# Logic Mathematics (Math 132)

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# Chapter 6: Cardinality of Sets

# Cardinality of Sets

In Chapter 2,

- We defined the cardinality of a finite set as the number of elements in the set.
- We use the cardinalities of finite sets to tell us when they have the same size, or when one is bigger than the other.

In this Chapter,

- We extend this notion to infinite sets. That is, we will define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.
- We will be particularly interested in countably infinite sets, which are sets with the same cardinality as the set of positive integers.
- We will establish the surprising result that the set of rational numbers is countably infinite.
- We will also provide an example of an uncountable set when we show that the set of real numbers is not countable.

# Cardinality of Sets

## Definition 1

*The sets $A$ and $B$ have the same cardinality if and only if there is a one-to-one correspondence from $A$ to $B$. When $A$ and $B$ have the same cardinality, we write $|A| = |B|$.*

## Definition 2

*If there is a one-to-one function from $A$ to $B$, the cardinality of $A$ is less than or the same as the cardinality of $B$ and we write $|A| \leqslant |B|$. Moreover, when $|A| \leqslant |B|$ and $A$ and $B$ have different cardinality, we say that the cardinality of $A$ is less than the cardinality of $B$ and we write $|A| < |B|$.*

# Cardinality of Sets

**Countable Sets:** We will now split infinite sets into two groups, those with the same cardinality as the set of natural numbers and those with a different cardinality.

### Definition 3

*A set that is either finite or has the same cardinality as the set of positive integers is called countable. A set that is not countable is called uncountable. When an infinite set $S$ is countable, we denote the cardinality of $S$ by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that $S$ has cardinality "aleph null".*

# Cardinality of Sets

## Example 1

Show that the set of odd positive integers is a countable set.
**Solution:** To show that the set of odd positive integers is countable, we will exhibit a one-to-one correspondence between this set and the set of positive integers. Consider the function $f(n) = 2n - 1$.
From $\mathbb{Z}^+$ to the set of odd positive integers. We show that f is a one-to-one correspondence by showing that it is both one-to-one and onto. To see that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n - 1 = 2m - 1$, so $n = m$. To see that it is onto, suppose that $t$ is an odd positive integer. Then $t$ is $1$ less than an even integer $2k$, where k is a natural number. Hence $t = 2k - 1 = f(k)$. We display this one-to-one correspondence in Figure 1.

# Cardinality of Sets

## Example 2

Show that the set of all integers is countable.

**Solution:** We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers: $0, 1, -1, 2, -2, \ldots$. Alternatively, we could find a one-to-one correspondence between the set of positive integers and the set of all integers.We leave it to the reader to show that the function $f(n) = \dfrac{n}{2}$ when $n$ is even and $f(n) = -\dfrac{(n-1)}{2}$ when $n$ is odd is such a function. Consequently, the set of all integers is countable.

# Cardinality of Sets

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers). The reason for this is that a one-to-one correspondence $f$ from the set of positive integers to a set $S$ can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ , where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$.

## Example 3

Show that the set of all integers is countable.

**Solution:** We can list all integers in a sequence by starting with $0$ and alternating between positive and negative integers: $0, 1, -1, 2, -2, \ldots$ .
Alternatively, we could find a one-to-one correspondence between the set of positive integers and the set of all integers. We can show that the function $f(n) = n/2$ when $n$ is even and $f(n) = -(n-1)/2$ when $n$ is odd is such a function. Consequently, the set of all integers is countable.

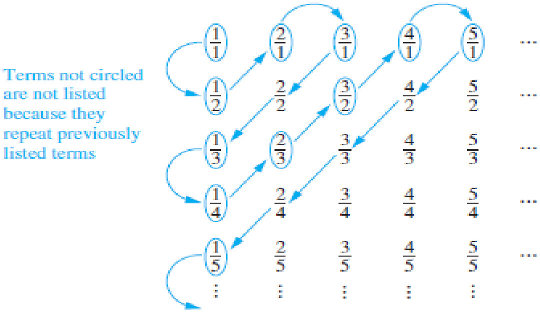
## Example 4

Show that the set of positive rational numbers is countable.

**Solution:** It may seem surprising that the set of positive rational numbers is countable, but we will show how we can list the positive rational numbers as a sequence $r_1, r_2, \ldots, r_n, \ldots$ . First, note that every positive rational number is the quotient $p/q$ of two positive integers. We can arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on, as displayed in Figure below. The key to listing the rational numbers in a sequence is to first list the positive rational numbers $p/q$ with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on, following the path shown in Figure below. Whenever we encounter a number $p/q$ that is already listed, we do not list it again. For example, when we come to $2/2 = 1$ we do not list it because we have already listed $1/1 = 1$.

# Cardinality of Sets

The initial terms in the list of positive rational numbers we have constructed are $1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4/5$, and so on. These numbers are shown circled; the uncircled numbers in the list are those we leave out because they are already listed. Because all positive rational numbers are listed once, as the reader can verify, we have shown that the set of positive rational numbers is countable.



**An Uncountable Sets:** We have seen that the set of positive rational numbers is a countable set. Do we have a promising candidate for an uncountable set? The first place we might look is the set of real numbers. In the Example below, we show that the set of real numbers is not countable.

### Example 5

Show that the set of real numbers is an uncountable set.

# Cardinality of Sets

**Theorem**

If A and B are countable sets, then $A \cup B$ is also countable.

**Theorem**

If A is a countable sets, then all subsets of $A$are also countable.

**SCHRÖDER-BERNSTEIN THEOREM**

If A and B are sets with $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

**Theorem**

We can show that the power set of $\mathbb{Z}^+$ and the set of real numbers $\mathbb{R}$ have the same cardinality.
In other words, we know that $|P(\mathbb{Z}^+)| = |\mathbb{R}| = c$, where $c$ denotes the cardinality of the set of real numbers.