Ch 2: Linear Time-Invariant System

A system is said to be *Linear Time-Invariant* (LTI) if it possesses the basic system properties of linearity and time-invariance.

Consider a system with an output signal $y(t)$ corresponding to an input signal $x(t)$. The system will be called a time-invariant system, if for an arbitrary time shift T_0 in the input signal, i.e., $x(t + T_0)$, the output signal is time-shifted by the same amount T_0, i.e., $y(t + T_0)$.

If $x_k[n]$, $k = 1, 2, 3, \dots$, are a set of inputs to a discrete-time linear system with corresponding outputs $y_k[n]$, $k = 1, 2, 3, \dots$, then we get

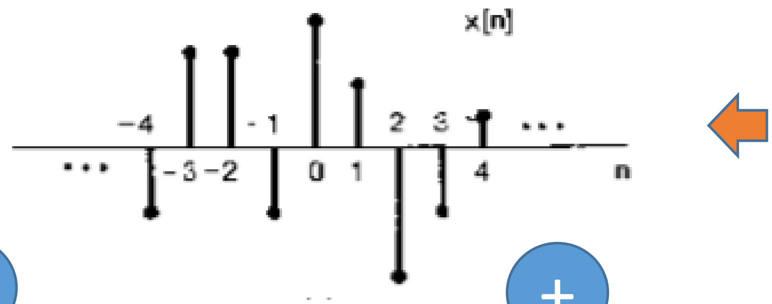
INPUT: $$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \cdots$$

OUTPUT: $$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \cdots$$

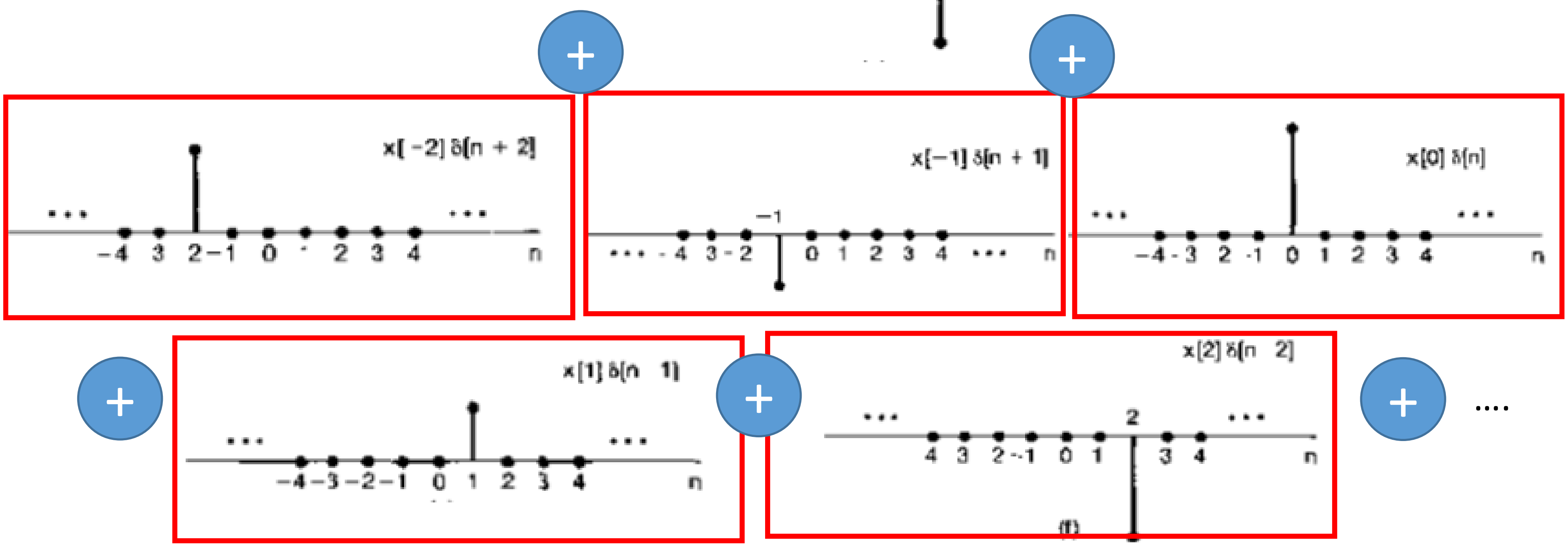
2. Discrete-Time LTI Systems: the Convolution Sum

Any discrete-time signal x[n] can be represented as a function of shifted unit impulses δ[n-k], where the weights in this linear combination are x[k].

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



Original Signal



The Convolution Sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

← Scaled impulses

For a particular case of unit step function:

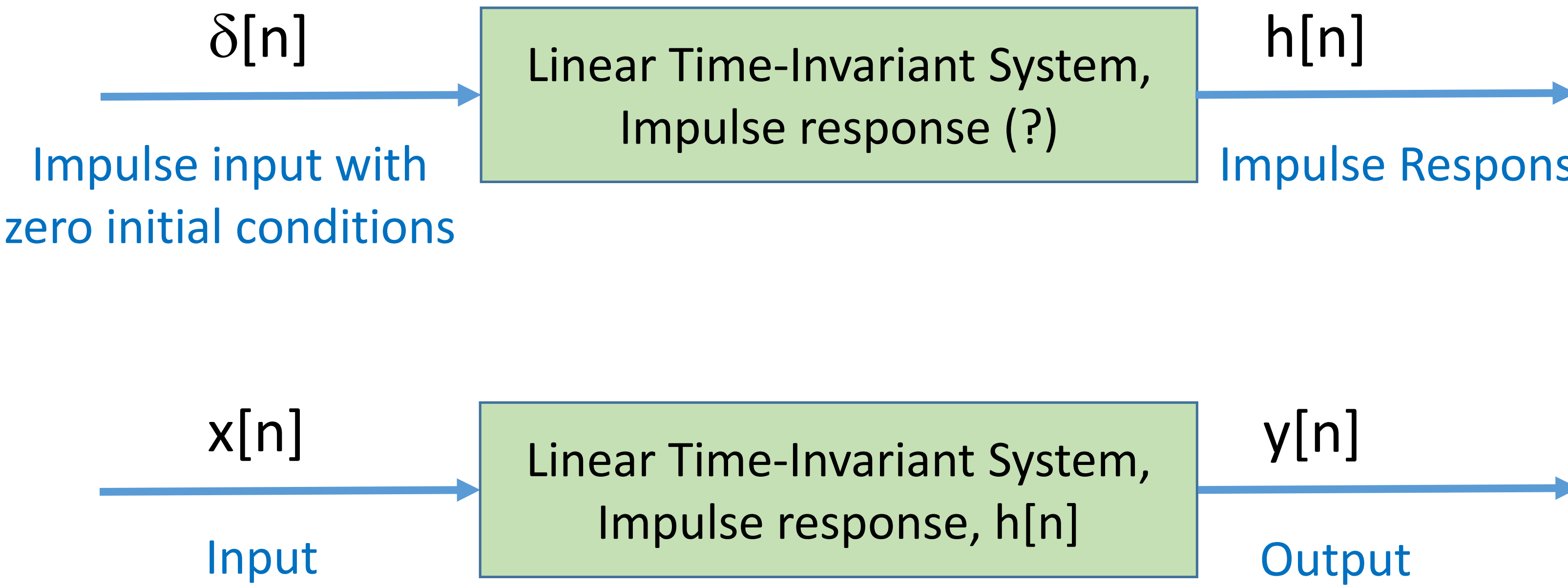
$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

We can write:

$$u[n] = \sum_{k=0}^{\infty} u[k]\delta[n-k]$$

← Shifting property of the discrete-time unit impulse

The Convolution Sum - contd.



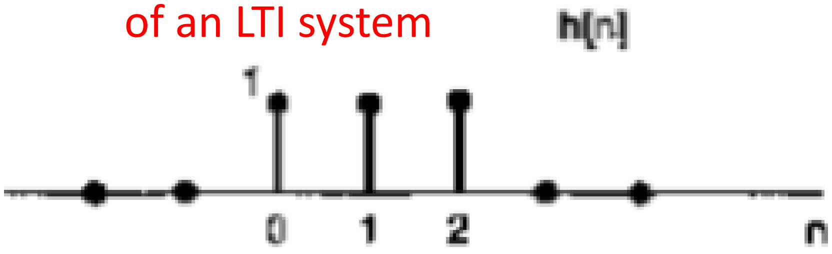
$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

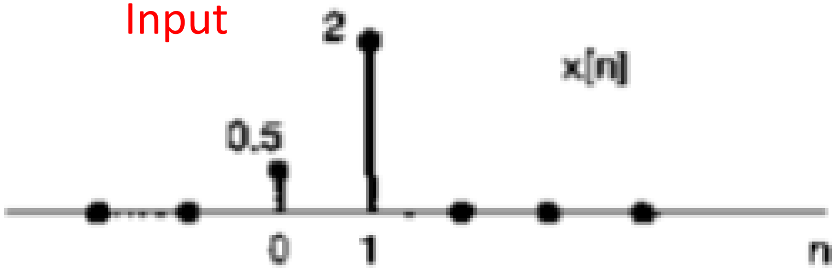
Convolution

Example: Convolution – (1)

Impulse response of an LTI system



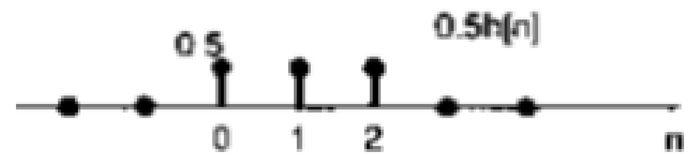
Input

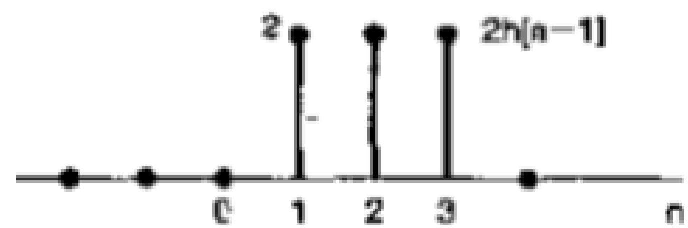


Find output.

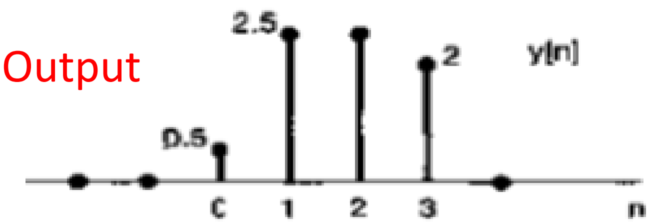
There are only two non-zero values for the input.

$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$
$$= 0.5h[n] + 2h[n-1]$$

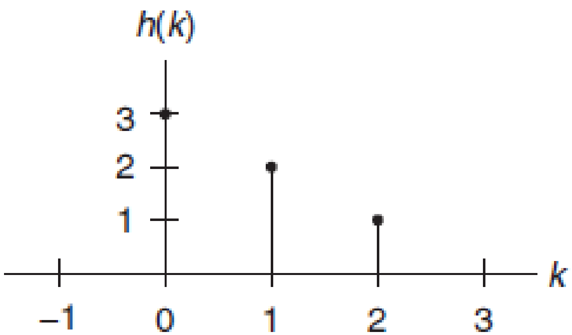




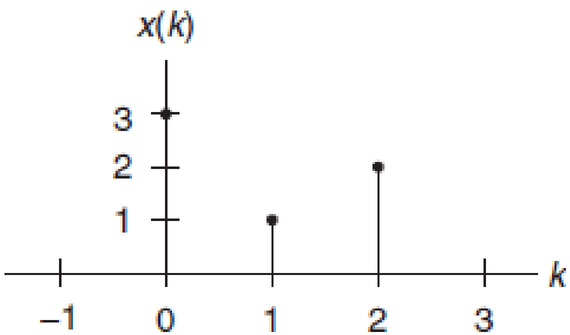
Output



Example: Convolution - (2)



Length = 3



Length = 3

Solution:

Convolution sum using the table method.

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Convolution length = 3 +3 – 1 = 5

Example: Convolution - (3)

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & otherwise \end{cases} \text{ and } h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

Length = 3

Length = 2

Solution:

k:	-2	-1	0	1	2	3	4	5	...	
$x(k)$:			**1**	**1**	**1**				...	
$h(-k)$:	1	1	0							$y(0) = 0$ (no overlap)
$h(1-k)$		1	1	0						$y(1) = 1 \times 1 = 1$
$h(2-k)$			1	1	0					$y(2) = 1 \times 1 + 1 \times 1 = 2$
$h(3-k)$				1	1	0				$y(3) = 1 \times 1 + 1 \times 1 = 2$
$h(4-k)$					1	1	0			$y(4) = 1 \times 1 = 1$
$h(n-k)$						1	1	0		$y(n) = 0,\ n \geq 5$ (no overlap)
										Stop

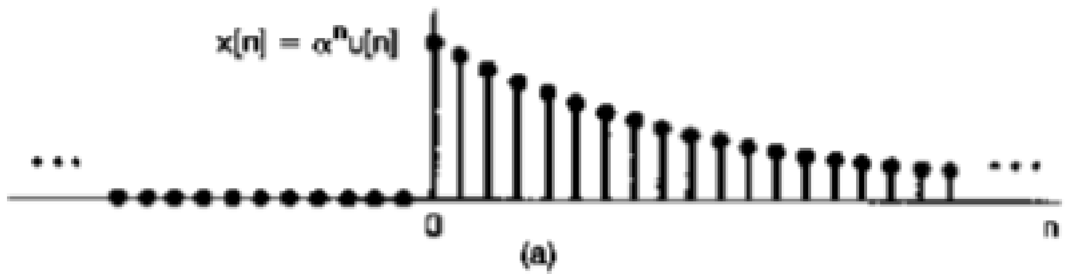
Convolution length = 3 + 2 – 1 = 4

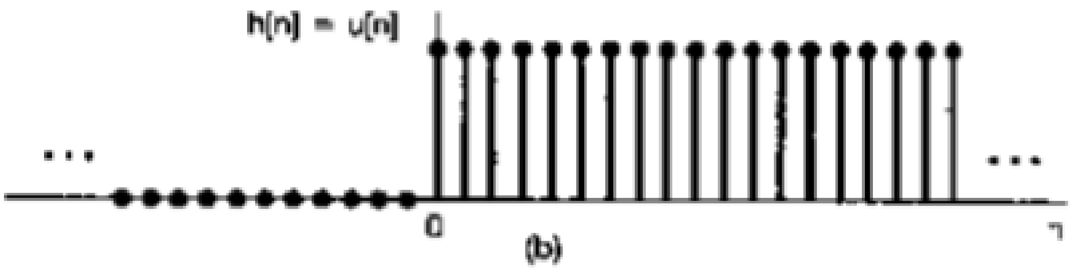
Example: Convolution – (4)

Find the output of an LTI system having unit impulse response, h[n], for the input, x[n], as given below.

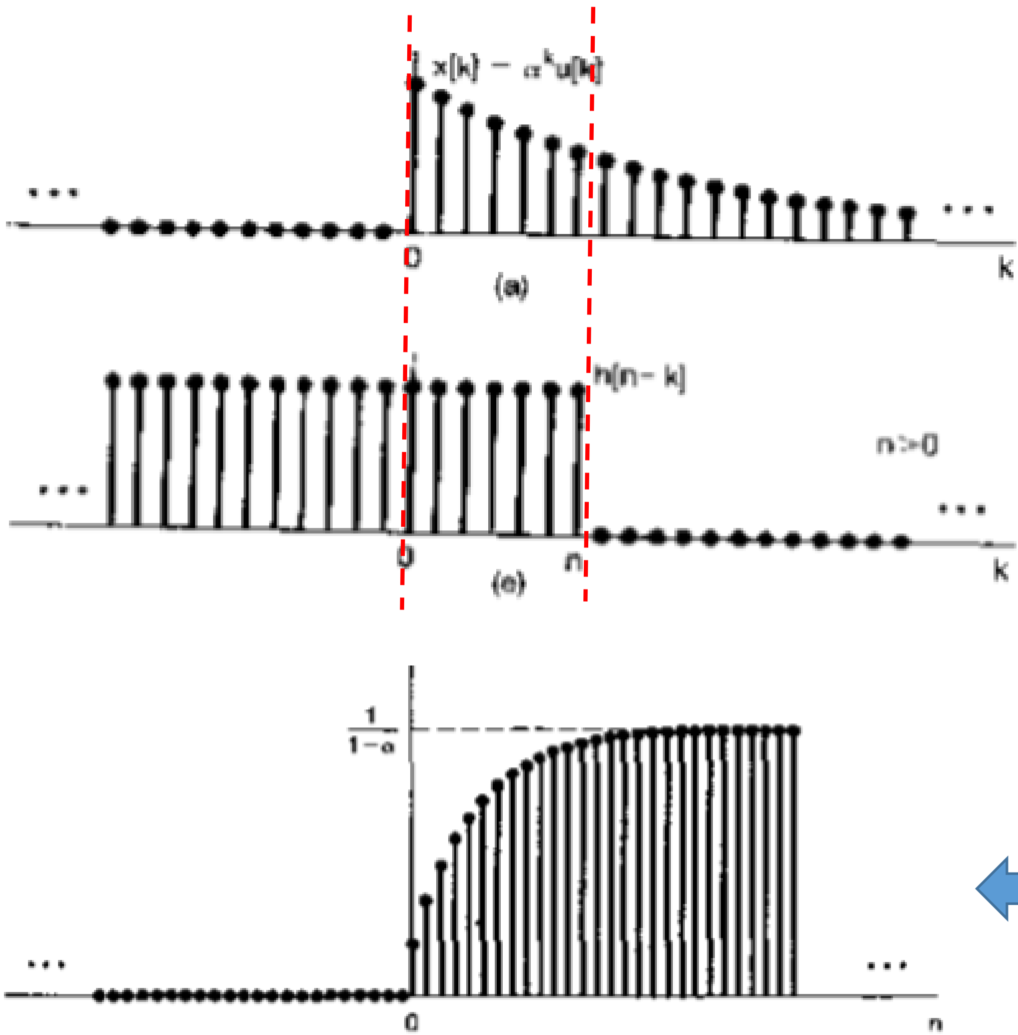
$$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$$

$$h[n] = u[n]$$





Example: Convolution - (4) - contd.



$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

Thus, for n ≥ 0,

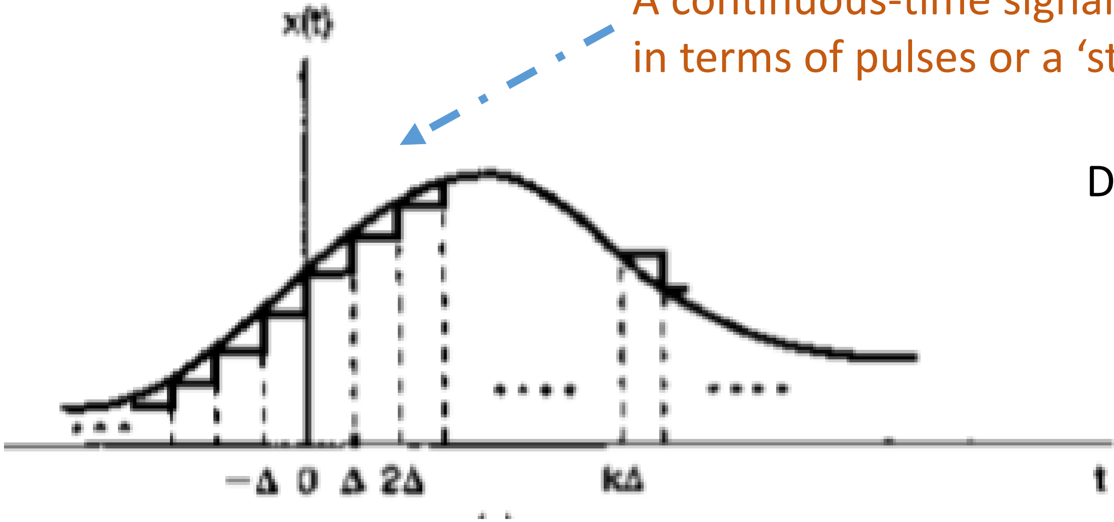
$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}, \quad \text{for } n \ge 0$$

Thus, for all *n*,

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n].$$

Representation of Continuous-Time Signals in Terms of Impulses

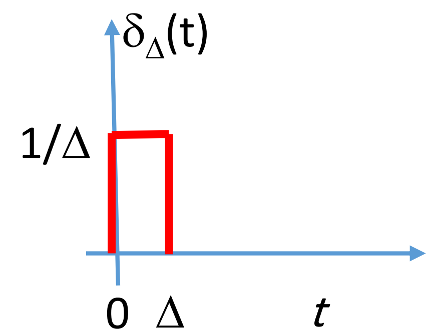
A continuous-time signal, x(t), is approximated in terms of pulses or a ‘staircase’.



Defining:

$$\delta_\Delta(t) = \begin{cases} 1/\Delta, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \Delta\delta_\Delta(t) = \begin{cases} 1, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases}$$

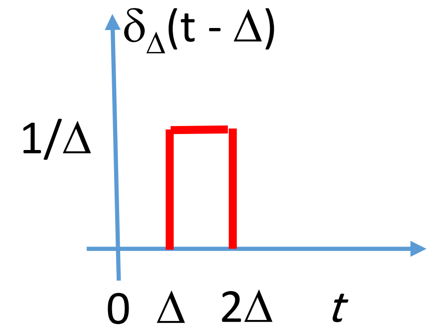


Pulse or ‘staircase’ approximation of x(t) at t = 0:

$$\hat{x}(0) = x(0)\Delta\delta_\Delta(t) = \begin{cases} x(0), & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases}$$

Continuous-Time Signals in Terms of Impulses - contd.

Going one step further, shifted δ_Δ can be written as:



$$\delta_\Delta(t-\Delta) = \begin{cases} 1/\Delta, & \Delta \le t \le 2\Delta \\ 0, & \text{otherwise} \end{cases}$$

Pulse or 'staircase' approximation of x(t) at t = Δ:

$$\hat{x}(\Delta) = x(\Delta)\Delta\delta_\Delta(t-\Delta) = \begin{cases} x(\Delta), & \Delta \le t \le 2\Delta \\ 0, & \text{otherwise} \end{cases}$$

Continuous-Time Signals in Terms of Impulses - contd.

In general, for an arbitrary *k*, we write

$$\delta_\Delta(t-k\Delta)=\begin{cases}1/\Delta, & k\Delta\le t\le(k+1)\Delta\\ 0, & \text{otherwise}\end{cases}$$

Pulse or 'staircase' approximation of x(t) at t = kΔ:

$$\hat{x}(k\Delta)=x(k\Delta)\Delta\delta_\Delta(t-k\Delta)=\begin{cases}x(k\Delta), & k\Delta\le t\le(k+1)\Delta\\ 0, & \text{otherwise}\end{cases}$$

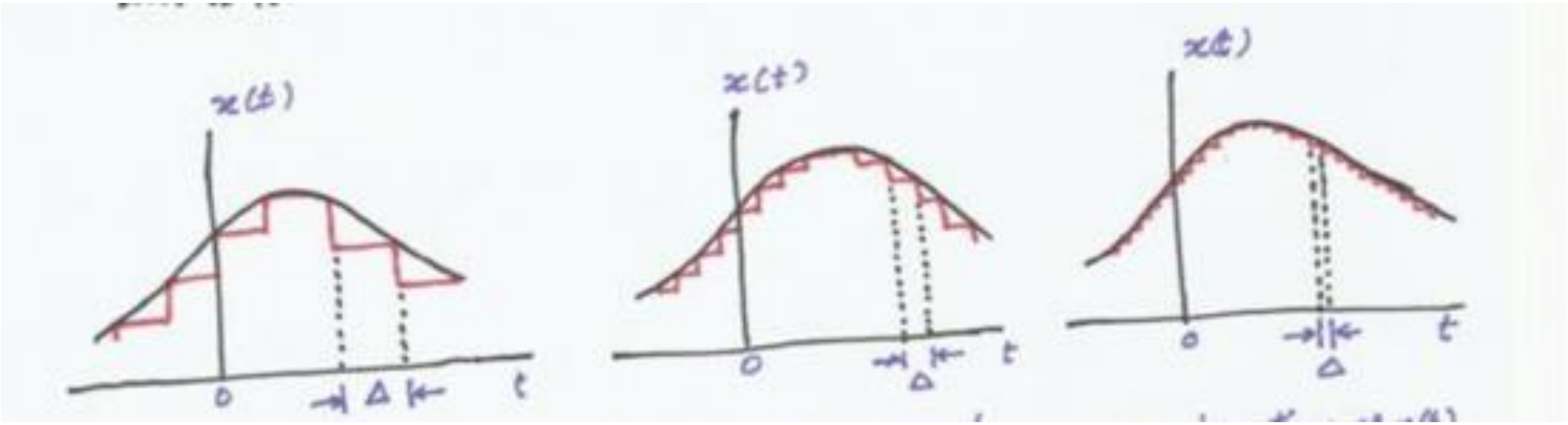
Combining all individual approximations, we get the complete pulse/staircase approximation of x(t) as:

$$\hat{x}(t)=\ldots+\hat{x}(-\Delta)+\hat{x}(0)+\hat{x}(\Delta)+\ldots=\sum_{k=-\infty}^{\infty}\hat{x}(k\Delta)=\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Continuous-Time Signals in Terms of Impulses - contd.

If we keep on reducing the value of Δ, the approximation becomes closer and closer to the original value.

$$x(t) = \lim_{\Delta \to 0} \hat{x}(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$



In the limiting case:

$$\Delta \to 0; \quad \delta_{\Delta}(t) \to \delta(t)$$

$$\sum_{k=-\infty}^{\infty} \ldots\Delta \to \int_{\tau=-\infty}^{\infty} \ldots d\tau$$

Consequently,

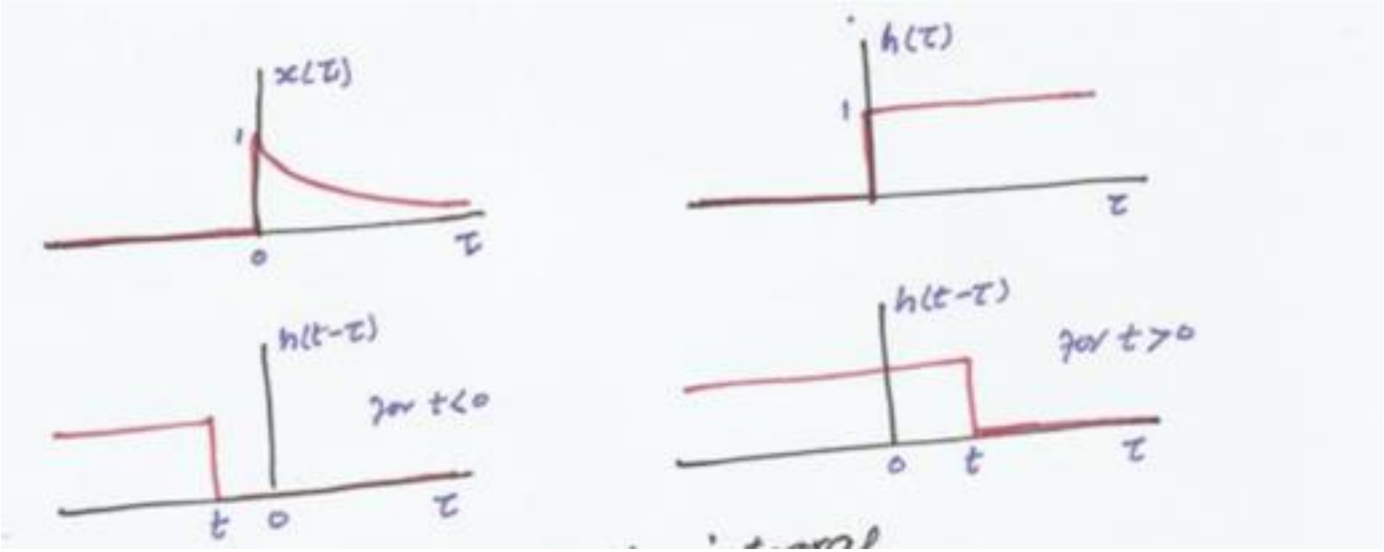
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Let, the input x(t) to an LTI system with unit impulse response h(t) be given as **$x(t) = e^{-at}u(t)$** for a > 0 and **h(t) = u(t).**

Find the output y(t) of the system.



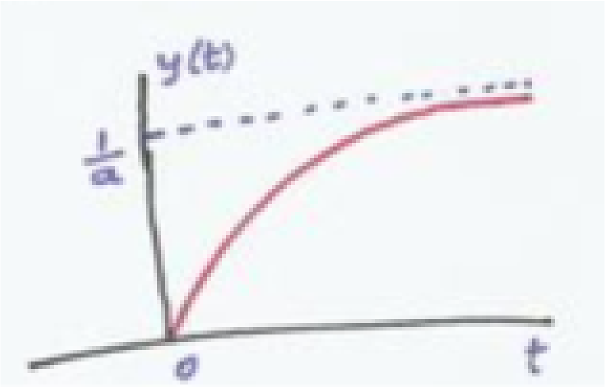
The Convolution Integral - contd.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} e^{-a\tau} h(t-\tau)d\tau, \quad \text{for } t > 0$$

$$= \int_{0}^{t} e^{-a\tau} d\tau = e^{-a\tau} . \frac{-1}{a} \Big|_{0}^{t} = \frac{1}{a}\left(1 - e^{-at}\right)$$

Thus, for all *t*, we can write

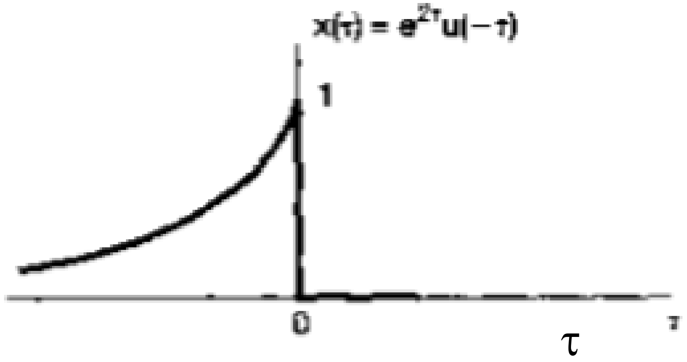
$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$$



Example: The Convolution Integral

Find y(t) = x(t)*h(t), where $x(t) = e^{2t}u(-t)$

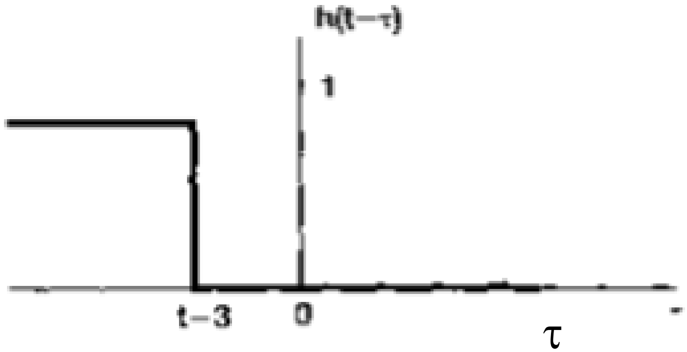
$h(t) = u(t-3)$



We observe that there is nonzero overlap.

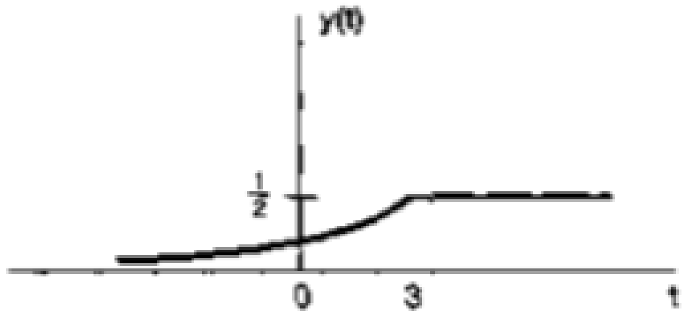
t-3 ≤ 0:

For $-\infty < \tau < t-3, \; y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}$

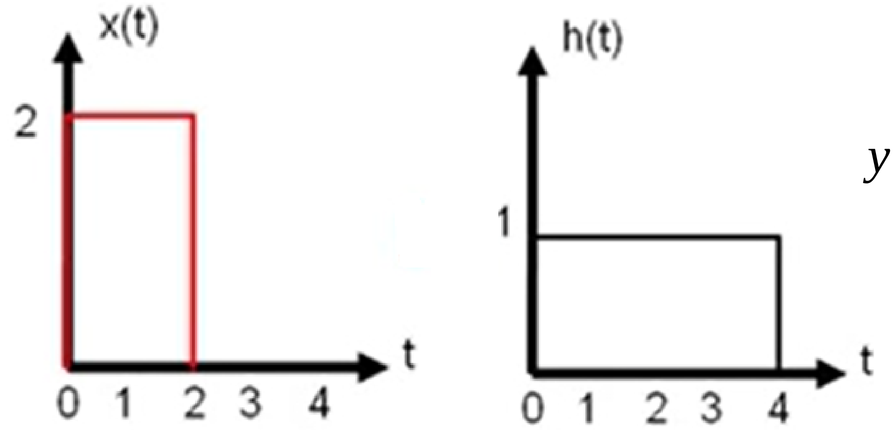


t-3 > 0:

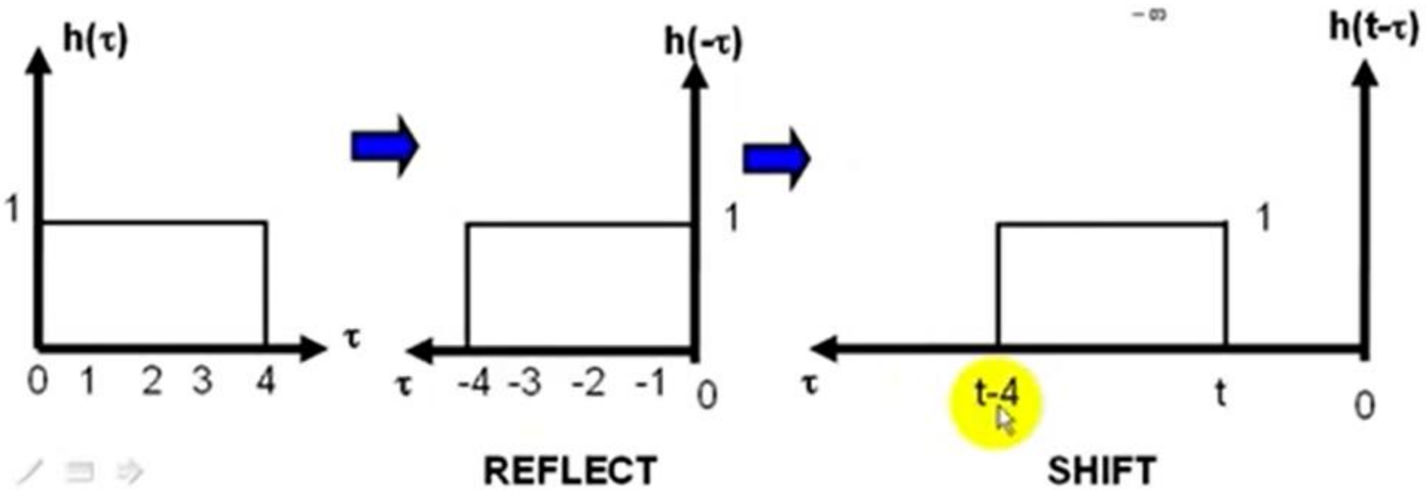
For $-\infty < \tau < 0, \; y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}$



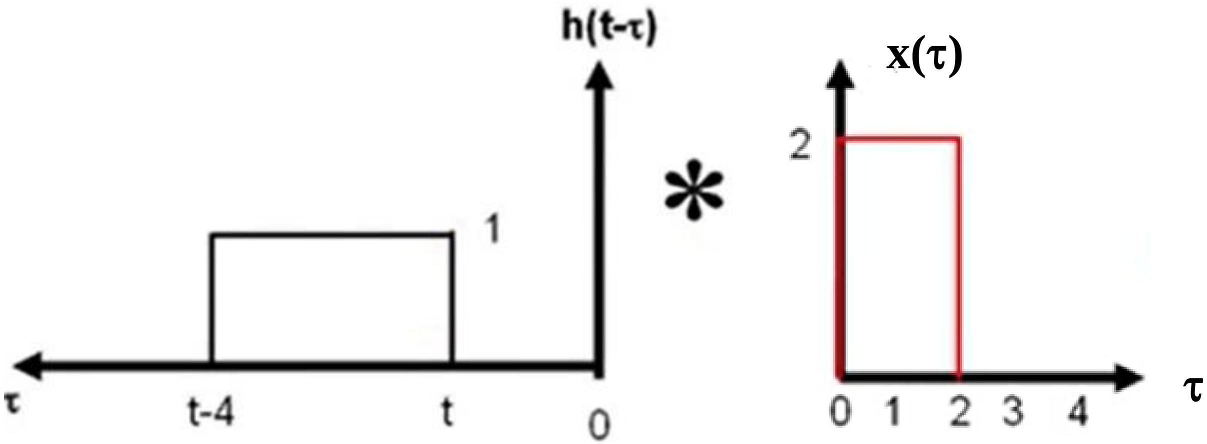
Example: The Convolution Integral

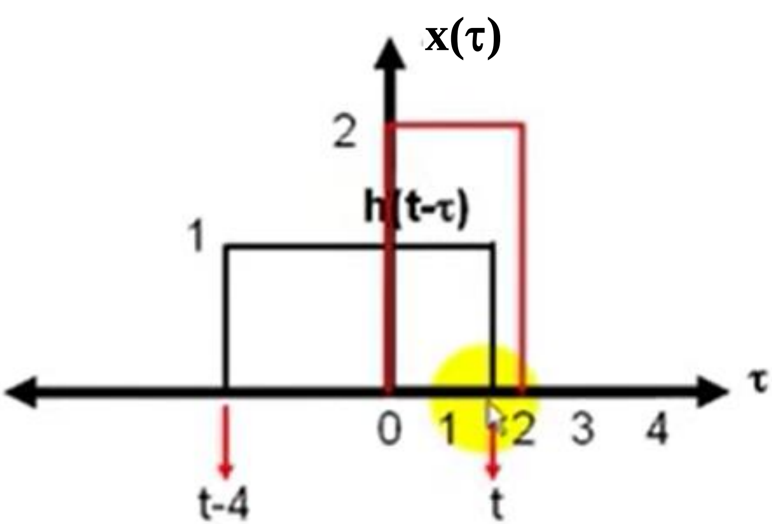


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$



Example: The Convolution Integral - contd.

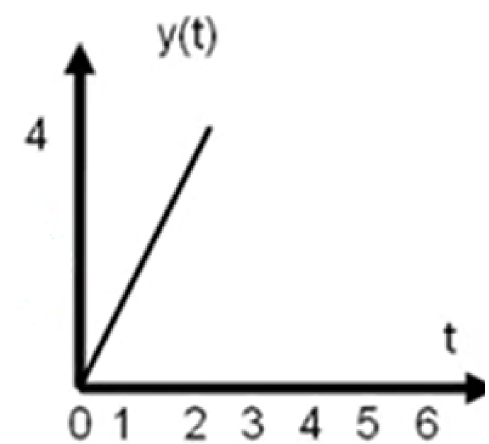




$$y(t) = \int_0^t 2 \cdot 1 d\tau \quad 0 < \tau < t$$

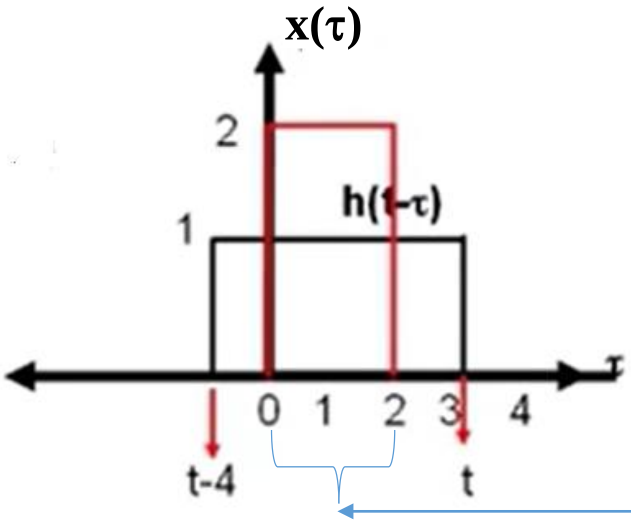
$$y(t) = 2 \cdot (\tau |_0^t)$$

$$y(t) = 2t$$



$t < 2$

Example: The Convolution Integral - contd.



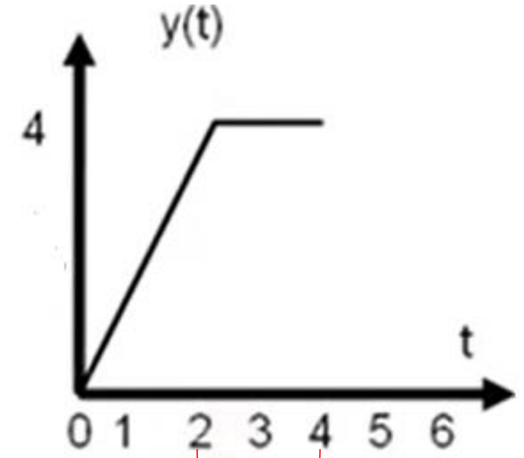
$t-4<0 \qquad t>2$

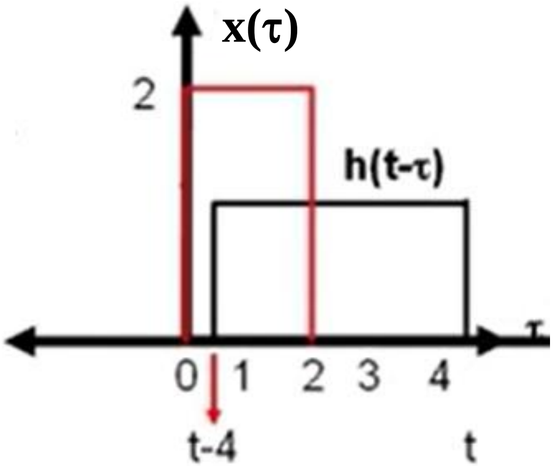
$2<t<4$

$0<\tau<2$

$$y(t)=\int_{0}^{2} 2\,d\tau$$

$$y(t)=4$$





$$y(t)=\int_{t-4}^{2} 2\cdot 1\,d\tau$$

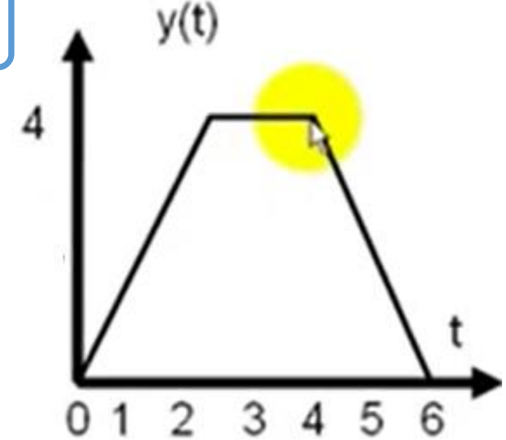
$t-4<\tau<2$

$t<6$

$$y(t)=2\cdot\left(\tau\Big|_{t-4}^{2}\right)$$

$$y(t)=2(2-(t-4))$$

$$y(t)=-2t+12$$



Properties of LTI Systems

- The characteristics of an LTI system are completely determined by its impulse response. This property holds in general for LTI systems only.

- The unit impulse response of a nonlinear system does not completely characterize the behavior of the system.

Consider a discrete-time system with unit impulse response: $h[n] = \begin{cases} 1, & n = 0,1 \\ 0, & otherwise \end{cases}$

If the system is LTI, we get (by convolution): $y[n] = x[n] + x[n-1]$

There is only one such LTI system for the given h[n].

However, there are many nonlinear systems with the same response, h[n].

$$y[n] = \left(x[n] + x[n-1]\right)^2$$

$$y[n] = \max\left(x[n], x[n-1]\right)$$

Commutative Property

$$x(t) * h(t) = h(t) * x(t)$$

$$x[n] * h[n] = h[n] * x[n]$$

Proof: (discrete domain)

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

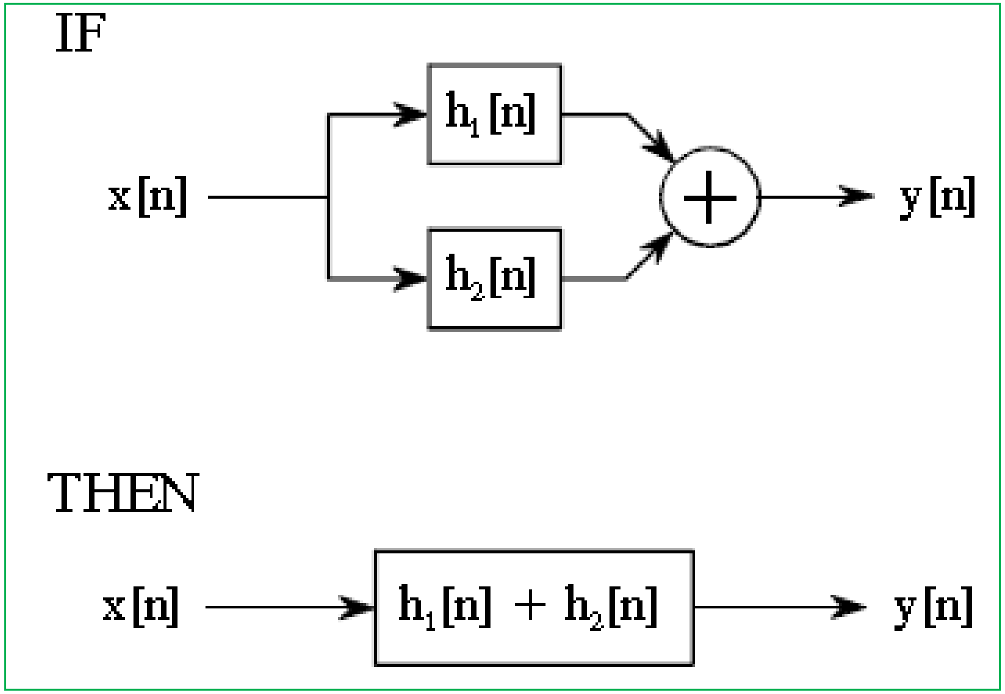
Put r = n – k $\Rightarrow$ k = n - r

$$x[n] * h[n] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = \sum_{r=-\infty}^{\infty} h[r]x[n-r] = h[n] * x[n]$$

Similarly, we can prove it for continuous domain.

Distributive Property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$



Example: Distributive Property

y[n] = x[n] * h[n]

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

$$h[n] = u[n]$$

x[n] in nonzero for entire *n*, so direct convolution is difficult. Therefore, we will use commutative property.

$$y[n] = x[n] * h[n] = \left(x_1[n] + x_2[n]\right) * h[n] = \left(x_1[n] * h[n] + x_2[n] * h[n]\right) = y_1[n] + y_2[n]$$

$$y_1[n] = x_1[n] * h[n] = \sum_{k=-\infty}^{\infty} x_1[k] h[n-k] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] u[n-k]$$

$$= \left(\frac{1-(1/2)^{n+1}}{1-(1/2)}\right) u[n] = 2\left(1-(1/2)^{n+1}\right) u[n]$$

$$y_2[n] = x_2[n] * h[n] = \sum_{k=-\infty}^{\infty} 2^k u[-k] u[n-k] = 2^{n+1}$$

$$y[n] = y_1[n] + y_2[n] = 2\left(1-(1/2)^{n+1}\right) u[n] + 2^{n+1}$$

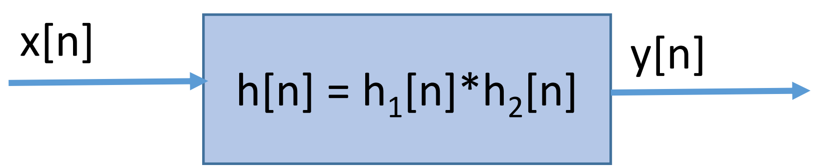
Associative Property

$$x(t)*\left[h_1(t)*h_2(t)\right]=\left[x(t)*h_1(t)\right]*h_2(t)$$

$$x[n]*\left[h_1[n]*h_2[n]\right]=\left[x[n]*h_1[n]\right]*h_2[n]$$

(A)

(B)



Proof: From (A), $y[n]=w[n]*h_2[n]=\left(x[n]*h_1[n]\right)*h_2[n]$

From (B), $y[n]=x[n]*h[n]=x[n]*(h_1[n]*h_2[n])$

LTI Systems With and Without Memory

A discrete-time LTI system can be memoryless if only: $h[n] = 0, \quad \text{for } n \neq 0$

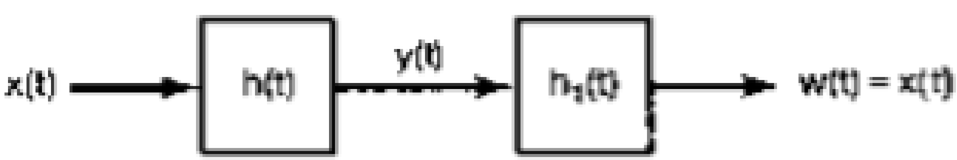
Thus, the impulse response have the form: $h[n] = K\,\delta[n], \quad K \text{ is a constant}$

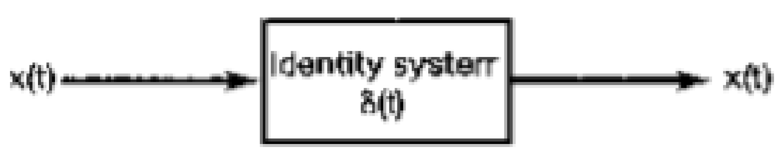
$$y[n] = Kx[n]$$

If K = 1, then the system is called identity system.

Similarly for continuous LTI systems.

Invertibility of LTI Systems:





The system with impulse response $h_1[n]$ is inverse of the system with impulse response h(t), if

$$h(t) * h_1(t) = \delta(t)$$

Example: LTI Systems Properties

Consider, the following LTI system with pure time-shift.

$$y(t) = x(t - t_0)$$

- Such a system is a 'delay' if $t_0 > 0$, and an 'advance' if $t_0 < 0$.
- If $t_0 = 0$, the system is an identity system and is memoryless.
- For $t_0 \neq 0$, the system has memory.
- The impulse response of the system is $h(t) = \delta(t - t_0)$, therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0)$$

The convolution of a signal with a shifted impulse simply shifts the signal.

To recover the input (i.e. to invert the system), we simply need to shift the output back.

The impulse response of the inverted system: $h_1(t) = \delta(t + t_0)$

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

Example: LTI Systems Properties

Consider, an LTI system with impulse response: $h[n] = u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

- The system is a summer or accumulator.
- The system is invertible and its inverse is given by: $y[n] = x[n] - x[n-1]$, which is a first difference equation.

By putting, $x[n] = \delta[n]$, we find the impulse response of the inverse system: $h_r[n] = \delta[n] - \delta[n-1]$

To check that $h[n]$ and $h_r[n]$ are impulse responses of the systems that are inverse of each other, we do the following calculation:

$$h[n]*h_1[n] = u[n]*\{\delta[n] - \delta[n-1]\}$$
$$= u[n]*\delta[n] - u[n]*\delta[n-1] = u[n] - u[n-1] = \delta[n]$$

Therefore, the two systems are inverses of each other.

Causality of LTI Systems

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$y[n]$ must not depend on $x[k]$ for $k > n$, to be causal.

Therefore, for a discrete-time LTI system to be causal: **h[n] = 0, for n < 0**.

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

Similarly, for a continuous-time LTI system to be causal:

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Both the accumulator ($h[n] = u[n]$) and its inverse ($h[n] = \delta[n] - \delta[n-1]$) are causal.

Stability of LTI Systems

Consider, an input x[n] to an LTI system that is bounded in magnitude:

$$|x[n]| < B, \quad \text{for all } n$$

Suppose that we apply this to the LTI system with impulse response h[n].

$$|y[n]| = |\sum_{k=-\infty}^{\infty} h[k]x[n-k]|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$|x[n-k]| < B$, for all n and k

$$\leq B \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n$$

Therefore, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then $|y[n]| < \infty$

Similar case in continuous-time LTI system.

If the impulse response is absolutely summable, then y[n] is bounded in magnitude, and hence the system is stable.

Example: Stability of LTI Systems

- An LTI system with pure time shift is stable.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n-n_0]| = 1$$

- An accumulator (DT domain) system is unstable.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} |u[n]| = \infty$$

- Similarly, an integrator (CT domain) system is unstable.

Unit Step Response of An LTI System

Discrete-time domain

$$s[n] = u[n] * h[n] = h[n] * u[n]$$

$$\Rightarrow s[n] = \sum_{k=-\infty}^{n} h[k]$$ ← Running Sum

$$\Rightarrow h[n] = s[n] - s[n-1]$$ ← First Difference

Continuous-time domain

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$ ← Running Integral

$$h(t) = \frac{ds(t)}{ds} = s'(t)$$ ← First Derivative

Linear Constant-Coefficient Differential Equation

Consider an LTI system described by the following differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Solve for y(t).

where the input to the system is: $x(t) = Ke^{3t}u(t)$

Particular solution

$$y(t) = y_p(t) + y_h(t)$$

Homogeneous solution

Determine

$$y_p(t) = Ye^{3t}$$

From differential equation:

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t} \Rightarrow 3Y + 2Y = K \Rightarrow Y = \frac{K}{5}$$

$$y_p(t) = \frac{K}{5}e^{3t}, \quad \text{for } t > 0$$

$$y_h(t) = Ae^{st}$$

$$Ase^{st} + 2Ae^{st} = 0 \Rightarrow A(s+2)e^{st} = 0 \Rightarrow s = -2$$

$$y_h(t) = Ae^{-2t}$$

Complete solution:

$$y(t) = \frac{K}{5}e^{3t} + Ae^{-2t}$$

Solution - contd.

- Still, the value of A is unknown. We can find it by using the auxiliary condition. Different auxiliary conditions lead to different solutions of y(t).

- Suppose that the auxiliary condition is y(0) = 0, i.e., at t = 0, y(t) = 0.

  Using this condition into the complete solution, we get:

$$0 = \frac{K}{5} + A \Rightarrow A = -\frac{K}{5}$$

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right], \quad t > 0$$
$$= \frac{K}{5}\left[e^{3t} - e^{-2t}\right]u(t)$$

A general N-th order linear constant-coefficient differential equation is given by:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

A particular case when N = 0: →

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

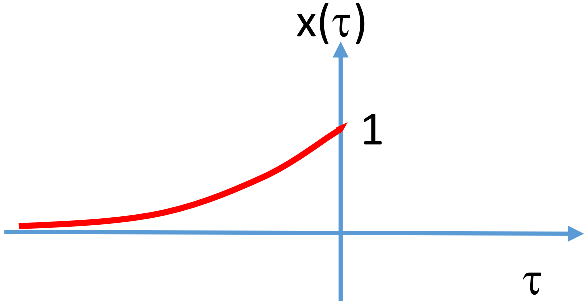
Workout (1)

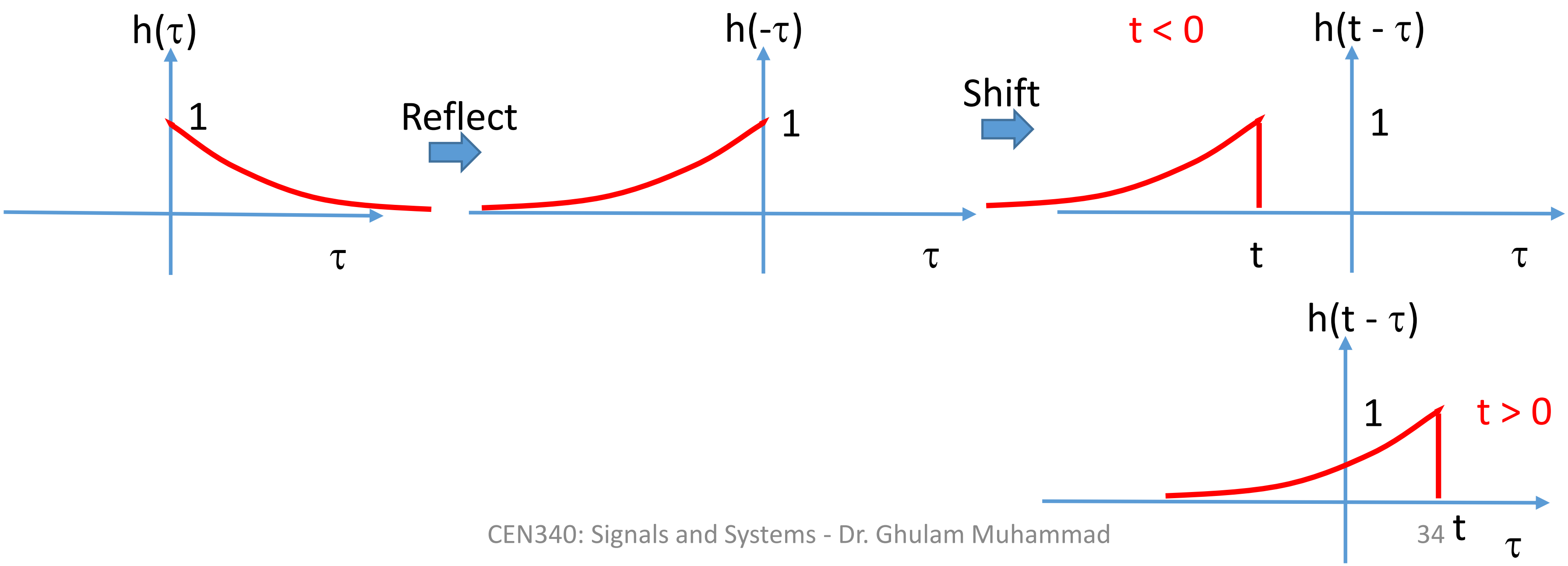
Convolution Integral:

$$x(t) = e^{at}u(-t)$$

$$h(t) = e^{-at}u(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$





Workout (1) - contd.

For t < 0, the overlap between x(τ) and h(t - τ) is between the range τ = -∞ and τ = t.

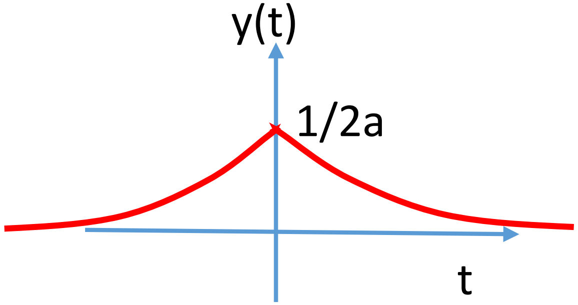
$$y(t) = \int_{-\infty}^{t} e^{a\tau} e^{-a(t-\tau)} d\tau = e^{-at} \int_{-\infty}^{t} e^{2a\tau} d\tau = e^{-at} \times \frac{e^{2a\tau}}{2a}\Bigg|_{-\infty}^{t} = \frac{e^{at}}{2a}$$

For t > 0, the overlap between x(τ) and h(t - τ) is between the range τ = -∞ and τ = 0.

$$y(t) = \int_{-\infty}^{0} e^{a\tau} e^{-a(t-\tau)} d\tau = e^{-at} \int_{-\infty}^{t} e^{2a\tau} d\tau = e^{-at} \times \frac{e^{2a\tau}}{2a}\Bigg|_{-\infty}^{0} = \frac{e^{-at}}{2a}$$

By combining,

$$y(t) = \frac{1}{2a} e^{-a|t|}$$



Acknowledgement

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- Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab, *Signals & Systems*, 2nd Edition, Prentice-Hall, Inc., 1997.

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