Ch 4: The Continuous-Time Fourier Transform

Fourier Transform of x(t) →

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier Transform →

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Example 4.1

Consider the signal $x(t) = e^{-at}u(t), \quad a > 0$

Find its Fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

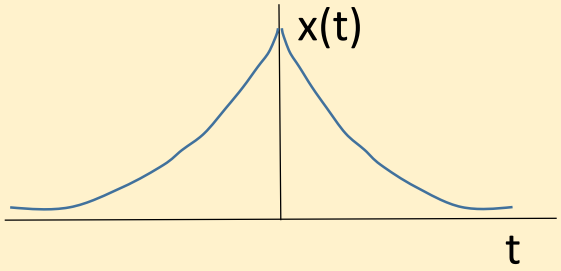
$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)t}\right|_{0}^{\infty} = \frac{1}{a+j\omega}$$

$$|X(j\omega)| = \left|\frac{1}{a+j\omega}\right| = \left|\frac{a-j\omega}{a^2+\omega^2}\right| = \sqrt{\frac{a^2}{\left(a^2+\omega^2\right)^2} + \frac{\omega^2}{\left(a^2+\omega^2\right)^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle X(j\omega) = \tan^{-1}\left(\frac{-\omega}{a^2+\omega^2} / \frac{a}{a^2+\omega^2}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

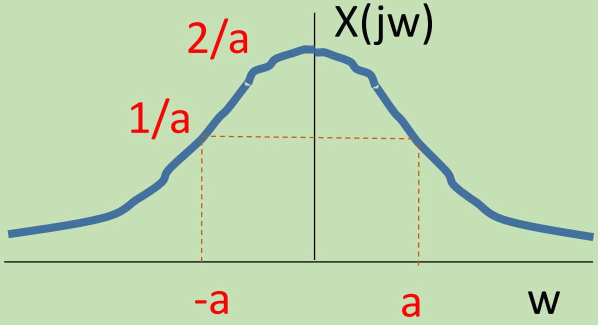
Example 4.2

$$x(t) = e^{-a|t|}, \quad \text{for } a > 0$$



Time-domain

frequency-domain



The Fourier transform of the signal is:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

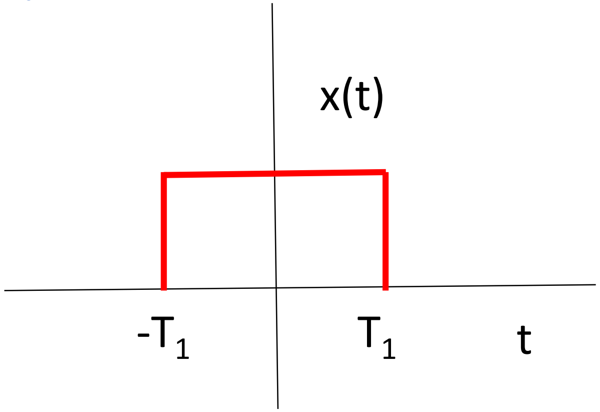
$$= \left.\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+j\omega+a-j\omega}{a^2+\omega^2}$$

$$= \frac{2a}{a^2+\omega^2}$$

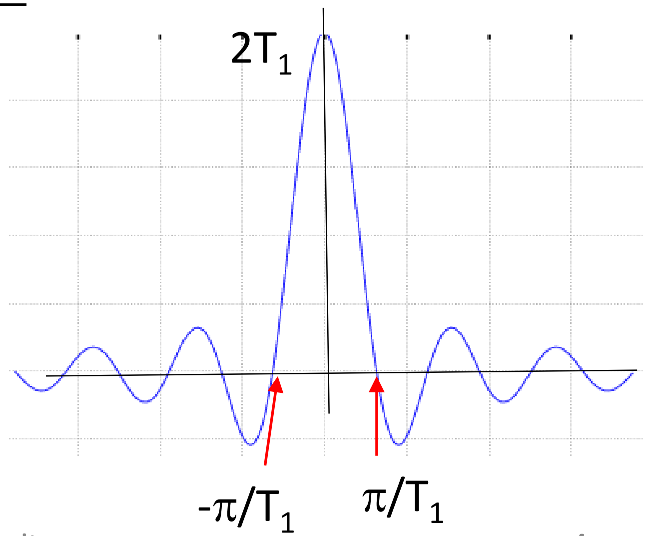
Example 4.4

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



$$X(j\omega) = \int_{-T_1}^{T_1} 1.e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T_1}^{T_1} = -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

$$= \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right] = \frac{2j\sin(\omega T_1)}{j\omega} = \frac{2\sin(\omega T_1)}{\omega}$$



Example 4.5

$$X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

Complete by yourself.

Draw the signals.

$$x(t)=\frac{1}{2\pi}\int_{-W}^{W}1.e^{j\omega t}d\omega=\frac{\sin(Wt)}{\pi t}$$

Problem 4.1

Use the Fourier transform analysis equation to calculate the Fourier transforms of

(a) $e^{-2(t-1)}u(t-1)$ (b) $e^{-2|t-1|}$

(a)

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-2(t-1)}u(t-1)e^{-j\omega t}dt$$

τ = t-1

$$=\int_{-\infty}^{\infty}e^{-2\tau}u(\tau)e^{-j\omega(\tau+1)}d\tau=e^{-j\omega}\int_{-\infty}^{\infty}e^{-2\tau}u(\tau)e^{-j\omega\tau}d\tau$$

$$=e^{-j\omega}\int_{0}^{\infty}e^{-(2+j\omega)\tau}d\tau=e^{-j\omega}\left.\frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)}\right|_{0}^{\infty}$$

$$=\frac{e^{-j\omega}}{2+j\omega}$$

(b)

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-2|t-1|}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}e^{-2|\tau|}e^{-j\omega(\tau+1)}d\tau=e^{-j\omega}\int_{-\infty}^{\infty}e^{-2|\tau|}e^{-j\omega\tau}d\tau$$

$$=e^{-j\omega}\int_{-\infty}^{0}e^{2\tau}.e^{-j\omega\tau}d\tau+e^{-j\omega}\int_{0}^{\infty}e^{-2\tau}.e^{-j\omega\tau}d\tau$$

$$=e^{-j\omega}\left[\left.\frac{e^{(2-j\omega)\tau}}{(2-j\omega)\tau}\right|_{-\infty}^{0}+\left.\frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)\tau}\right|_{0}^{\infty}\right]$$

$$=\frac{4e^{-j\omega}}{4+\omega^2}$$

Problem 4.2

Use the Fourier transform analysis equation to calculate the Fourier transform of

(a) $\delta(t+1)+\delta(t-1)$

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}[\delta(t+1)+\delta(t-1)]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}\delta(t+1)e^{-j\omega t}dt+\int_{-\infty}^{\infty}\delta(t-1)e^{-j\omega t}dt$$

$$=e^{-j\omega(-1)}+e^{-j\omega(1)}=e^{j\omega}+e^{-j\omega}$$

$$=2\times\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)=2\cos\omega$$

$$|X(j\omega)|=2|\cos(\omega)|$$

Some useful Fourier transform:

$$\boxed{\begin{aligned}&x(t)=\delta(t)\\&\Rightarrow X(j\omega)=1\end{aligned}}$$

$$x(t)=\delta(t+1)$$

$$\Rightarrow X(j\omega)=e^{j\omega}$$

$$x(t)=\delta(t-1)$$

$$\Rightarrow X(j\omega)=e^{-j\omega}$$

Problem 4.2

(b) $\frac{d}{dt}\{u(-2-t)+u(t-2)\}$

$$X(j\omega)=\int_{-\infty}^{\infty}\frac{d}{dt}\{u(-2-t)+u(t-2)\}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}-\delta(-2-t)e^{-j\omega t}dt+\int_{-\infty}^{\infty}\delta(t-2)e^{-j\omega t}dt$$

$$=I_1+I_2$$

$$I_1=\int_{-\infty}^{\infty}-\delta(-2-t)e^{-j\omega t}dt,\quad \text{Let } \tau=-2-t, d\tau=-dt$$

$$=\int_{\infty}^{-\infty}-\delta(\tau)e^{j\omega(\tau+2)}(-d\tau)=-e^{2j\omega}\int_{-\infty}^{\infty}\delta(\tau)e^{j\omega\tau}d\tau=-e^{2j\omega}$$

$$I_2=\int_{-\infty}^{\infty}\delta(t-2)e^{-j\omega t}dt=e^{-2j\omega}$$

$$X(j\omega)=-e^{2j\omega}+e^{-2j\omega}$$

$$=-2j\times\left(\frac{e^{2j\omega}-e^{-2j\omega}}{2j}\right)$$

$$=-2j\sin(2\omega)$$

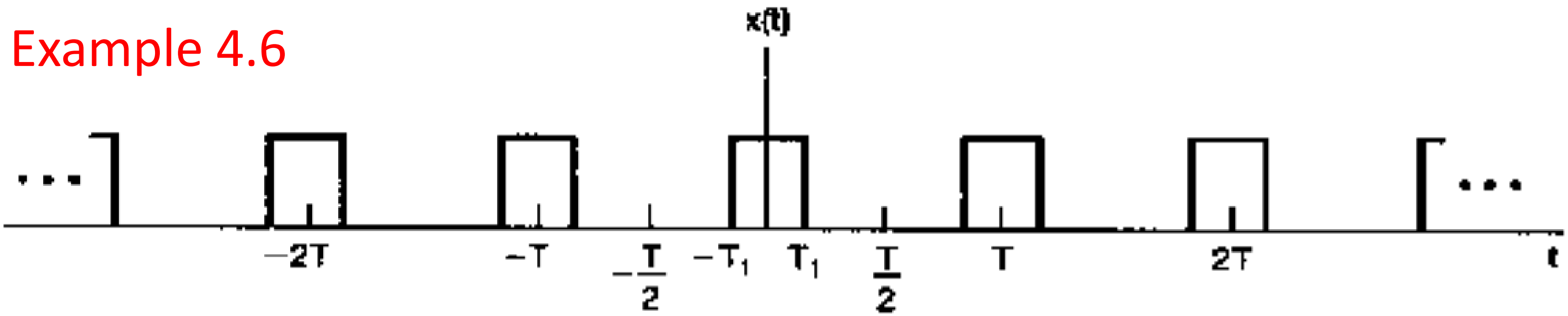
$$|X(j\omega)|=2|\sin(2\omega)|$$

Fourier Transform for Periodic Signals

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \, 2\pi\delta(\omega - k\omega_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Example 4.6

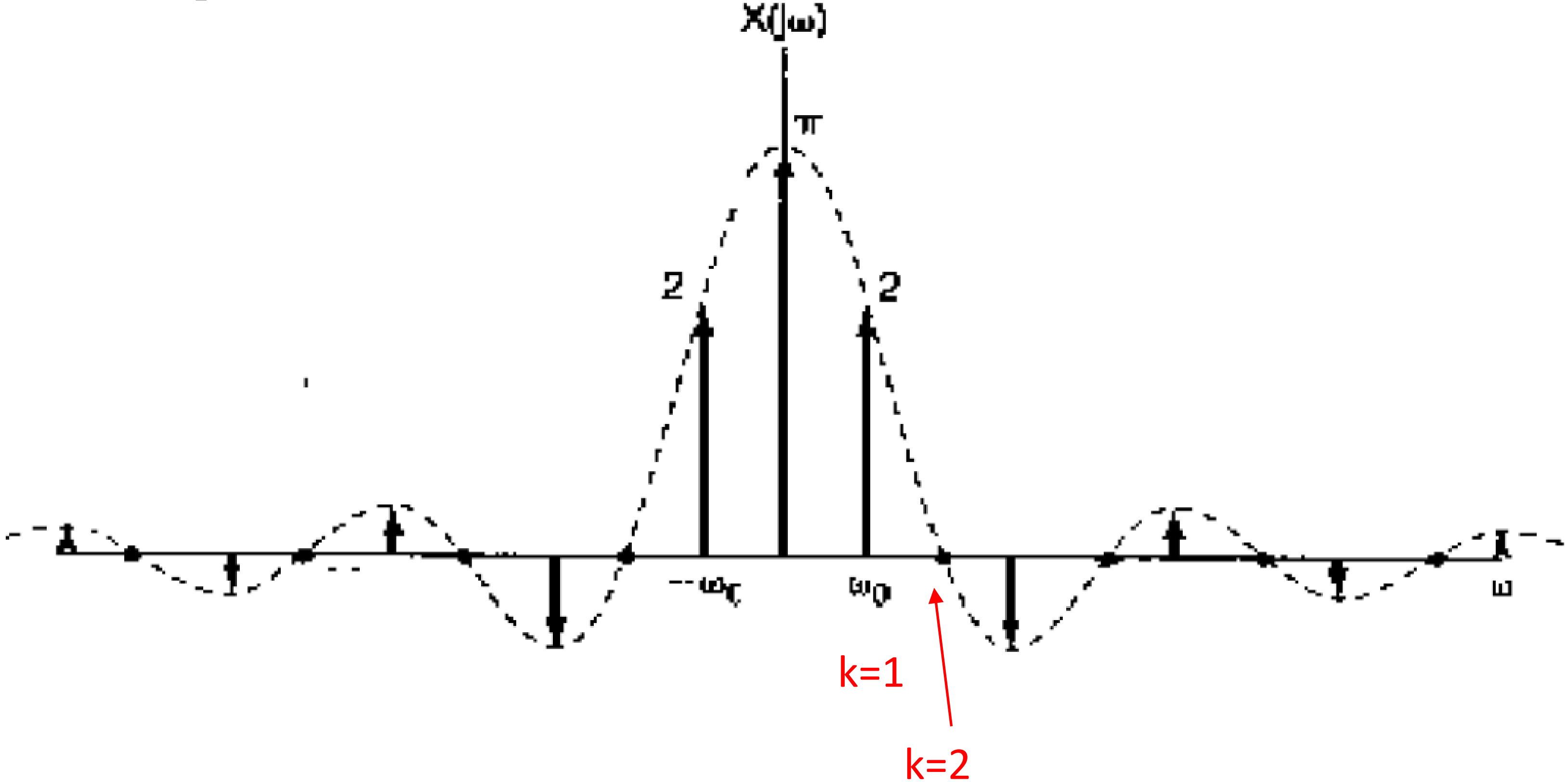


The Fourier series coefficients of the above periodic signal: $a_k = \dfrac{\sin(k\omega_0 T_1)}{\pi k}$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \frac{\sin(k\omega_0 T_1)}{\pi k} \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} 2 \frac{\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$

Example 4.6 - continued

Using $T = 4T_1$



Example 4.7

$$x(t) = \sin(\omega_0 t)$$

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

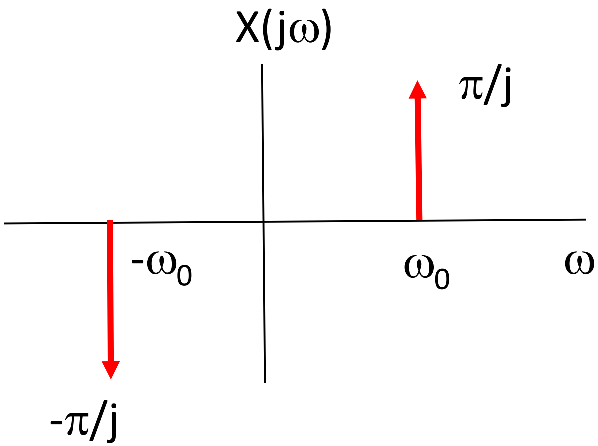
Fourier series coefficients: $a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}, a_k = 0 \text{ for } |k| \neq 1$

Find Fourier transform: X(jw).

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{2j}\delta(\omega - \omega_0) - \frac{2\pi}{2j}\delta(\omega + \omega_0)$$

$$= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

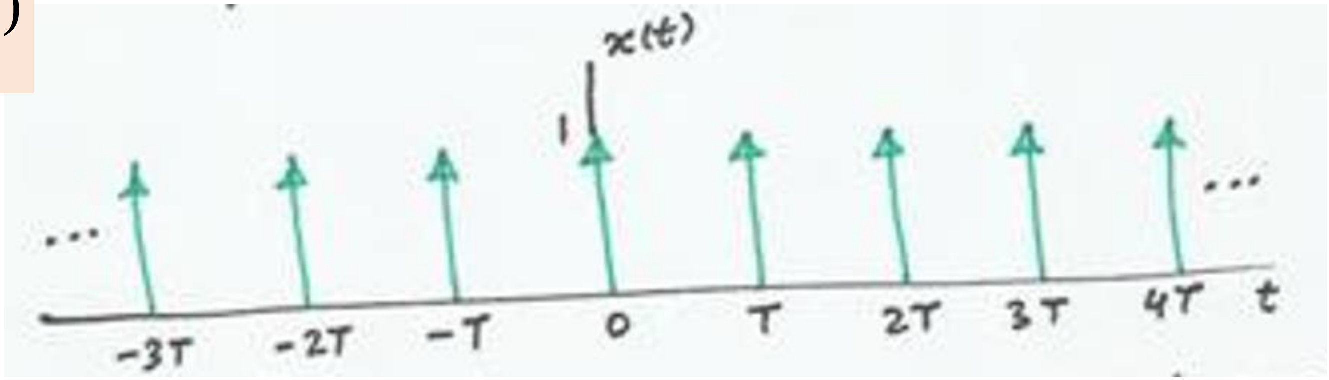


Do for $x(t) = \cos(\omega_0 t)$

Example 4.8

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Impulse train with a period of T.



The Fourier series coefficients for this signal are:

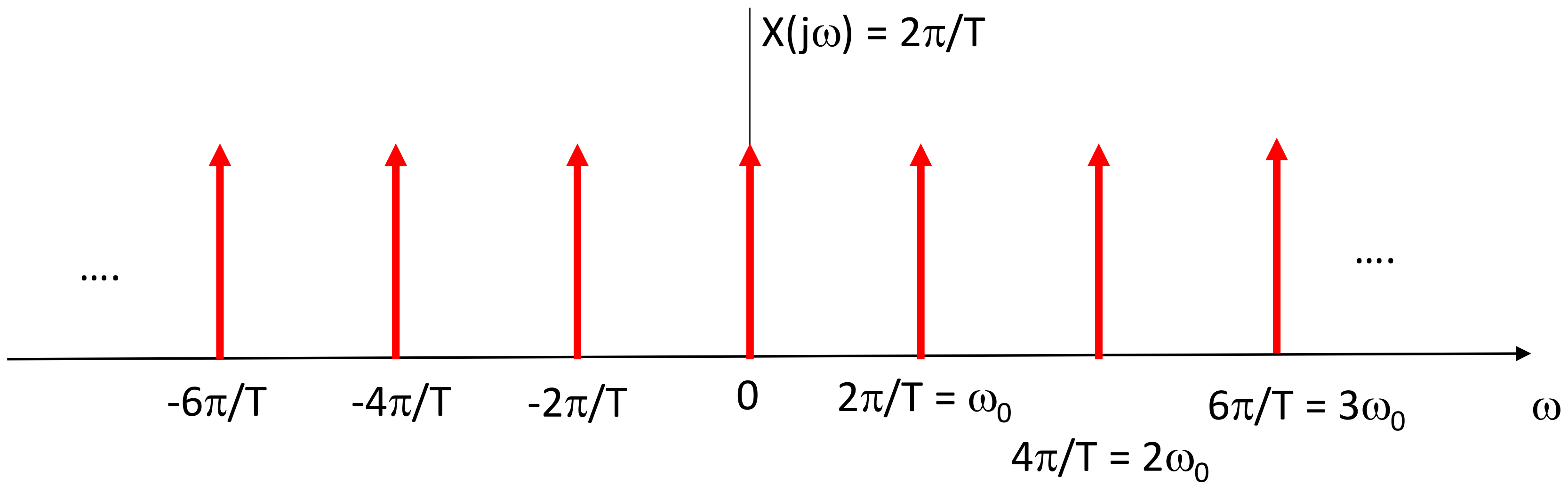
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T}e^0 = \frac{1}{T}$$

Therefore, the Fourier transform is:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}k)$$

Example 4.8 - contd.



Problem 4.3

Determine the Fourier transform of each of the following periodic signals.

(a) $\sin\left(2\pi t + \frac{\pi}{4}\right)$

(b) $1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$

Solution:

(a)

$$x(t) = \sin\left(2\pi t + \frac{\pi}{4}\right); \text{ comparing with } x(t) = \sin\left(\omega_0 t + \frac{\pi}{4}\right), \text{ and } T = \frac{2\pi}{\omega_0}$$

$$\omega_0 = 2\pi; T = 1$$

$$x(t) = \frac{e^{j(\omega_0 t + \pi/4)} - e^{-j(\omega_0 t + \pi/4)}}{2j} = \underbrace{\frac{e^{j\pi/4}}{2j}}_{a_1} e^{j\omega_0 t} - \underbrace{\frac{e^{-j\pi/4}}{2j}}_{a_{-1}} e^{-j\omega_0 t}$$

$a_k = 0$, for all other k

Problem 4.3-contd.

The Fourier transform of the periodic signal

$$X(j\omega)=\sum_{k=-\infty}^{\infty} a_k\, 2\pi\delta(\omega-k\omega_0)=\frac{\pi}{j}e^{j\pi/4}\delta(\omega-2\pi)-\frac{\pi}{j}e^{-j\pi/4}\delta(\omega+2\pi)$$

(b)

$$x(t)=1+\cos\left(6\pi t+\frac{\pi}{8}\right)=e^{0}+\frac{e^{j(6\pi t+\pi/8)}+e^{-j(6\pi t+\pi/8)}}{2}$$

$\omega_0=6\pi$

$$=\underbrace{1}_{a_0}.e^{j(0)t}+\underbrace{\frac{1}{2}e^{j\pi/8}}_{a_1}e^{j6\pi t}+\underbrace{\frac{1}{2}e^{-j\pi/8}}_{a_{-1}}e^{-j6\pi t}$$

All other a_k = 0

$$X(j\omega)=\sum_{k=-\infty}^{\infty} a_k\, 2\pi\delta(\omega-k\omega_0)$$

$$=2\pi\delta(\omega)+\pi e^{j\pi/8}\delta(\omega-6\pi)+\pi e^{-j\pi/8}\delta(\omega+6\pi)$$

Problem 4.4

Use the Fourier synthesis equations to determine the Fourier transform of

(a) $X_1(j\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$

(b) $X_2(j\omega) = \begin{cases} 2, & 0 \le \omega < 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 0 \end{cases}$

Solution:

(a)

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)\right]e^{j\omega t}d\omega$$

$$= \int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega - 4\pi)e^{j\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega + 4\pi)e^{j\omega t}d\omega$$

$$= e^{0} + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t}$$

$$= 1 + \cos(4\pi t)$$

Problem 4.4 - contd.

Solution:

(b)

$$x_2(t) = \frac{1}{2\pi}\left[\int_0^2 2e^{j\omega t}d\omega + \int_{-2}^0 -2e^{j\omega t}d\omega\right]$$

$$= \frac{1}{\pi}\left[\left.\frac{e^{j\omega t}}{jt}\right|_0^2 - \left.\frac{e^{j\omega t}}{jt}\right|_{-2}^0\right] = \frac{1}{\pi jt}\left[(e^{j2t}-1)-(1-e^{-j2t})\right]$$

$$= \frac{1}{\pi jt}\left[\left(e^{jt}\right)^2 + \left(e^{-jt}\right)^2 - 2e^{jt}e^{-jt}\right] = \frac{1}{\pi jt}\left(e^{jt}-e^{-jt}\right)^2$$

$$= \frac{-4}{\pi jt}\left(\frac{e^{jt}-e^{-jt}}{2j}\right) = \frac{-4}{j\pi t}\sin^2(t)$$

4.3 Properties of Cont. Time Fourier Trans.

Linearity

If
$$x(t) \overset{\Im}{\leftrightarrow} X(j\omega)$$
$$y(t) \overset{\Im}{\leftrightarrow} Y(j\omega)$$

Then
$$ax(t) + by(t) \overset{\Im}{\leftrightarrow} aX(j\omega) + bY(j\omega)$$

Time shifting

$$x(t - t_0) \overset{\Im}{\leftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Proof:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

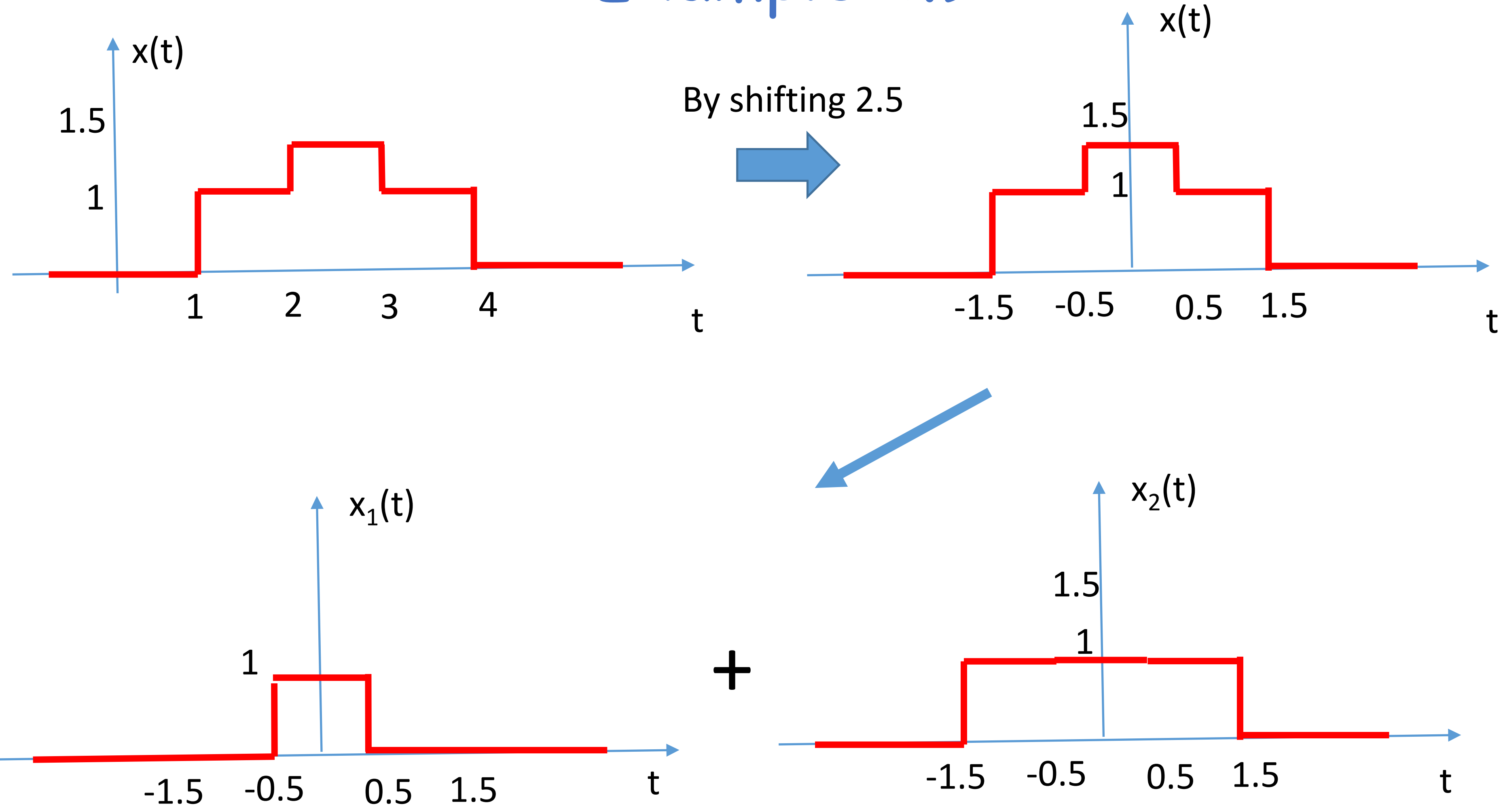
$$\Rightarrow x(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t - t_0)} d\omega$$

$$\Rightarrow x(t - t_0) = e^{-j\omega t_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\Rightarrow \Im(x(t - t_0)) = e^{-j\omega t_0} X(j\omega)$$

If a signal is time-shifted, the magnitude of the Fourier transform does not change; only there is a phase-shift in the Fourier transform.

Example 4.9



$x(t) = ½\, x_1(t - 2.5) + x_2(t - 2.5)$

Example 4.9 - contd.

From Example 4.4 and the signals of the previous slide, we get:

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \quad \text{and} \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

By linearity and time shifting properties:

$$X(j\omega) = \frac{1}{2}\Im\{x_1(t-2.5)\} + \Im\{x_2(t-2.5)\}$$

$$= \frac{1}{2}e^{-2.5j\omega}\Im\{x_1(t)\} + e^{-2.5j\omega}\Im\{x_2(t)\}$$

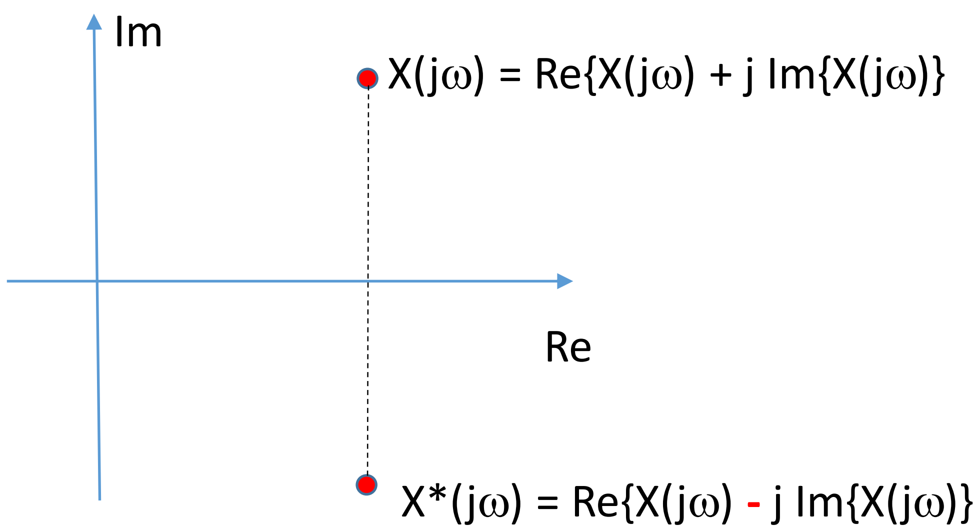
$$= \frac{1}{2}e^{-2.5j\omega}\frac{2\sin(\omega/2)}{\omega} + e^{-2.5j\omega}\frac{2\sin(3\omega/2)}{\omega}$$

$$= e^{-2.5j\omega}\left(\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right)$$

Conjugation and Conjugate Symmetry -1

If $x(t) \overset{\mathfrak{I}}{\leftrightarrow} X(j\omega)$ then $x^*(t) \overset{\mathfrak{I}}{\leftrightarrow} X^*(-j\omega)$

If x(t) is real, x(t) = x*(t), hence $X^*(j\omega) = X(-j\omega)$



[Re{X(jω) + j Im{X(jω)}]*
= Re{X(-jω)} + j Im{X(-jω)}

Conjugation and Conjugate Symmetry -2

For real x(t):

$$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$ $\Rightarrow$ Even function of ω

$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$ $\Rightarrow$ Odd function of ω

In polar form:

$$|X(j\omega)| = |X(-j\omega)|$$ $\Rightarrow$ Even function of ω

$$\angle X(j\omega) = -\angle X(-j\omega)$$ $\Rightarrow$ Odd function of ω

For real and even x(t):

$$x(t) \overset{\mathfrak{I}}{\leftrightarrow} X(j\omega)$$

$$\mathrm{Even}\{x(t)\} \overset{\mathfrak{I}}{\leftrightarrow} \mathrm{Re}\{X(j\omega)\}$$

$$\mathrm{Odd}\{x(t)\} \overset{\mathfrak{I}}{\leftrightarrow} j\,\mathrm{Im}\{X(j\omega)\}$$

Example 4.10

Evaluate Fourier transform of $x(t) = e^{-a|t|}$ for a > 0

Example 4.1:

$$x(t) = e^{-at}u(t), \quad a > 0 \quad \Rightarrow \quad e^{-at}u(t) \overset{\Im}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2\mathrm{Even}\{e^{-at}u(t)\}$$

t > 0   t < 0

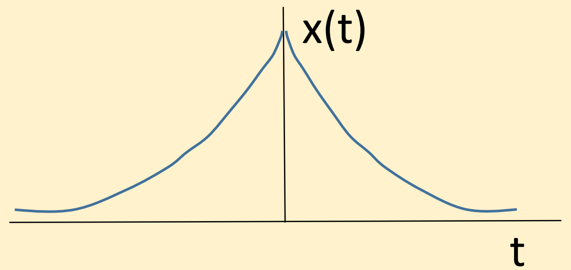
$e^{-at}u(t)$ is real; from symmetric property,

$$2\mathrm{Even}\{e^{-at}u(t)\} \overset{\Im}{\longleftrightarrow} 2\,\mathrm{Re}\{e^{-at}u(t)\}$$

$$\Rightarrow X(j\omega) = 2\,\mathrm{Re}\left\{\frac{1}{a + j\omega}\right\} = 2\,\mathrm{Re}\left\{\frac{a - j\omega}{a^2 + \omega^2}\right\}$$

$$= \frac{2a}{a^2 + \omega^2}$$

$$x(t) = e^{-a|t|}, \quad \text{for } a > 0$$



Differentiation and Integration

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \longrightarrow \quad x(t) \overset{\Im}{\leftrightarrow} X(j\omega)$$

Differentiating both sides w.r.t. time *t*

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)j\omega e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(j\omega X(j\omega)\right)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} \overset{\Im}{\leftrightarrow} j\omega X(j\omega)$$

Similarly,

$$\int_{-\infty}^{\infty} x(\tau)d\tau \overset{\Im}{\leftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Example 4.11

Determine the Fourier transform of the unit step function.

$$x(t) = u(t) \Rightarrow X(j\omega) = ?$$

$$\frac{du}{dt} = \delta(t); \quad \text{For unit impulse}, g(t) = \delta(t) \Rightarrow G(j\omega) = 1$$

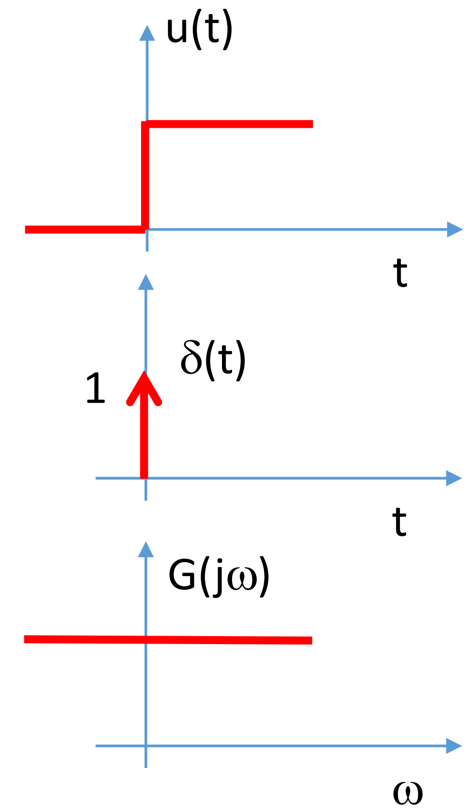
Now,

$$x(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \Rightarrow X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi \overset{1}{G(0)}\delta(\omega)$$

$$\Rightarrow X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Also, we observe that

$$\delta(t) = \frac{du}{dt} \overset{\Im}{\leftrightarrow} j\omega\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\} = 1 + \pi \overset{0}{\cancel{j\omega\delta(\omega)}} = 1$$



Time and Frequency Scaling

$$\text{If } \; x(t) \overset{\mathfrak{I}}{\leftrightarrow} X(j\omega)$$

$$\text{Then } \; x(at) \overset{\mathfrak{I}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Where *a* is real and nonzero

Proof:

$$\mathfrak{I}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

Let, τ = at, and we get two cases:

i) When a > 0:

$$\tau = at \Rightarrow d\tau = adt \Rightarrow dt = \frac{1}{a} d\tau$$

$$\text{At } t = -\infty, \tau = -\infty; \quad \text{at } \; t = \infty, \tau = \infty$$

$$\mathfrak{I}\{x(at)\} = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau/a} (1/a) d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau = \frac{1}{a} X\left(\frac{j\omega}{a}\right)$$

Time and Frequency Scaling - contd.

ii) When a > 0:

$$\tau = at \Rightarrow d\tau = adt \Rightarrow dt = \frac{1}{a} d\tau$$

At $t = -\infty, \tau = \infty$; at $t = \infty, \tau = -\infty$

$$\Im\{x(at)\} = \int_{\infty}^{-\infty} x(\tau) e^{-j\omega\tau/a} (1/a) d\tau$$

$$= \frac{-1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau = \frac{-1}{a} X\left(\frac{j\omega}{a}\right)$$

Therefore,

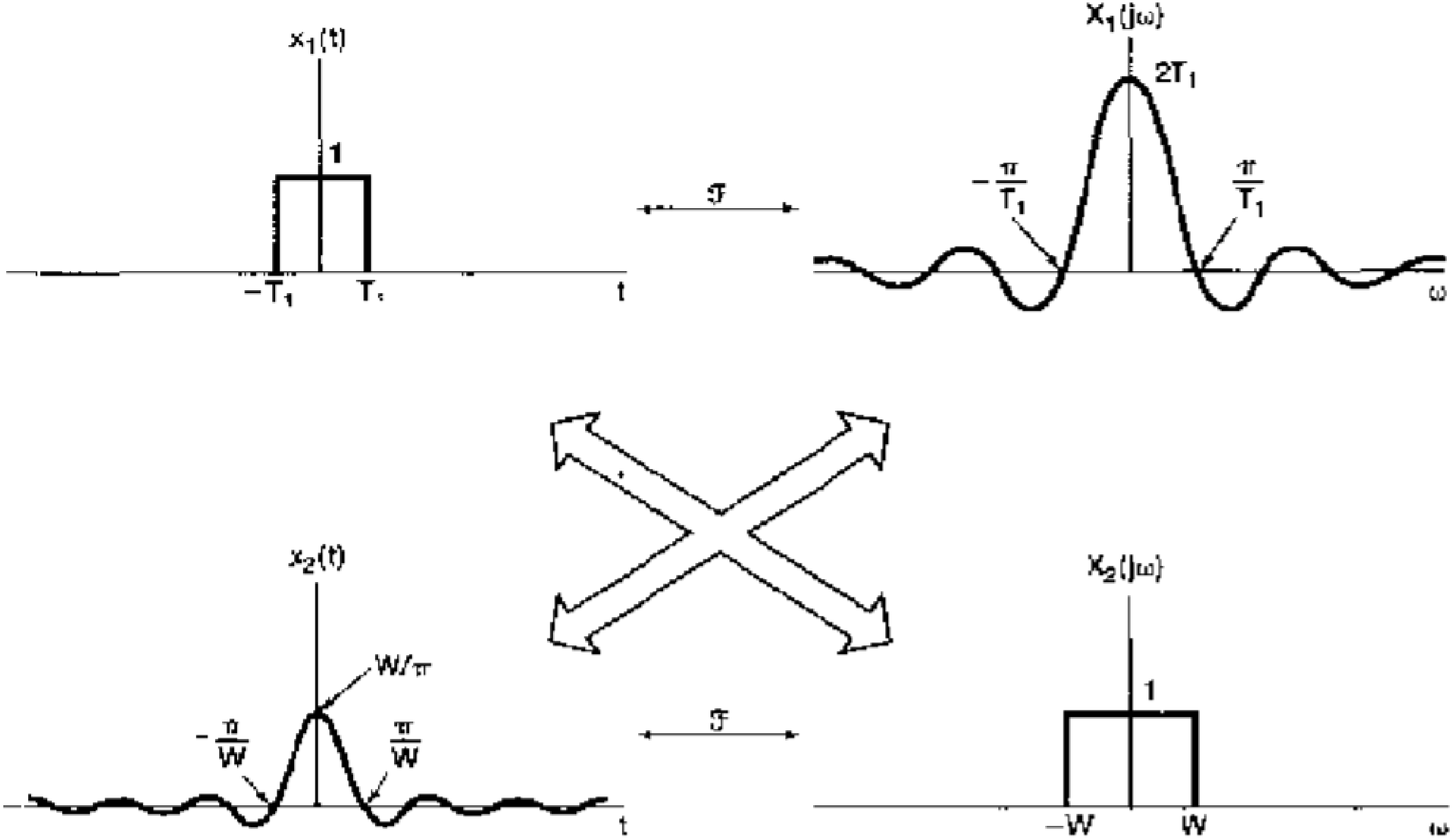
$$\Im\{x(at)\} = \begin{cases} \frac{1}{a} X\left(\frac{j\omega}{a}\right), & \text{for } a > 0 \\ \frac{-1}{a} X\left(\frac{j\omega}{a}\right), & \text{for } a < 0 \end{cases} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

In particular,

$$x(-t) \overset{\Im}{\leftrightarrow} X(-j\omega) \Rightarrow$$

Reversing a signal in time reverses its Fourier transform also.

Duality



$x_1(t)$
1
$-T_1$
T_1
t
$\mathfrak{F}$
$X_1(j\omega)$
$2T_1$
$-\frac{\pi}{T_1}$
$\frac{\pi}{T_1}$
ω
$x_2(t)$
W/π
$-\frac{\pi}{W}$
$\frac{\pi}{W}$
t
$\mathfrak{F}$
$X_2(j\omega)$
1
$-W$
W
ω

Example 4.13

What is the Fourier transform, $G(j\omega)$ of $g(t) = 2 / (1 + t^2)$?

From example 4.2: $x(t) = e^{-a|t|}, a > 0 \;\Rightarrow X(j\omega) = \dfrac{2a}{a^2 + \omega^2}$

For a = 1, $x(t) = e^{-|t|} \;\Rightarrow X(j\omega) = \dfrac{2}{1+\omega^2}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \Rightarrow \quad 2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega$$

Interchanging *t* and ω, and then substituting *t* by $-t$,

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{j\omega t}dt = \int_{\infty}^{-\infty}\left(\frac{2}{1+(-t)^2}\right)e^{-j\omega t}(-dt) = \boxed{\int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt}$$

g(t)

G(jω)

$$G(j\omega) = 2\pi e^{-|\omega|}$$

Convolution

For an LTI system:

$$y(t) = h(t) * x(t) \overset{\Im}{\longleftrightarrow} Y(j\omega) = X(j\omega)H(j\omega)$$

A convolution in time domain implies a multiplication in Fourier domain.

Example 4.15

An impulse response of an LTI system: $h(t) = \delta(t - t_0)$

The frequency response of the system: $H(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$

The Fourier transform of the output: $Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0} X(j\omega)$

This is consistent with the time shifting property.

$$y(t) = x(t - t_0)$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt$$

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$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega t} d\tau = e^{-j\omega t_0} X(j\omega)$$

Examples 4.16, 4.17

$$x(t) \rightarrow \boxed{\text{LTI System}} \rightarrow y(t)$$

$$y(t) = \frac{dx(t)}{dt}$$

Differentiator

From differential property,

$$Y(j\omega) = j\omega X(j\omega)$$

This implies that

$$\boxed{H(j\omega) = j\omega}$$

Frequency response of a differentiator.

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Integrator

The impulse response of the system is a unit step, u(t).

$$h(t) = u(t) \overset{\Im}{\Rightarrow} H(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt$$

From example 4.11

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)X(j\omega)$$

$$= \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(j\omega)$$

$$= \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(0)$$

Example 4.19

Find the response, y(t) of an LTI system, if $x(t) = e^{-bt}u(t),\ \ h(t) = e^{-at}u(t);\ \ a > 0, b > 0$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{a+j\omega}$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-bt}u(t)e^{-j\omega t}dt = \frac{1}{b+j\omega}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right)$$

$$\left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega}$$

$$\Rightarrow 1 = A(b+j\omega) + B(a+j\omega)$$

$$\Rightarrow 1 = Ab + Ba + j\omega(A+B)$$

$$Ab + Ba = 1;\ \ A + B = 0 \Rightarrow A = \frac{1}{b-a} = -B$$

$$Y(j\omega) = \frac{1}{b-a}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right)$$

Example 4.19 - contd.

$$e^{-at}u(t) \overset{\Im}{\leftrightarrow} \frac{1}{a + j\omega} \quad \text{and} \quad e^{-bt}u(t) \overset{\Im}{\leftrightarrow} \frac{1}{b + j\omega}$$

This implies, $\boxed{y(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t), \quad b \neq a}$

If b = a, $\quad Y(j\omega) = \left(\frac{1}{a + j\omega}\right)\left(\frac{1}{b + j\omega}\right) = \frac{1}{(a + j\omega)^2}$

We know,

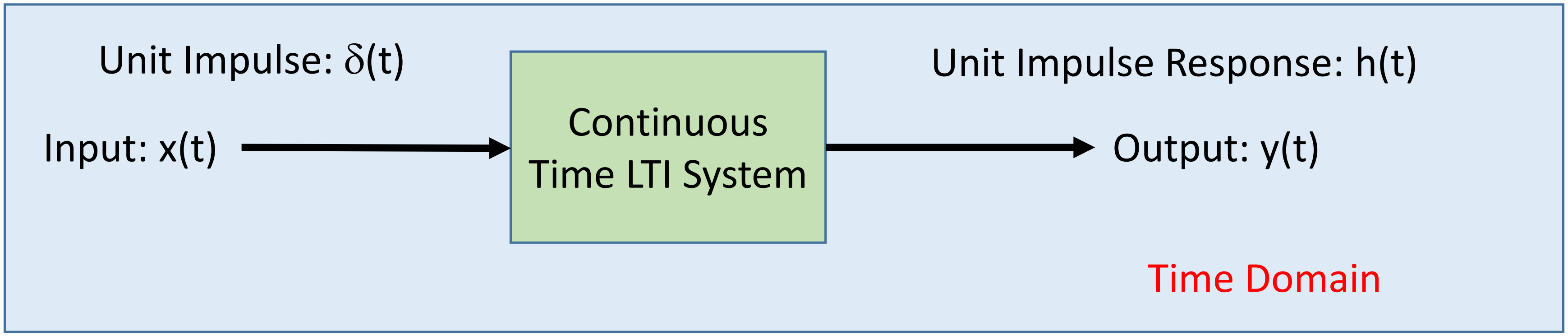
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)}{v^2} \Rightarrow \frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right) = \frac{\left((a + j\omega)(0) - (1)(j)\right)}{(a + j\omega)^2}$$

$$\Rightarrow j\frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right) = \frac{1}{(a + j\omega)^2} = Y(j\omega)$$

Table 4.1

$$tx(t) \overset{\Im}{\leftrightarrow} j\frac{d}{d\omega}X(j\omega) \Rightarrow t\left(e^{-at}u(t)\right) \overset{\Im}{\leftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right) \Rightarrow \boxed{y(t) = te^{-at}u(t), \quad a = b}$$

Some Important Cont.-Time Relationship



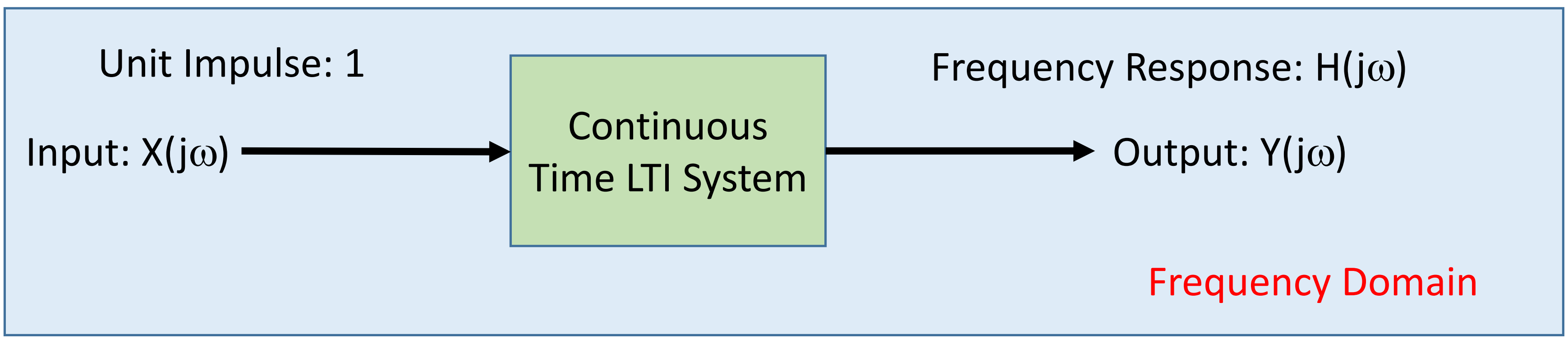


TABLE 4.1 PROPERTIES OF FOURIER TRANSFORM

Property	Aperiodic Signal	Fourier Transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\, Y\big(j(\omega - \theta)\big)d\theta$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega) = X^*(-j\omega)$ $\mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\}$ $\mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\}$ $\|X(j\omega)\| = \|X(-j\omega)\|$ $\sphericalangle X(j\omega) = -\sphericalangle X(-j\omega)$
Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\}, & x(t) \text{ real} \\ x_o(t) = \mathcal{O}d\{x(t)\}, & x(t) \text{ real} \end{cases}$	$\mathcal{Re}\{X(j\omega)\}$ $j\,\mathcal{Im}\{X(j\omega)\}$
Parseval's Relation for Aperiodic Signals		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Selected

Signal	Fourier Transform
$x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2\sin(\omega T_1)}{\omega}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}\,u(t), \quad \mathcal{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}\,u(t), \quad \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}\,u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$