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** Linear Depenedency

Definition

Let $\{v_1, v_2, \dots, v_n\}$ be a set of vectors of a Linear space.

(i) If $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ iff $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ then $\{v_1, v_2, \dots, v_n\}$ is Linear independente where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$

(ii) If $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ has a non-zero values of $\lambda_1, \dots, \lambda_n$ then $\{v_1, \dots, v_n\}$ is Linear dependente

Remark

To examine $\{v_1, \dots, v_n\}$ is Linear independente or not,

STEP1 : suppose $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$.

STEP2 : Deduce Homogenous system

STEP3 : If it has unique (zero) solution, i.e., $\lambda_1 = \dots = \lambda_n = 0$ then $\{v_1, \dots, v_n\}$ is Linear independte. otherwise, The system has non-zero solution, and then it is Linear Dependonte.

(Ex) Let $\{(6,2,1), (-1,3,2)\}$ be a set of vectors of $\mathbb{R}^3$. Does the set be Linear independte ?

Solution suppose that $\lambda_1(6,2,1) + \lambda_2(-1,3,2) = (0,0,0)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\left.\begin{aligned} 6\lambda_1 - \lambda_2 &= 0 \\ 2\lambda_1 + 3\lambda_2 &= 0 \\ \lambda_1 + 2\lambda_2 &= 0 \end{aligned}\right\}$$ rectangular - Homogenous system

$$\left[\begin{array}{cc|c} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{REEF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has unique solution $\lambda_1 = \lambda_2 = 0$

$\therefore$ $\{(6,2,1), (-1,3,2)\}$ is Linear independent.

(ex) Let $S = \{ (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}), (3, 4, 7/2), (-\frac{3}{2}, 6, 2) \}$ be a set of vectors of $\mathbb{R}^3$. Determine, if S is Linear independent or dependent ?

<u>Solution</u>

suppose that $\lambda_1 (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}) + \lambda_2 (3, 4, 7/2) + \lambda_3 (-\frac{3}{2}, 6, 2) = (0,0,0)$

then,

$$\left.\begin{array}{lll} \frac{3}{4}\lambda_1 + 3\lambda_2 - \frac{3}{2}\lambda_3 = 0 & \Longleftrightarrow & 3\lambda_1 + 12\lambda_2 - 6\lambda_3 = 0 \\ \frac{5}{2}\lambda_1 + 4\lambda_2 + 6\lambda_3 = 0 & \Longleftrightarrow & 5\lambda_1 + 8\lambda_2 + 12\lambda_3 = 0 \\ 3/2\,\lambda_1 + 7/2\,\lambda_2 + 2\lambda_3 = 0 & \Longleftrightarrow & 3\lambda_1 + 7\lambda_2 + 4\lambda_3 = 0 \end{array}\right\}$$

It is square - Homogenous system.

So, $|A| = \begin{vmatrix} 3 & 12 & -6 \\ 5 & 8 & 12 \\ 3 & 7 & 4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 4 & -2 \\ 5 & 8 & 12 \\ 3 & 7 & 4 \end{vmatrix} \quad \frac{-5R_1 + R_2}{-3R_1 + R_3}$

$$= 3 \begin{vmatrix} 1 & 4 & -2 \\ 0 & -12 & 22 \\ 0 & -5 & 10 \end{vmatrix}$$

$$= 3\,(1) \begin{vmatrix} -12 & 22 \\ -5 & 10 \end{vmatrix} \neq 0$$

So, A^{-1} is existed $\Rightarrow$ the zero - solution is the unique solution

$\Rightarrow$ S is Linear independent.

(ex) Let $S = \{ 1 + x^2 , \; 2 + x + x^2 \}$ be a set of vectors of $P_2(x)$. Determine, if S is Linear independent or not ?

<u>Solution</u> suppose that $\lambda_1 (1 + x^2) + \lambda_2 (2 + x + x^2) = 0 + 0x + 0x^2$

$$\Rightarrow \left.\begin{array}{r} \lambda_1 + 2\lambda_2 = 0 \\ \lambda_2 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{array}\right\}$$

It is clear that $\lambda_1 = \lambda_2 = 0$ is the unique solution.

Hence, S is Linear independent ■

Basis and Dimension

Definition Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors of Linear space V. Then

S is a basis of V iff $\begin{cases} ① \ S \text{ spans } V. \\ ② \ S \text{ is linear independent.} \end{cases}$

Remarks

① If S is a basis of a linear space V, then $|S|$ is called the dimension of V, $\mathrm{Dim}(V)$, .

② If $\mathrm{Dim}(V) < \infty$, then V is finite dimensional.
If $\mathrm{Dim}(V) = \infty$, then V is infinite dimensional.

③ Every vector space has atleast a basis which is not necessarily unique

standered basis of some famous Linear spaces :

Linear space	Basis	Dim
$\mathbb{R}^2$	$B = \{(1,0), (0,1)\}$	2
$\mathbb{R}^3$	$B = \{(1,0,0), (0,1,0), (0,0,1)\}$	3
$M_2(\mathbb{R})$	$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$	4
$P_1(x)$	$B = \{1, x\}$	2
$P_2(x)$	$B = \{1, x, x^2\}$	3

★★ Some Properties

Property (1) Let B_1 and B_2 be two basis of ~~the~~ a Linear space V, then

$$|B_1| = |B_2| = Dim(V)$$

Property (2) Let S be a set of vectors of a linear space V such that $0 \in S$, then S is Linear dependent which implies that S is not basis

Property (3) Let S be a set of vectors of a Linear space V where there exists a vector $v \in S$ can be written as Linear Combination of the remain vectors in S. Then S is Linear dependent which implies that S is not basis

for example $\{ \overset{v_1}{(2,-4)}, \overset{v_2}{(-1,2)} \} \subseteq \mathbb{R}^2$.
Notice that $(2,-4) = -2(-1,2)$. So,
$\{(2,-4), (-1,2)\}$ is Linear dependent.

Property (4) Let S be a set of vector of a Linear space V. If S is Linear dependent then there exists $v \in S$ such that v is a Linear Combination of the remain vectors in S

for example $S = \{ 2x, x^2, 5 \} \subseteq P_2(x)$.

Notice that
$$5 \neq \lambda_1(2x) + \lambda_2(x^2)$$
$$2x \neq \lambda_1(5) + \lambda_2(x^2)$$
$$x^2 \neq \lambda_1(5) + \lambda_2(2x)$$

So, S is Linear indepent.

Property 5 Let W be a linear subspace of V then
$Dim(W) \leq Dim(V)$.

Property 6 Let B be a spanning set of a vector space V. If S is Linear independent then $|S| \leq |B|$

for example

Let $Dim(V) = 3$ and S be a set of vectors where $|S| = 4$. Then S is Linear dependent (because B_v is spanning set of V and $|B_v| = 3$)

Remark If $|S| > Dim(V)$ then S is Linear dependent
If S is Linear indepent then $|S| \leq Dim(V)$

Property 7 Let S be a set of vectors of a Linear space V. Then

How to prove S is a basis if you know $Dim(V)$

(1) If $|S| = Dim(V)$ and S is Linear indepent then S is a basis.

(2) If $|S| = Dim(V)$ and S is spanning set of V then S is a basis.

Property 8 Let S be Linear indepent set ~~of ve.~~ of a Linear space V where $|S| < Dim(V)$. Then there exists a basis B of V such that $S \subseteq B$.

(Ex) find a basis of $\mathbb{R}^2$ contains $v = (1,1)$?
Notice that $Dim(V) = Dim(\mathbb{R}^2) = 2$.
So, choose $v' = (0,1)$. It is clear that
$v' \neq \lambda v$ and $v \neq \lambda' v'$ for any $\lambda, \lambda' \in \mathbb{R}$
So, $\{v, v'\}$ is Linear indepent $\Rightarrow$ $\{v, v'\}$ is basis.

(Ex) find a basis of $\mathbb{R}^3$ contains $v = (1,1,1)$.

Solution

we know that $\text{Dim}(\mathbb{R}^3) = 3$

so, choose $v_1 = (1,0,0)$

$v_2 = (0,1,0)$

then $S = \{(1,1,1), (1,0,0), (0,1,0)\}$ is Linear independent because

(After forming $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$)

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

so, the system has unique solution (zero solution).

Theorem A squar matrix is invertible iff it is Columns are Linear independent. iff it is rows are Linear independent

Theorem If A is row-echelon form matrix then the non-zero rows is Linear independent

for example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is row-echelon form

Then, the set $\{(1,2,3,4), (0,0,1,1)\}$ is Linear independent in $\mathbb{R}^4$.

** Rank of matrix

$\text{Rank}(A) = \text{Dim}(V)$ where V is the Linear space spanned by columns of A.

How to find rank A ?

To find Rank(A) : step 1 : A in row-echolon form

step 2 : $\text{Rank}(A) =$ number of non-zero rows.

** Row space, column space and Null space:

Def : Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be $n \times m$ matrix.

The vectors

$r_1 = (a_{11}, a_{12}, \ldots, a_{1n})$

$\vdots$

$r_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$

are vectors in R^n which called row-vectors

The vectors

$c_1 = \begin{bmatrix} a_{11} \\ | \\ a_{m1} \end{bmatrix} \cdots \quad c_n = \begin{bmatrix} a_{1n} \\ | \\ a_{mn} \end{bmatrix}$ are vectors in R^m

which called colum vectors

(1) The subspace which spanned by row-vectors is called row space of A.

(2) The subspace which spanned by column-vectors is called column space of A

(3) The space of solutions of $AX = 0$ is called the null space of A

** Finding a basis of null space of matrix

(ex) find a basis of null space of

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

solution we have to find the solution of $AX = 0$ by gauss method

$$S = \left\{ \begin{bmatrix} -3s-4t-2k \\ s \\ -2t \\ t \\ k \\ 0 \end{bmatrix} ; \ s, t \text{ and } k \in IR \right\}$$

~~note~~

$$= \begin{bmatrix} -3s \\ s \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4t \\ 0 \\ -2t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2k \\ 0 \\ 0 \\ 0 \\ k \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} k \Rightarrow S = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

the basis

Remark

(1) Let A be a matrix. Then the row factors which has Leaders 1 (as echolon form) is the basis of the row space of A.

(2) Let A be a matrix. Then the column with leaders 1 (as echolon form) is the basis of the column space.

(Ex) If $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. It is clear that A is in Row echolon form. So,

$$B_{\text{row-space}} = \{ (1,-2,5,0,3),\ (0,1,3,0,0),\ (0,0,0,1,0) \}$$

$$B_{\text{colum-space}} = \{ (1,0,0,0),\ (-2,1,0,0),\ (0,0,1,0) \}$$

** Basis of the space spanned by a set of vectors

Example Let $S = \{ v_1 = (1,2,2,-1),\ v_2 = (-3,-6,-6,3),$
$v_3 = (4,9,9,-4),\ v_4 = (-2,-1,-1,2),$
$v_5 = (5,8,9,-5),\ v_6 = (4,2,7,-4) \}$

find the Basis of Span(S)?

Answer: write the vectors as columns

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

write A as row echolon form $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$B = \{ (1,0,0,0),\ (4,1,0,0),\ (5,-2,1,0) \}$$

Rule Let A be a matrix. Then

$$\text{Dim}(\text{row space of } A) = \text{Dim}(\text{Column space of } A)$$

Definition

① $\text{Dim}(\text{null space of } A) = \text{nullity}(A)$

② $\text{Dim}(\text{row space}) = \text{Dim}(\text{column space}) = \text{rank}(A)$

Example find the rank and nullity of the following matrix :

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution

STEP1 : write A on the row-echolon form.

$$A = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

STEP2 : To find nullity (A) :

(i) find the solutions of $AX=0$.

$N(E) = 2$
$N(V) = 6$ $\Rightarrow N(P) = 6-2 = 4$

we have two equations

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

Let $x_3 = r$ $\quad x_4 = s$ $\quad x_5 = t$ $\quad x_6 = u$

$$\Rightarrow x_1 = 4r + 28s + 37t - 13u$$
$$x_2 = 2r + 12s + 16t - 5u$$

(ii) Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + s\underbrace{\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_2} + t\underbrace{\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3} + u\underbrace{\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_4}$$

$$B_{\text{null-space}} = \{v_1, v_2, v_3, v_4\} \Rightarrow \text{nullity}(A) = 4$$

STEP3 To find Rank (A) :

$$B_{\text{row-space}} = \{ (-1, 0, -4, -28, -37, 13), \; (0, 1, -2, -12, -16, 5) \}$$

Hence rank (A) = Dim (row-space) = 2

Rule

Let $A_{m \times n}$, then

$$\text{rank}(A) + \text{nullity}(A) = n$$
(number of columns)

Rule

Let A be matrix

$$\text{rank}(A) = \text{rank}(A^t)$$

example

Let $A_{5 \times 10}$ where rank (A) = 4 . Find nullity (A^t)?

solution

Notice that size $(A^t) = 10 \times 5$

$\text{rank}(A^t) = \text{rank}(A) = 4$

Now

$\text{rank}(A^t) + \text{nullity}(A^t) = 5 \Rightarrow \text{nullity}(A^t) = 5 - 4 = 1$

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II Some Excersices :

① Let $M = \{ (x,y,z) : 2x - y + z = 0 \}$.

(i) Prove M is Linear subspace of $\mathbb{R}^3$?

(ii) find $Dim(M)$

Solution

(i) Homework !

(ii) notice that every $(x,y,z) \in M$ is a solution of $2x - y + z = 0$

$$\left.\begin{array}{l} N(E) = 1 \\ N(V) = 3 \end{array}\right\} \Rightarrow N(P) = 2$$

Let $x = t$, $y = s$

then $z = -2t + s$

So, $M = \left\{ \begin{bmatrix} t \\ s \\ -2t+s \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

$= \left\{ t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

Hence $B = \{ (1,0,-2) , (0,1,1) \}$ spans M

since $(1,0,-2) \neq \lambda (0,1,1) \quad \forall \lambda \in \mathbb{R}$

then B is Linear independant

Therefore, B is basis to $M \Rightarrow Dim(M) = 2$

② Let $M = \{ a + bx + cx^2 + dx^3 : a + b = c - 2d = 0 \}$

(i) Prove M is Linear subspace ?

(ii) find $Dim(M)$?

Solution

(i) Home work!

(ii) We have $a + b = 0 \Rightarrow a = -b$ and
$c - 2d = 0 \Rightarrow c = 2d$

So, any Polynomial belongs to M will be written as :

$-b + bx + 2dx^2 + dx^3$
$= b \underbrace{(1-x)}_{\boxed{v_1}} + d \underbrace{(2x^2 - x^3)}_{\boxed{v_2}}$

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So, $B = \{ v_1 = -1 + x \;,\; v_2 = 2x^2 + x^3 \}$ spans M

Notice that $v_1 \neq \lambda v_2 \quad \forall \lambda \in \mathbb{R}$

Hence, B is Linear independent

Therefore, B is basis of $M \Rightarrow Dim(M) = 2$

3. Does $S = \{ (1,3,-1), (0,1,5), (2,2,3) \}$ be a basis of $\mathbb{R}^3$?

Solution

Notice that $|S| = 3 = Dim(\mathbb{R}^3)$. So, it is enough to study if S is Linear independent or not.

suppose that

$\lambda_1 (1,3,-1) + \lambda_2 (0,1,5) + \lambda_3 (2,2,3) = (0,0,0)$. Then, we have

$$\left.\begin{array}{l} \lambda_1 + \quad + 2\lambda_3 = 0 \\ 3\lambda_1 + \lambda_2 + 2\lambda_3 = 0 \\ -\lambda_1 + 5\lambda_2 + 3\lambda_3 = 0 \end{array}\right\} \quad \text{square + Homogenous system}$$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ -1 & 5 & 3 \end{vmatrix} \xrightarrow[R_1 + R_3]{-3R_1 + R_2} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 \\ 5 & 5 \end{vmatrix} = 25 \neq 0$$

So, we have only the zero solution

Hence, S is Linear independent $\Rightarrow$ S is a basis of $\mathbb{R}^3$.

Definition

Let $B = \{ v_1, \dots, v_n \}$ be a basis of V. If $v \in V$ where $v = a_1 v_1 + \dots + a_n v_n$ then $(a_1, \dots, a_n)$ is called the coordinate of v.

Notice that : the coordinate of v by using the basis B is unique.

(4) Let $S = \{(1,3), (0,1)\}$ be a basis of $\mathbb{R}^2$. Find the coordinate of $(2,5)$?

solution

since $(2,5) \in \mathbb{R}^2$, $(2,5) = \lambda_1 (1,3) + \lambda_2 (0,1)$

So, $\left.\begin{array}{l} \boxed{\lambda_1 = 2} \\ 3\lambda_1 + \lambda_2 = 5 \end{array}\right\} \Rightarrow \boxed{\lambda_2 = -1}$

Hence $(2,-1)$ is the coordinat of $(2,5)$ respects to the basis S.

** Coordinates and change of basis

Let $B = \{v_1, \dots, v_n\}$ be a basis of V. If $v \in V$ then $v = a_1 v_1 + \dots + a_n v_n$ where $a_1, \dots, a_n \in \mathbb{R}$.
The coordinate vector ~~of~~ of v respects to B is $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = (a_1, a_2, \dots, a_n)$

Example Let $\{(1,2), (-1,4)\} = S$ be a set of vectors of $\mathbb{R}^2$. (1) Prove that S is a basis ?
(2) find $[(5,6)]_B$?

Solution 1. Homework!
2. Suppose that $(5,6) = \lambda_1 (1,2) + \lambda_2 (-1,4)$
then, we have $\lambda_1 - \lambda_2 = 5$
$2\lambda_1 + 4\lambda_2 = 6$

Therefore $\lambda_1 = \frac{13}{3}$, $\lambda_2 = -\frac{2}{3}$

Hence $[(5,6)]_B = \begin{bmatrix} \frac{13}{3} \\ -\frac{2}{3} \end{bmatrix}$

Example Let $B = \{v_1 = 3,\ v_2 = -1 + x,\ v_3 = x^2\}$ be a set of vetors of $P_2(x)$. (1) Prove that B is basis.
(2) find $[1 - x^2]_B$?

Solution (1) Notice that $|B| = 3 = \mathrm{Dim}(P_2(x))$
So, It is enough to prove that B is Linear indep. to show that B is basis. For that

suppose that

$$\lambda_1 (3) + \lambda_2 (-1+x) + \lambda_3 (x^2) = 0 + 0x + 0x^2$$

Then, we have

$$\left.\begin{aligned} 3\lambda_1 - \lambda_2 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \end{aligned}\right\}$$

Hence $\lambda_1 = \lambda_2 = \lambda_3 = 0$
Therefore, $\mathcal{B}$ is Linear indep- which implies that B is basis.

2. suppose that $1-x^2 = \lambda_1 (3) + \lambda_2 (-1+x) + \lambda_3 (x^2)$

Then $\left.\begin{aligned} 3\lambda_1 - \lambda_2 &= 1 \\ \boxed{\lambda_2 = 0} \\ \boxed{\lambda_3 = -1} \end{aligned}\right\} \Rightarrow \boxed{\lambda_1 = \frac{1}{3}}$

Hence $[(1-x^2)]_B = \begin{bmatrix} 1/3 \\ 0 \\ -1 \end{bmatrix}$

** Transmission matrix

Let B_1 and B_2 be two basis of a Linear space V. The following example will show you how to find transmission matrix from B_1 to B_2 and the converse

Example: Let $B = \{(2,1), (0,3)\}$ and $B' = \{(-1,0), (3,3)\}$ be two basis of $\mathbb{R}^2$. Find the transmission matrix from B into B'?

Solution step 1: find $[(2,1)]_{B'}$. For that suppose that $(2,1) = \lambda_1 (-1,0) + \lambda_2 (3,3)$.

Then $\left.\begin{aligned} -\lambda_1 + 3\lambda_2 &= 2 \\ 3\lambda_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_2 &= 3 \\ \lambda_1 &= -1 \end{aligned}$

Hence $[(2,1)]_{B'} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

step 2: find $[(0,3)]_{B'}$. By the same method, we will deduce that

$$[(0,3)]_{B'} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

There fore ;

$$\underset{B' \; B}{P} = \text{transmission matrix from } B \text{ into } B' = \begin{bmatrix} -1 & 3 \\ \frac{1}{3} & 1 \end{bmatrix}$$

Rule

Let B_1 and B_2 be two basis of a Linear space V and $v \in V$. Then

$$[v]_{B_2} = {}_{B_2}P_{B_1} \, [v]_{B_1}$$

Example Let $S = \{ \frac{1}{2} , -x , 2x^2 \}$ be a basis of $P_2(x)$. If $[f(x)]_S = (1,2,-1)$ and ${}_{S^*}P_S = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, find $[f(x)]_{S^*}$ where S^* is another basis of $P_2(x)$. Find $f(x)$?

solution

(i) $$[f(x)]_{S^*} = {}_{S^*}P_S \, [f(x)]_S = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

(ii) $$f(x) = \boxed{1} \left(\frac{1}{2}\right) + \boxed{2}(-x) + \boxed{-1}(2x^2) = \frac{1}{2} - 2x - 2x^2 .$$

Example Let $S = \{ (2,1) , (0,3) \}$ be a basis of $\mathbb{R}^2$. If S^* is another basis of $\mathbb{R}^2$ such that ${}_{S^*}P_S = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. find S^*

solution From ${}_{S^*}P_S$, we can deduce that

$$[(2,1)]_{S^*} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} , \text{ and}$$

$$[(0,3)]_{S^*} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

- So, if $S^* = \{(a,b), (c,d)\}$ we have that

$$\left.\begin{aligned} (2,1) &= \frac{1}{\sqrt{5}}(a,b) - \frac{2}{\sqrt{5}}(c,d) \\ (0,3) &= \frac{2}{\sqrt{5}}(a,b) + \frac{1}{\sqrt{5}}(c,d) \end{aligned}\right\}$$

which implies that

$$\frac{1}{\sqrt{5}}a - \frac{2}{\sqrt{5}}c = 2 \quad \cdots (1)$$

$$\frac{1}{\sqrt{5}}b - \frac{2}{\sqrt{5}}d = 1 \quad \cdots (2)$$

$$\frac{2}{\sqrt{5}}a + \frac{1}{\sqrt{5}}c = 0 \quad \cdots (3)$$

$$\frac{2}{\sqrt{5}}b + \frac{1}{\sqrt{5}}d = 3 \quad \text{---} (4)$$

- Solve the system (1) and (3) and the system (2), (4) to get

$$(a,b) = \left(\frac{2}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$$

$$(c,d) = \left(\frac{4}{\sqrt{5}}, {}^{-1}/\sqrt{5}\right) \quad \blacksquare$$

Rules

(1) $P_{S'S^*} \cdot P_{S^*S} = P_{S'S}$

(2) $P_{S^*S} \cdot P_{SS^*} = I \iff P_{SS^*} = \left(P_{S^*S}\right)^{-1}$