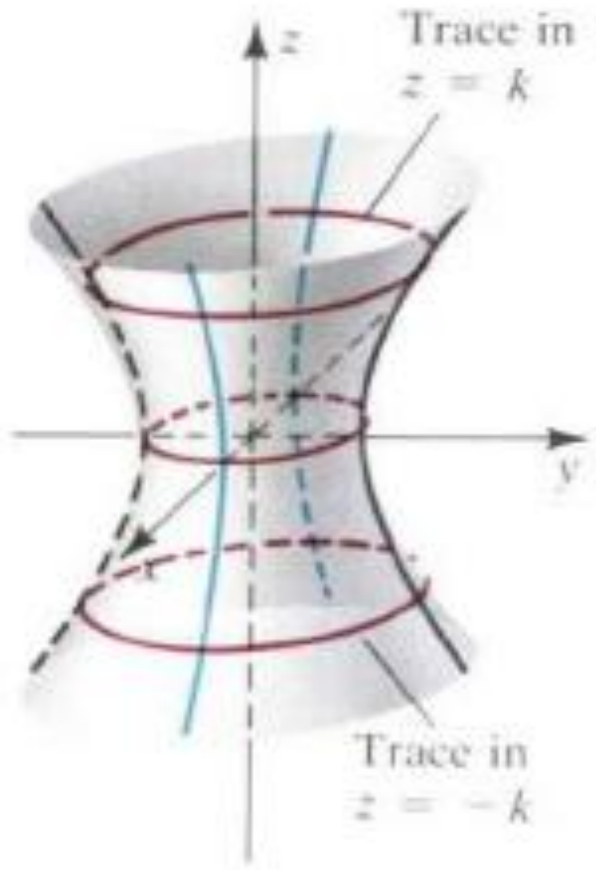
The z-axis is the axis of the hyperboloid,

The trace in a plane z = k parallel to the xy-plane is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}+1$ and hence is an ellipse.

As k increases through positive values, the lengths of the axes of the ellipse increase.

FIGURE 14.65

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$



The graphs of

$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

are also hyperboloids of one sheet; however, in the first case the axis of the hyperboloid is the y-axis,
and in the second case the axis of the hyperboloid coincides with the x-axis.
Thus, the term that is negative in these equations indicates the axis of the hyperboloid.

HYPERBOLOID OF TWO SHEETS (14.43) $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

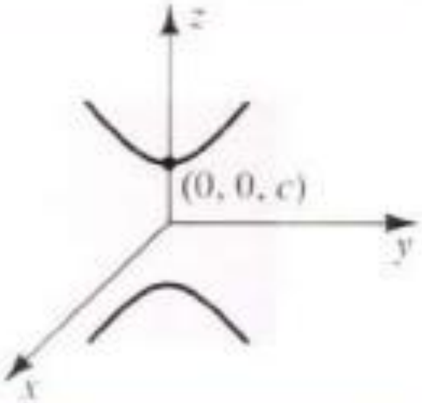
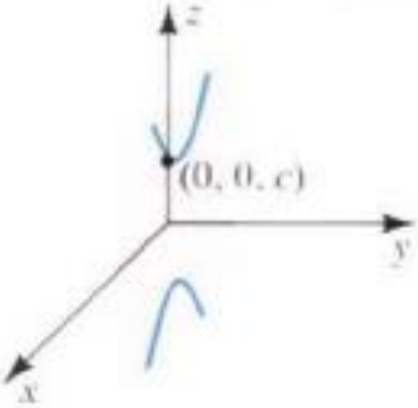
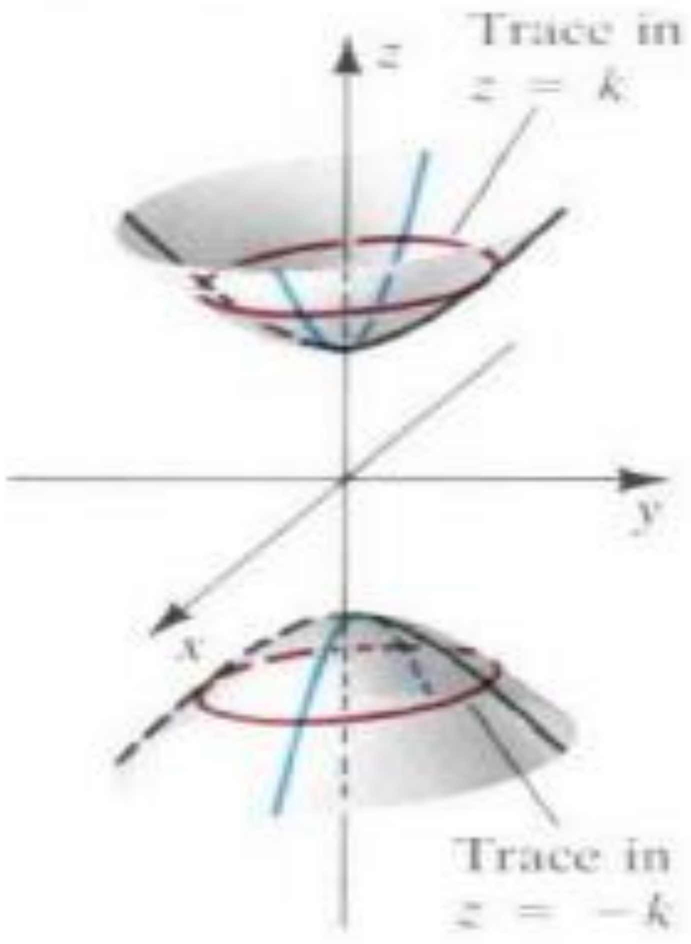
Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	None	No graph
yz-trace	$-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Hyperbola	
xz-trace	$-\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$	Hyperbola	

FIGURE 14.66

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

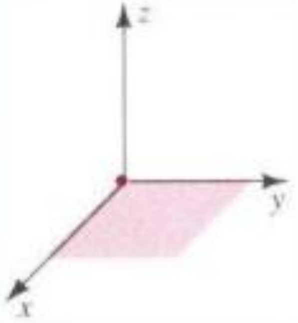
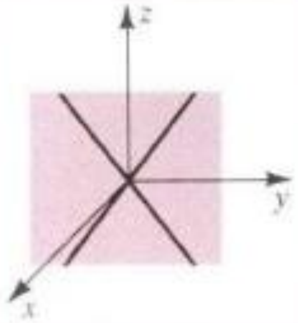
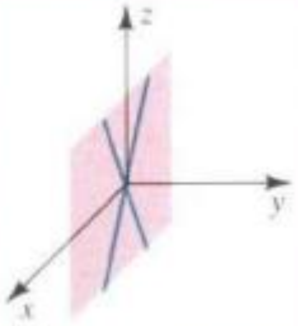


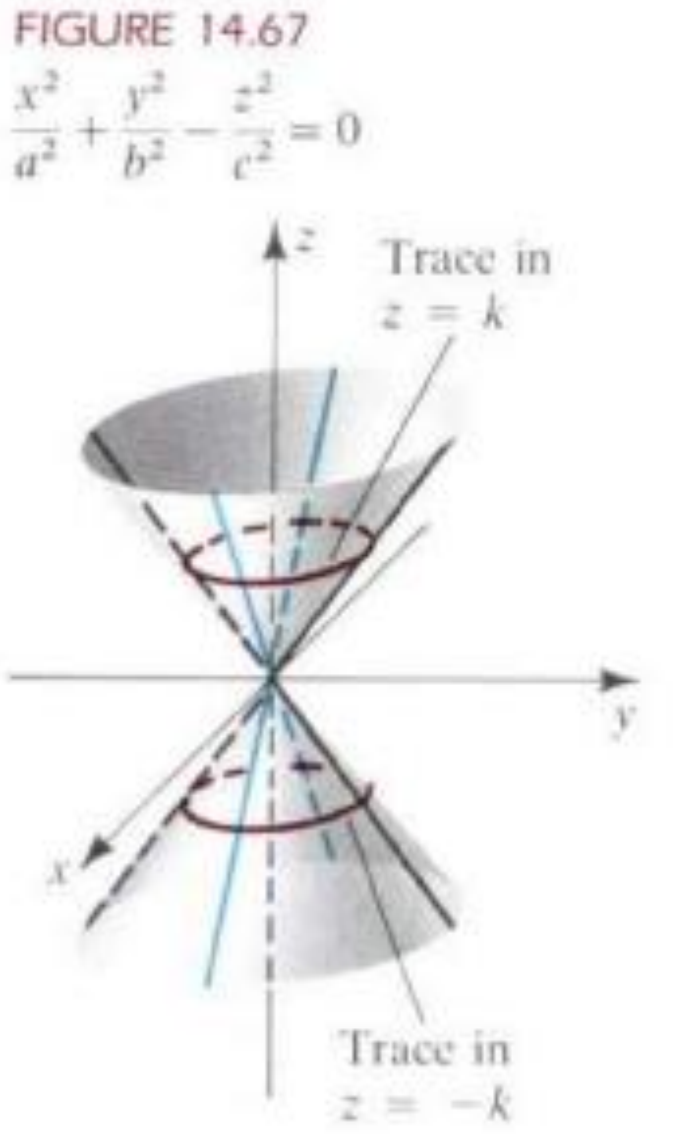
Traces in planes parallel to the yz-plane or xz-plane are hyperbolas.
The z-axis is the axis of the hyperboloid.

By using minus signs on different terms, we can obtain a hyperboloid of two sheets whose axis is the x-axis or the y-axis.

CONE (14.44)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	Origin	
yz-trace	$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Two intersecting lines	
xz-trace	$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$	Two intersecting lines	



The trace in a plane z = k parallel to the xy-plane has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$$

and hence is an ellipse .

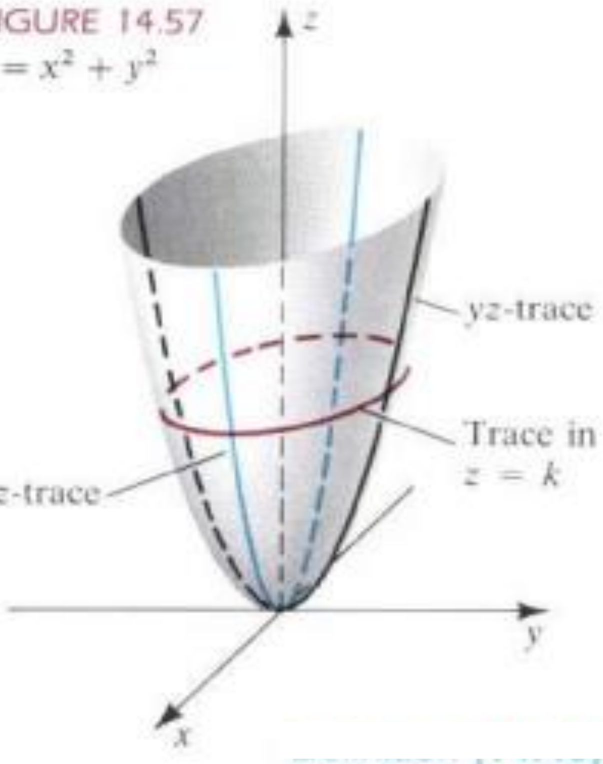
Traces in planes parallel to the other coordinate axes are hyperbolas

The z-axis is the axis of the cone.

PARABOLOID (14.45) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\text{CZ}$

Example 1 is the special case of (14.45) with a = b = c = 1.
If c > 0, then the graph of (14.45) is similar to that shown in Figure 14.57,
except that if $a \neq b$, then traces in planes z = k parallel to the xy-plane are ellipses instead of circles.
If c < 0. then the paraboloid opens downward.
The z-axis is the axis of the paraboloid.
The graphs of the equations $\frac{x^2}{a^2}+\frac{z^2}{b^2}=\text{cy}$ and $\frac{y^2}{a^2}+\frac{z^2}{b^2}=\text{cx}$ are paraboloids whose axes are the y-axis and x-axis, respectively.

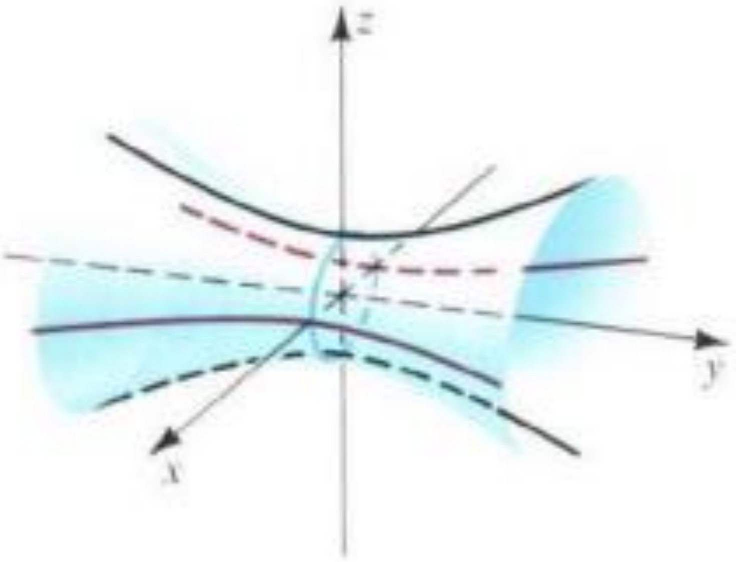
FIGURE 14.57
$z = x^2 + y^2$



EXAMPLE 4: SKETCH THE GRAPH OF $16x^2 - 9y^2 + 36z^2 = 144$, AND IDENTIFY THE SURFACE.

Trace	Equation of trace	Description of trace
xy-plane	$\frac{x^2}{9} - \frac{y^2}{16} = 1$	Hyperbola
yz-plane	$\frac{z^2}{4} - \frac{y^2}{16} = 1$	Hyperbola
xz-plane	$\frac{x^2}{9} + \frac{z^2}{4} = 1$	Ellipse

FIGURE 14.69
$16x^2 - 9y^2 + 36z^2 = 144$

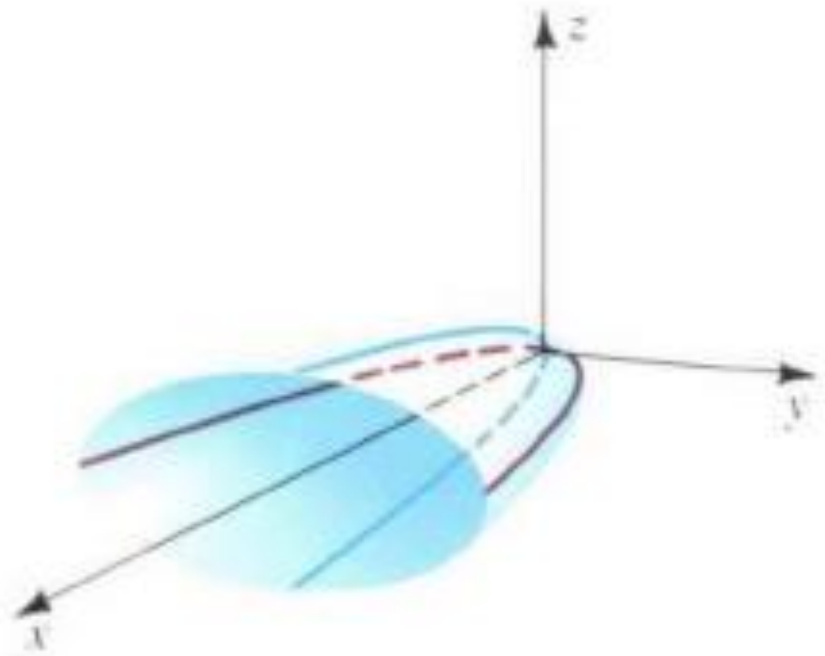


The graph, a hyperboloid of one sheet with the y-axis as its axis, is sketched in Figure 14.69.
Traces in planes parallel to the xz-plane are ellipses,
and traces in planes parallel to the xy- or yz-planes are hyperbolas.

EXAMPLE 5: SKETCH THE GRAPH OF $y^2 + 4z^2 = x$, AND IDENTIFY THE SURFACE.

Trace	Equation of trace	Description of trace
xy-plane	$y^2 = x$	Parabola
yz-plane	$y^2 + 4z^2 = 0$	Origin
xz-plane	$4z^2 = x$	Parabola

FIGURE 14.70
$y^2 + 4z^2 = x$



The trace in a plane x = k parallel to the yz-plane has equation $y^2 + 4z^2 = k$, which is an ellipse if k > O.
Traces in planes parallel to the xz- or xy-planes are parabolas.
The surface, a paraboloid having the x-axis as its axis, is sketched in Figure 14.70.

EXAMPLE 6: THE GRAPH OF $9x^2 + 4y^2 = 36$ IS REVOLVED ABOUT THE Y-AXIS. FIND AN EQUATION FOR THE RESULTING SURFACE.

We find an cquation for the surface by substituting $x^2 + z^2$ for x^2 .

This gives us $9(x^2 + z^2) + 4y^2 = 36$

The surface is an ellipsoid of revolution.

If we divide both sides by 36 and rearrange terms,

we obtain which is of the form in (14.41)(Ellipsoid).