Chapter 5: Basis of a vector space

Def: Let V be a vector space and

$S = \{v_1, \ldots, v_n\}$ be a set o vectors in V. Then S is called a basis of V if and only if

1- S spans V, i.e., every vector of V can be written as a linear combination of the vectors in S.
2- S is linear independent.

Some properties:

1- if $S = \{v_1, \ldots, v_n\}$ is a basis of V then every vector of V can be written as a linear combination of the vectors in S in (unique method).
2- Let $S = \{e_1, \ldots, e_n\}$ be a set of the following vectors in $\mathbb{R}^n$:
   $e_1 = (1, 0, \ldots, 0), \ldots, e_k = (0, \ldots, 0, 1, 0, \ldots 0), \ldots, e_n = (0, \ldots, 0, 1)$.
   Then S is a basis of $\mathbb{R}^n$ which is called natural basis of $\mathbb{R}^n$.

Ex: Prove the set of vectors $S = \{1, X, \ldots, X^n\}$ is a basis of $\mathcal{P}_n$?

Example: Let .$v_3 = (1, 1, \lambda)$ و $v_2 = (1, \lambda, 1)$,$v_1 = (\lambda, 1, 1)$ be vectors in $\mathbb{R}^3$. Find the value of $\lambda \in \mathbb{R}$ that makes $\{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$?

Solution: The set $\{v_1, v_2, v_3\}$ should be a linear independent, i.e., the equation

$xv_1 + yv_2 + zv_3 = 0$ has a unique solution x=y=z=0. So,

$$A = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}$$ has an inverse, i.e., |A| is not equal to zero.

After calculation, $\lambda \notin \{-2, 1\}$.

As |A| is not equal to zero, the system to prove the linear combination will be with unique solution.

REMARK:

Notice that the basis of a vector space is not necessarily to be unique.

Some properties:

1- Let S be a basis of a vector space where $S = \{v_1, \ldots, v_n\}$. If $T = \{u_1, \ldots, u_m\}$ is a set of vectors in V where $m > n$, i.e., |T|>|S| then T is linear dependant.
2- If S and T are two basis of a vector space V then |S|=|T|.

Definition: Dimension of a vector space V, Dim(V) , is equal to |S| where S is a basis of V.

For example:

1- Dim($\mathbb{R}^n$)= n
2- Dim($\mathcal{P}_n$)= n+1

Remark: if V is a vector space where Dim(V)=n then to prove the set of vectors $S = \{v_1, \ldots, v_n\}$ in V is a basis, it is enough to prove that S is a linear independent or it is a spanning set of V.

** How to find the dimension of a subspace Span(S) where S is the set of Vectors in a vector space V?

We will do the following steps:

Step 1: Let A be the matrix where every vector in S is a column in A.

Step 2: Eliminate the matrix A.

Step 3: the vector which meet the columns with the 1 leader will be the basis of Span(S).

Theorem:

1- If V is a vector space and S is a spanning set of V then S contains T where T is a basis of V.
2- If S is a linear independent set in a vector space V then there is a basis T of V where T contains S.

Example: Let W= Span (S) be a subspace of $\mathbb{R}^5$ where S= $\{v_1, v_2, v_3, v_4\}$ where

$v_1 = (1, 0, 2, -1, 2)$, $v_2 = (2, 0, 4, -2, 4)$, $v_3 = (1, 2, -1, 2, 0)$, $v_4 = (1, 4, -4, 5, -2)$.

a- Find a basis of W?
b- Find a basis of V contains {v1 , v3}?

Answer:

a- Firstly, we will construct A as follows:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 2 & 4 & -1 & -4 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

After elimination, we get

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, the basis of Span(S) is {V1, V3}

b- The basis T of $\mathbb{R}^5$ that contains {V1, V3} is $\{v_1, v_3, e_1, e_2, e_3\}$

Where $e_3 = (0,0,1,0,0)$, $e_2 = (0,1,0,0,0)$, $e_1 = (1,0,0,0,0)$

Question from previous exam:

Let $W = \{(x,y,z,t) \in \mathbb{R}^4;\ 2x + y + z = 0,\ x - y + z = 0\}$.

1- Prove that W is a subspace of $\mathbb{R}^4$.
2- Find a basis of W?

Answer: 1- you can answer by regular method through satisfying the two conditions of subspace. Another method can be done as follows:

W is the set of sollutions of the system 2x+y+z=0 and x-y+Z=0. The system is homogenous AX=0. Therefore, the set of answers is subspace.

2- we have (x,y,z,t)= (x,y,z,0)+ (0, 0, 0, t)……(#)

(0,0,0,t)= t(0,0,01) ………..(*)

(x,y,z,0) are solutions to the system 2x+y+z=0 and x-y+Z=0

$$\begin{cases} 2x + y + z = 0 \\ x - y + z = 0 \end{cases} \iff \begin{cases} x = -2y \\ z = 3y \end{cases}$$

So, the solution will be written by parameter y

$$y\begin{pmatrix} -2 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$ ………………(**)

Hence by (#) , (*) and (**), W is generated by

$$\{\begin{pmatrix}-2\\1\\3\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}$$ which is the basis of W.

Question :

Let $V=\mathbb{R}^3$. Give example on

1- Linear independent set which is not a basis?

2- Spanning set of V which is not a basis?

Answer:

1- we can take S={(1,0,0)}. |S|= 1 which is not equal Dim($V=\mathbb{R}^3$)=3. So it is not a basis.

If a(1,0,0)=(0,0,0) then (a,0,0)=(0,0,0). Therefore, a=0. Hence S is linear independent.

2- Let T={(1,0,0) , (0,1,0) , (0,0,1) , (1,1,1)}

If (a,b,c) is a vector then

(a,b,c)= a(1,0,0)+ b(0,1,0)+c(0,0,1)+ 0(1,1,1)

So, T spans $V=\mathbb{R}^3$ but |T| > Dim($V=\mathbb{R}^3$). Hence, T is not a basis.

** Change of a basis:

Definition: Let $S=\{v_1,\ldots,v_n\}$ be a basis of a vector space V and $v\in V$، where

$v=x_1v_1+\ldots x_nv_n$. Then $(x_1,\ldots x_n)$ is called the coordinate of v related to the basis S.

We write

$$[v]_S=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

to denote the coordinate vector of v related to S.

Theorem:

Let V be a vector space which has tow basis

$C=\{u_1,\ldots,u_n\}$ و $B=\{v_1,\ldots,v_n\}$

Then the matrix ${}_CP_B$ is a matrix with column

$[v_1]_C,\ldots,[v_n]_C$. The matrix ${}_CP_B$ is called the transition matrix from the basis B into the basis C. This matrix has an inverse and for every vector v in V, we have

$$[v]_C={}_CP_B[v]_B$$

.

Practise:

Suppose that $\mathbb{R}^3$ has two basis

$B=\{v_1=(0,1,1),v_2=(1,0,-2),v_3=(1,1,0)\}$

And $C=\{u_1=(1,0,0),u_2=(0,1,0),u_3=(0,0,1)\}$

1- Find ${}_BP_C$ و ${}_CP_B$.

2- If

$[v]_C=\begin{pmatrix}1\\0\\-1\end{pmatrix}$, find $[v]_B$?

Solution:

$$.{}_{B}P_{C}=\begin{pmatrix}-2 & 2 & -1\\ -1 & 1 & -1\\ 2 & -1 & 1\end{pmatrix}\quad {}_{C}P_{B}=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & -2 & 0\end{pmatrix}$$ [1]

$$.[v]_B={}_{B}P_{C}[v]_C=\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$ [2]

(Try to write the full answer for the above answer)

Question from one of previous exams:

In $\mathbb{R}^3$, prove that $w=(-2,1,-2)$ و $v=(-1,-1,2)$, $u=(1,0,1)$ is a basis of $\mathbb{R}^3$, and then find the coordinate vector of (x,y,z) related to such basis:

Answer:

As Dim($\mathbb{R}^3$)=3, it is enough to show that w,v,u spans $\mathbb{R}^3$. Suppose for any vector (x,y,z), the equation

(x,y,z)= au+bv+cw. The coefficients matrix of such system is

$$A=\begin{pmatrix}1 & -1 & -2\\ 0 & -1 & 1\\ 1 & 2 & -2\end{pmatrix}$$

$$|A|=-3$$

So, the system has a unique solution

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix}=A^{-1}X=\begin{pmatrix}2y+z\\ \frac{-x+z}{3}\\ \frac{-x+3y+z}{3}\end{pmatrix}$$

EX) Prove that $S = \{(1,1,1),(-1,1,0),(1,0,-1)\}$ is a basis of $\mathbb{R}^3$ and find the coordinate vectors of $(0,0,1)$ و $(1,0,1)$,$(1,0,0)$ which is related to S.

Answer: As Dim($\mathbb{R}^3$)=3, it is enough to show that w,v,u spans $\mathbb{R}^3$. So, it is enough to show that S is a spanning set of $\mathbb{R}^3$. After doing the equation of linear combination for any vector in $\mathbb{R}^3$, we get

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3 \neq 0$$

So, there is a unique solution to the system that was induced from the equation of linear combination.

To find the coordinate vectors related to S,

It is easy to write $(1,0,0) = \frac{1}{3}(1,1,1) - \frac{1}{3}(-1,1,0) + \frac{1}{3}(1,0,-1)$

So, the coordinate vector is $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$.

Similarly,

$$(0,0,1) = \frac{1}{3}(1,1,1) - \frac{1}{3}(-1,1,0) - \frac{2}{3}(1,0,-1)$$

So, the coordinate vector is $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$.

** Row and Column spaces:

Let A be a matrix of size nXm .

1- Row(A) is the subspace generated by rows of A (notice that every row will be considered a vector)

2- Col(A) is the subspace generated by column of A (notice that every column will be considered a vector)

Properties:

1- Let A be a matrix. If B is a matrix induced from doing some elementary row operations then

Row(A)=Row(B)

2- Let A be a matrix. After elimination, the non-zero rows are linear independent.

(Do not forget that any set of vectors contains zero vector is linear dependent…. Why?)

Definition : Let A be a matrix. The rank of A, ranK(A)= Dim (Row(A))

Definition: Order of matrix A= the number of non zero rows (After elimination)

Rules: for any matrix A,

1- Rank(A)= Dim(Row(A))= Dim(Col(A)).

2- Rank(A)= Rank (transpose (A)).

Theorem

Let A be a matrix of size n X m. Then the following are equivalent

1- AX=0 has unique solution(zero solution)

2- The columns of A are independent

3- rank(A)=n (number of rows)

4- $A^T A$ has an inverse.

Theorem

Let A be a matrix of size n X m. Then the following are equivalent

1- AX=B is consistent

2- The columns of A are spanning set of $\mathbb{R}^m$

3- rank(A)=m (number of column)

4- A^TA has an inverse.

Definition :

Let A be a matrix of size n X m. Then

1- The subspace W= $\{X \in \mathbb{R}^n;\ AX = 0\}$ is called null subspace, N(A), and

Nullity (A)= Dim(N(A)).

2- $\operatorname{Im}(A)$ = $\{AX;\ X \in \mathbb{R}^n\}$.

Rules:

1- Let A be a matrix of size n X m. Then Im(A)=Col(A)

2- Let A be a matrix of size n X m. Then nullity(A)+ rank(A)= n (number of rows).

Example: Let

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

1- Find a basis of the null space, N(A)?

2-Find a basis of Col(A)?

3- find order of A?

Answer:

1- After Elimination, we get

$$\cdot\begin{pmatrix}1 & 0 & 3 & 5\\ 0 & 1 & -2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

So, the basis

$$\begin{pmatrix}-3\\ -2\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}5\\ 3\\ 0\\ 1\end{pmatrix}$$ Non zero rows.

Basis of N(A).

2- $$\begin{pmatrix}1\\ 0\\ 2\\ 1\end{pmatrix}, \begin{pmatrix}2\\ -1\\ 3\\ 1\end{pmatrix}$$ the basis of Col(A) (The vectors the meet the column with leaders 1)

3- Order A= 2