SIGNALS & SYSTEMS

GHULAM

INVERTIBLE & NON-INVERTIBLE SYSTEM

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

SYSTEM

Example 1: $y(t) = 3x(t) + 2$

suppose: $x(t) = 2 \Rightarrow y(t) = 8$
$x(t) = -2 \Rightarrow y(t) = -4$ → different

$\therefore$ INVERTIBLE

$2 \longrightarrow 8$
$-2 \longrightarrow -4$
one to one

Example 2: $y(t) = |x(t)|$

$x(t) = 2 \Rightarrow y(t) = 2$
$x(t) = -2 \Rightarrow y(t) = 2$ → same value

NON-INVERTIBLE

$2 \longrightarrow 2$
$-2 \nearrow$
many to one

Example 3: $y(t) = t\,x(t)$

if $t = 0$,
$x(t) = 2$
$x(t) = -2$ $\Rightarrow y(t) = 0$ → same value
$x(t) =$ any value

NON-INVERTIBLE

$2 \longrightarrow 0$
$-2 \nearrow$
$\vdots \nearrow$
many-to-one

Example 4: $y(t) = \frac{d}{dt}\{x(t)\}$

suppose: $x(t) = K$ (constant) $\Rightarrow y(t) = 0$
$x(t) = M$ (another constant) $\Rightarrow y(t) = 0$ → same value

$K \longrightarrow 0$
$M \nearrow$
many to one

NON-INVERTIBLE

INVERTIBLE / NON-INVERTIBLE SYSTEMS

GHULAM

Example 5: $y(t)=\int_{-\infty}^{t} x(\tau)d\tau$

suppose: $x(t)=K \Rightarrow y(t)=\int_{-\infty}^{t} K\,d\tau = K\int_{-\infty}^{t} d\tau$

$$= K[\tau]_{-\infty}^{t} = K[t+\infty]=\infty$$

$x(t)=M \Rightarrow y(t)=\int_{-\infty}^{t} M\,d\tau = M\int_{-\infty}^{t} d\tau$

$$= M[\tau]_{-\infty}^{t} = M[t+\infty]=\infty$$

→ same

many to one

NON-INVERTIBLE

Example 6: $y(t)=\int x(\tau)d\tau$

$x(t)=K \Rightarrow y(t)=K\int d\tau = Kt$

if $t=0$, $y(t)=0$

$x(t)=M \Rightarrow y(t)=M\int d\tau = Mt$

if $t=0$, $y(t)=0$

→ same value

many to one

NON-INVERTIBLE

Example 7: $y(t)=x(2t) \longrightarrow$ INVERTIBLE

Example 8: $y(t)=x(t)+x(t-1)$

$x(t)=u(t) \Rightarrow y(t)=u(t)+u(t-1)$

$x(t)=-u(t) \Rightarrow y(t)=-u(t)-u(t-1)$

$$=-[u(t)+u(t-1)]$$

→ different

one to one

INVERTIBLE

~~INVERTIBLE / NON-INVERTIBLE~~

$\boxed{\text{GHULAM}}$

BIBO STABLE / UNSTABLE

Example 1: $y(t) = 2\,x(t)$

let, $x(t) = K =$ constant (finite)

then if $y(t) < \infty$ BIBO stable ✓ $\boxed{\text{BIBO stable}}$

for $t = \begin{Bmatrix} \infty \\ 0 \\ -\infty \end{Bmatrix} \Rightarrow y(t) = 2K = \text{finite}$

Example 2: $y(t) = 2\,x(t) + 3 \longrightarrow$ BIBO stable

Example 3: $y(t) = \cancel{2}\, t\, x(t) + 2$

$x(t) = K \Rightarrow y(t) = tK + 2$

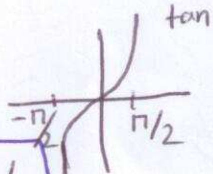
for $t = \infty,\ y(t) = \infty$
$t = -\infty,\ y(t) = -\infty$ $\longrightarrow \boxed{\text{BIBO unstable}}$

Example 4: $y(t) = \sin\{x(t)\}$

$x(t) = K \Rightarrow y(t) = \sin(K) \longrightarrow \boxed{\text{BIBO stable}}$

$-1 \le \ \ \le +1 \rightarrow$ finite

Example 5: $y(t) = \tan\{x(t)\}$



$x(t) = K \Rightarrow y(t) = \tan(K)$

if $K = \pi/2,\ y(t) = \infty$ $\rightarrow \boxed{\text{BIBO Unstable}}$
$K = -\pi/2,\ y(t) = -\infty$

Example 6: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$x(t) = K \Rightarrow y(t) = K \int_{-\infty}^{t} d\tau = K[\tau]_{-\infty}^{t}$

$= K[t + \infty] = \infty \longrightarrow \boxed{\text{BIBO UNSTABLE}}$

BIBO stable / Unstable

GHULAM

Example 7: $y(t) = \frac{d}{dt} x(t)$

$x(t) = K \Rightarrow y(t) = 0 \longrightarrow \boxed{\text{BIBO stable}}$

Example 8: $y(t) = \int_{-\infty}^{t} \cos\tau \; x(\tau)\, d\tau$

$x(t) = K \Rightarrow y(t) = K \int_{-\infty}^{t} \cos\tau \, d\tau$

$= K \left[\sin\tau\right]_{-\infty}^{t} = K\left[\sin t + \sin\infty\right]$

$-1 \leq \quad \leq +1 \qquad -1 \leq \quad \leq +1$

FINITE $\rightarrow \boxed{\text{BIBO stable}}$

Example 9: $h(t) = e^{-t} u(t)$

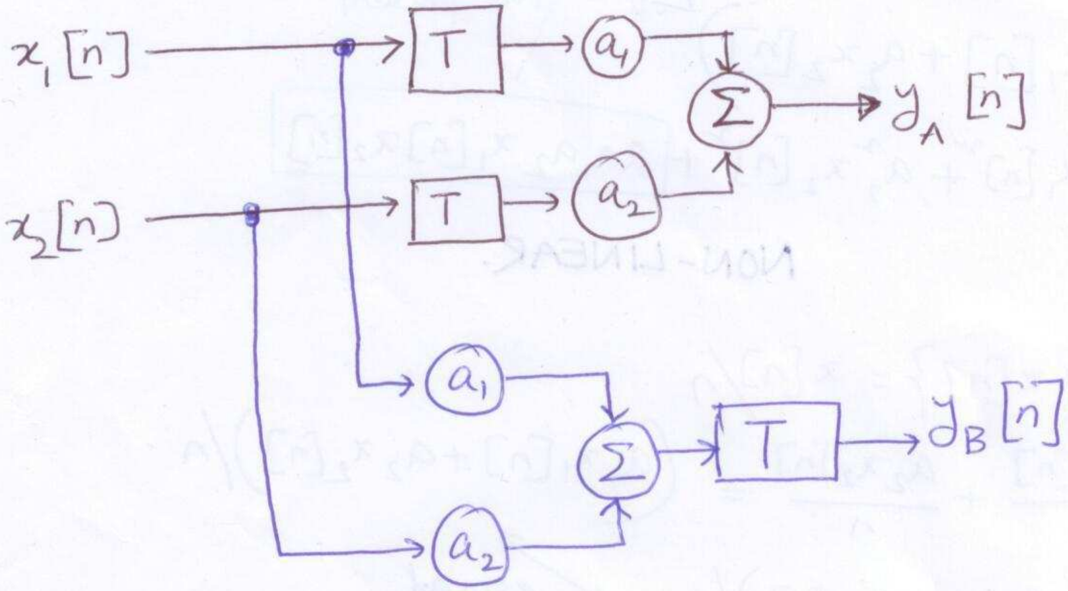
$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} |e^{-t}|\, dt = \left|(-1)\left[e^{-t}\right]_{0}^{\infty}\right|$ $\boxed{\text{BIBO stable}}$

$= \left|e^{-\infty} - e^{-0}\right| = |0-1| = 1 \rightarrow \text{finite}$

LINEAR AND NON-LINEAR SYSTEMS

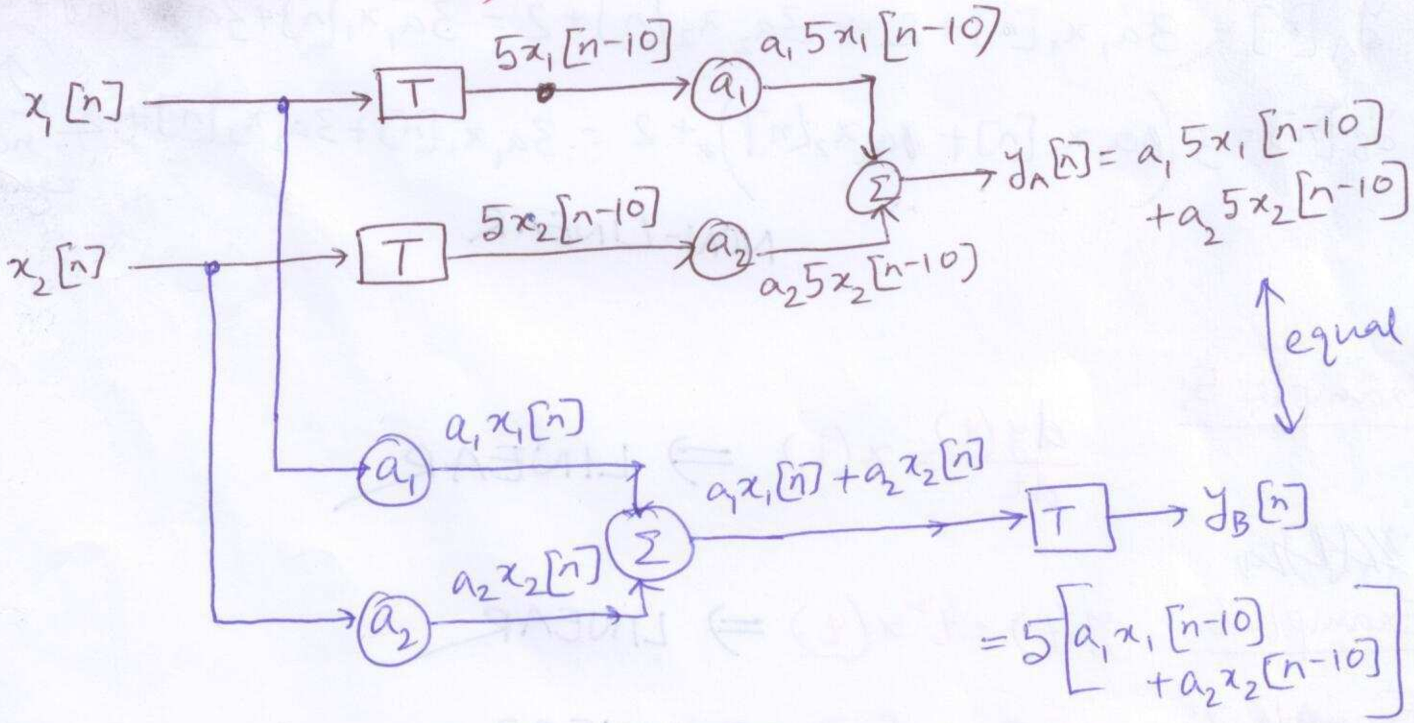
GHULAM

- Additivity
- Homogeneity



If $y_A[n] = y_B[n]$ for all n, T is a linear system.

Example 1 : $T\{x[n]\} = 5\,x[n-10]$ or $y[n] = 5\,x[n-10]$



$\therefore\ y_A[n] = y_B[n]$

$\boxed{\therefore \text{LINEAR}}$

Example 2: $T\{x[n]\} = x[n]^2$

$$y_A[n] = a_1 x_1[n]^2 + a_2 x_2[n]^2$$

Not present

$$y_B[n] = \left(a_1 x_1[n] + a_2 x_2[n]\right)^2$$

$$= a_1^2 x_1[n]^2 + a_2^2 x_2[n]^2 + \boxed{2a_1a_2\, x_1[n]\, x_2[n]}$$

NON-LINEAR.

Example 3: $T\{x[n]\} = x[n]/n$

$$y_A[n] = \frac{a_1 x_1[n]}{n} + \frac{a_2 x_2[n]}{n} = \left(a_1 x_1[n] + a_2 x_2[n]\right)/n$$

$$y_B[n] = \left(a_1 x_1[n] + a_2 x_2[n]\right)/n$$

equal

LINEAR

Example 4: $T\{x[n]\} = 3x[n] + 2$

$$y_A[n] = 3a_1 x_1[n] + 2 + 3a_2 x_2[n] + 2 = 3a_1 x_1[n] + 3a_2 x_2[n] + 4$$

$$y_B[n] = 3\left(a_1 x_1[n] + a_2 x_2[n]\right) + 2 = 3a_1 x_1[n] + 3a_2 x_2[n] + 2$$

not equal

NON-LINEAR

Example 5: $\dfrac{dy(t)}{dt} = x(t) \Rightarrow$ LINEAR

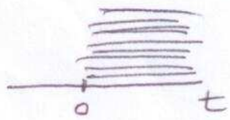
Example 6: $y(t) = t^2 x(t) \Rightarrow$ LINEAR

Example 7: $y[n] = x[2n] \Rightarrow$ LINEAR

TIME-INVARIANT / TIME-VARYING SYSTEMS $\boxed{\text{GHULAM}}$

Example 1: $y(t) = 3x^2(t)\,u(t)$

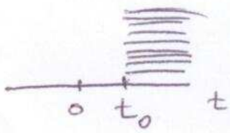
First shift the input $x(t)$ by $-t_0 \Rightarrow x(t-t_0)$



$$\therefore\ T\{x(t-t_0)\} = 3x^2(t-t_0)\,u(t)$$

Now shift the output $y(t)$ by $-t_0 \Rightarrow y(t-t_0)$

$$\therefore\ y(t-t_0) = 3x^2(t-t_0)\,u(t-t_0)$$



$$\neq 3x^2(t-t_0)\,u(t)$$

unless $t_0 = 0$

$\boxed{\text{TIME-VARYING}}$

Example 2: $y(t) = \int_0^t e^{-2\tau} x(\tau)\,d\tau$

$\tau - t_0 = M$

$$T\{x(t-t_0)\} = \int_0^t e^{-2\tau} x(\tau - t_0)\,d\tau =$$

$$= \int_{-t_0}^{t-t_0} e^{-2(\tau+t_0)}\, x(\tau)\,d\tau = e^{-2t_0}\int_{-t_0}^{t-t_0} e^{-2\tau} x(\tau)\,d\tau$$

$$y(t-t_0) = \int_0^{t-t_0} e^{-2\tau} x(\tau)\,d\tau$$

Not equal

$\boxed{\text{TIME-VARYING}}$

CONVOLUTION

Discrete-time Signals

GHULAM

(1) $x[n] = \left(-\frac{1}{2}\right)^n u[n-4]$

$h[n] = 4^n u[2-n]$

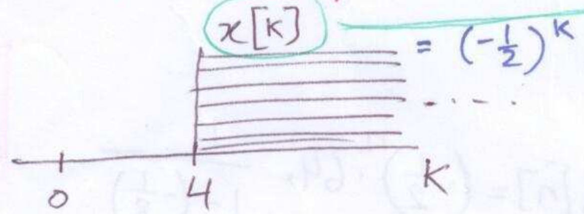
convolution

$$y[n] = x[n] * h[n]$$

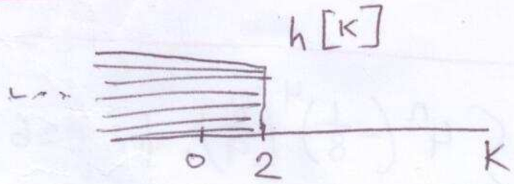
$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

STEP 1: Draw $x[k]$

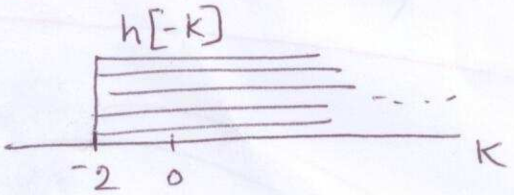
replace n by k in $x[n]$



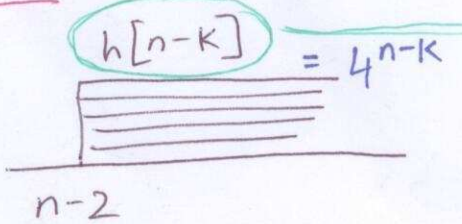
STEP 2: DRAW $h[k]$



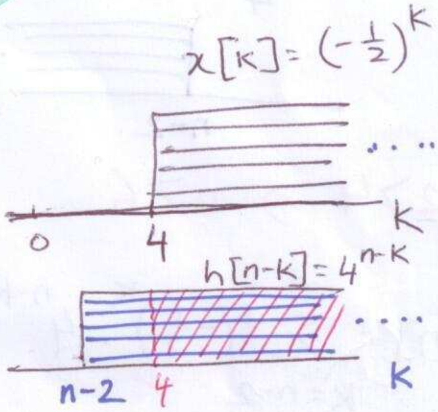
STEP 3: REFLECT $h[k] \rightarrow h[-k]$



STEP 4: ADD n to the above index



CASE I



$n-2 \leq 4 \Rightarrow n \leq 6$

$$y[n] = \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k (4)^{n-k}$$

$$= \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k 4^n\, 4^{-k}$$

$$= 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^k$$

not dependent on k

$$= 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{8}\right)^k$$

Let $m = k-4 \Rightarrow k = m+4$

if $k=4$, $m=0$

if $k=\infty$, $m=\infty$

$$\therefore y[n] = 4^n \sum_{m=0}^{\infty} \left(-\frac{1}{8}\right)^{m+4}$$

$$= 4^n \left(-\frac{1}{8}\right)^4 \sum_{m=0}^{\infty} \left(-\frac{1}{8}\right)^m$$

$\longrightarrow$

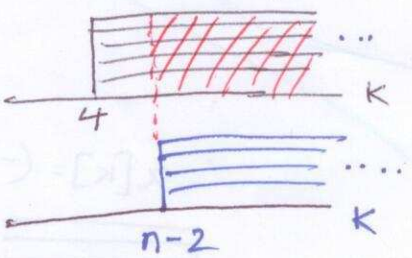
$$\sum_{m=0}^{\infty} (a)^m = \frac{1}{1-a} \quad \text{for } |a|<1$$

$$\rightarrow \quad y[n] = 4^n \left(-\tfrac{1}{8}\right)^4 \sum_{m=0}^{\infty} \left(-\tfrac{1}{8}\right)^m$$

$$= 4^n \left(-\tfrac{1}{8}\right)^4 \times \frac{1}{1-\left(-\frac{1}{8}\right)} = 4^n \left(-\tfrac{1}{8}\right)^4 \left(\tfrac{8}{9}\right) \quad \text{for } n \le 6$$

CASE II

$x[k] = \left(-\frac{1}{2}\right)^k$



$n-2 > 4 \Rightarrow n > 6$

$$y[n] = \sum_{k=n-2}^{\infty} \left(-\tfrac{1}{2}\right)^k \cdot 4^{n-k}$$

$$= 4^n \sum_{k=n-2}^{\infty} \left(-\tfrac{1}{2}\right)^k 4^{-k}$$

$$= 4^n \sum_{k=n-2}^{\infty} \left(-\tfrac{1}{8}\right)^k$$

Let $m = k-(n-2) \Rightarrow k = m+(n-2)$

if $k = n-2 \Rightarrow m = 0$

if $k = 0 \Rightarrow m = \infty$

$$\therefore y[n] = 4^n \sum_{m=0}^{\infty} \left(-\tfrac{1}{8}\right)^{m+(n-2)}$$

$$= 4^n \cdot \left(-\tfrac{1}{8}\right)^n \cdot \left(-\tfrac{1}{8}\right)^{-2} \sum_{m=0}^{\infty} \left(-\tfrac{1}{8}\right)^m$$

$$= \left(-\tfrac{4}{8}\right)^n (-8)^2 \sum_{m=0}^{\infty} \left(-\tfrac{1}{8}\right)^m$$

$$\rightarrow y[n] = \left(-\tfrac{1}{2}\right)^n \cdot 64 \cdot \frac{1}{1-\left(-\frac{1}{8}\right)}$$

$$= \left(-\tfrac{1}{2}\right)^n \cdot 64 \cdot \frac{8}{9} \quad \text{for } n > 6$$

$$\boxed{\therefore y[n] = \begin{cases} 4^n \left(-\frac{1}{8}\right)^4 \left(\frac{8}{9}\right), & \text{for } n \le 6 \\ \left(-\frac{1}{2}\right)^n \cdot 64 \cdot \frac{8}{9}, & \text{for } n > 6 \end{cases}}$$

CONVOLUTION

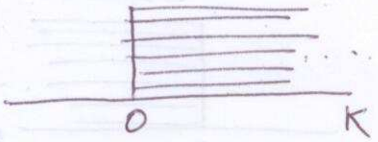
DISCRETE-TIME SIGNALS

GHULAM

$$\boxed{x[n] = h[n] = \alpha^n u[n]}$$

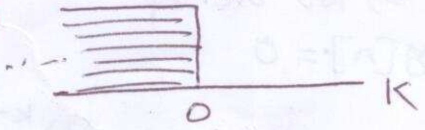
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

STEP 1: $x[k] = \alpha^k$

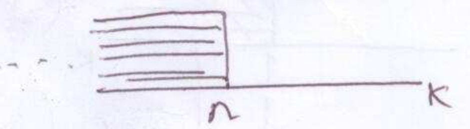


STEP 2: $h[k]$

STEP 3: $h[-k]$

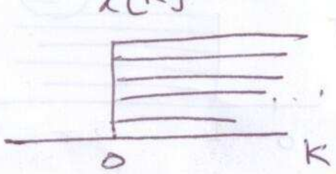


STEP 4: $h[n-k] = \alpha^{n-k}$

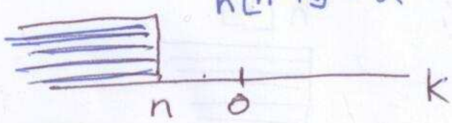


CASE I

$x[k] = \alpha^k$



$h[n-k] = \alpha^{n-k}$

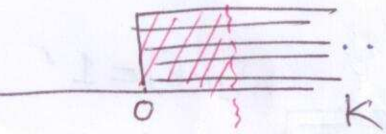


$n < 0 \rightarrow$ No overlap

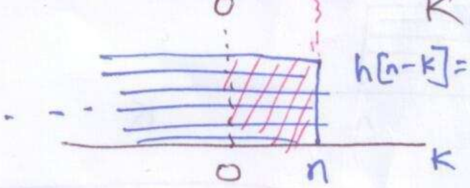
$\therefore y[n] = 0$

CASE II

$x[k] = \alpha^k$



$h[n-k] = \alpha^{n-k}$



$n \geq 0$

$$y[n] = \sum_{k=0}^{n} \alpha^k \alpha^{n-k}$$

$$= \alpha^n \sum_{k=0}^{n} \alpha^k \alpha^{-k}$$

$$= \alpha^n \sum_{k=0}^{n} 1 = \alpha^n (n+1)$$

$$\boxed{\therefore\ y[n] = \begin{cases} 0, & \text{for } n < 0 \\ \alpha^n (n+1), & \text{for } n \geq 0 \end{cases}}$$

CONVOLUTION

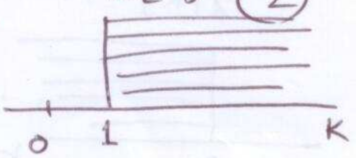
DISCRETE-TIME SIGNALS

$$x[n] = \left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

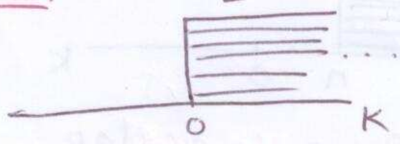
$$h[n] = u[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

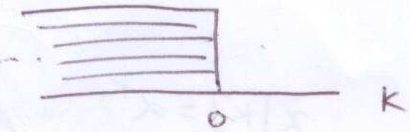
STEP 1: $x[k] = \left(\tfrac{1}{2}\right)^{k-1}$



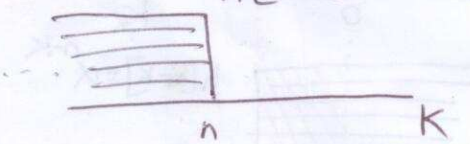
STEP 2: $h[k]$



STEP 3: $h[-k]$

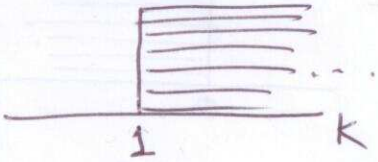


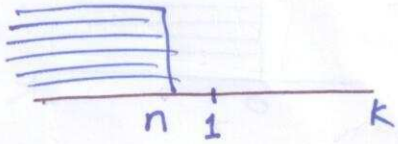
STEP 4: $h[n-k] = 1$



$$\boxed{\sum_{k=0}^{n} (\alpha)^k = \frac{1-(\alpha)^{n+1}}{1-\alpha} \quad \text{for } \alpha < 1}$$

CASE I: $x[k] = \left(\tfrac{1}{2}\right)^{k-1}$

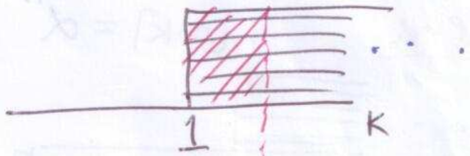


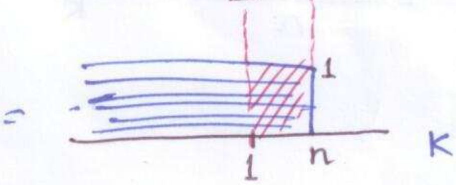


$n < 1 \Rightarrow$ No overlap

$y[n] = 0$

CASE II: $x[k] = \left(\tfrac{1}{2}\right)^{k-1}$





$n \geq 1$

$$y[n] = \sum_{k=1}^{n} \left(\tfrac{1}{2}\right)^{k-1} \cdot 1$$

let $m = k-1 \Rightarrow k = m+1$

if $k = 1 \Rightarrow m = 0$

if $k = n \Rightarrow m = n-1$

$$\therefore y[n] = \sum_{m=0}^{n-1} \left(\tfrac{1}{2}\right)^m$$

$$= \frac{1-\left(\frac{1}{2}\right)^{n-1+1}}{1-\left(\frac{1}{2}\right)} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$$

$$\therefore y[n] = \begin{cases} 0, & \text{for } n < 1 \\ \dfrac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}, & \text{for } n \geq 1 \end{cases}$$