General Mathematics II

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Contents

List of Abbreviations

Symbol	Symbol Name	Meaning / definition
$f(x)$	function of x	map x to $f(x)$
$D(f)$	domain	domain of a function f
$f \circ g$	composite	composition of functions f and g
$\int f(x)\,dx$	integral	the indefinite integral of a function f with respect to x
$\lim\limits_{x \to a} f(x)$	limit as x approaches a	limit value of a function f
$x \to a^+$	x approaches a from the right	x approaching a, and x is greater than a
$y' = \frac{dy}{dx}$	derivative	differentiate a function y with respect to x
$y'' = \frac{d^2y}{dx^2}$	second derivative	differentiate a function twice
$y^{(n)} = \frac{d^ny}{dx^n}$	nth derivative	differentiate a function n-times
$\frac{\partial f}{\partial x}$	partial derivative	partial derivative of f with respect to x
$\frac{\partial^2 f}{\partial x^2}$	second partial derivative	second partial derivative of f with respect to x
e	constant / Euler's number	$e \approx 2.718281828$
$[a,b]$	closed interval	$[a,b] = \{x \mid a \leq x \leq b\}$
(a,b)	open interval	$(a,b) = \{x \mid a < x < b\}$
$\{a,b,c,...\}$	set	a collection of elements
∞	lemniscate	infinity symbol
$\pm$	plus - minus	represents positive and negative number
$\in$	element of	set membership
$\mid x \mid$	absolute value	the absolute value of x is always either positive or zero
$a \cong b$	approximate	a is approximately equal to b
$\forall$	for all x	the statement is true for all values of x
a^n	power or exponent	times of using number a in a multiplication
$\sqrt{a}$	square root	$\sqrt{a}\ \sqrt{a} = a$
$\sqrt[n]{a}$	nth root (radical)	n times in a multiplication $\sqrt{a}\ \sqrt{a} ... \sqrt{a} = a$
$A = [a_{ij}]_{m \times n}$	matrix	a matrix A
I_n	identity matrix	$a_{ij} = 1$ if $i = j$ and 0 otherwise
A^t	transpose of a matrix A	if $A = [a_{ij}]_{m \times n}$ then $A^t = [a_{ji}]_{n \times m}$
$det(A) = \|A\|$	determinant of A	$\sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} (i = 1,...,n)$

Continued on next page

Symbol	Symbol Name	Meaning / definition
π	constant	$\pi \approx 3.141592654$
r	radius	radius of a circle or a disk
$\sum$	sigma	represents summation
$\mathbb{N}$	natural numbers set	$\mathbb{N} = \{1,2,3...\}$
(x_1,x_2)	ordered pair	$(x_1,x_2) \in mathbbR^2$
$\mathbb{Z}$	integer numbers set	$\mathbb{Z} = \{...,-3,-2,-1,0,1,2,3,...\}$
$\mathbb{Q}$	rational numbers set	$\mathbb{Q} = \{x \mid x = \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}^*\}$
$\mathbb{R}$	real numbers set	$\mathbb{R} = \{x \mid -\infty < x < \infty\}$
P	partition $P = \{x_0,x_1,...,x_n\}$	partition of an interval $[a,b]$
Δx_k	length of a subinterval $[x_{k-1},x_k]$	$\Delta x_k = x_k - x_{k-1}$
$\|\|P\|\|$	norm	the largest length among lengths of subintervals
ω	mark on P	$\omega = (\omega_1,\omega_2,...,\omega_n),\ \omega_k \in [x_{k-1},x_k]$
R_p	Riemann sum	$R_p = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(\omega_k)\Delta x_k$
$\log_a x$	general logarithmic function	logarithm with base $a > 0$
$\ln x$	natural logarithmic function	logarithm with base e
a^x	general exponential function	exponent with base $a > 0$
e^x	natural exponential function	exponent with base e
A	area	area of a region
V	volume	volume of solid of revolution
$L(f)$	arc length	arc length of a curve of f
$S.A$	area of revolution surface	
m	slope	the slope of the tangent line

Chapter 1

CONIC SECTIONS

1.1 Parabola

Definition 1.1 A parabola is a set of all points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line D (called the directrix) in the same plane.

Let the focus lies along the x-axis at $F = (a,0)$ and let the directrix be the line $x = -a$. From Definition 1.1, we have $|PF| = |PD|$. Then from the distance formula, we obtain

$$\sqrt{(x-a)^2+(y-0)^2} = \sqrt{(x+a)^2+(y-y)^2}$$
$$\Rightarrow \sqrt{(x-a)^2+y^2} = (x+a)$$
$$\Rightarrow (x-a)^2+y^2 = (x+a)^2$$
$$\Rightarrow y^2 = (x+a)^2-(x-a)^2$$
$$\Rightarrow y^2 = 4ax\ .$$

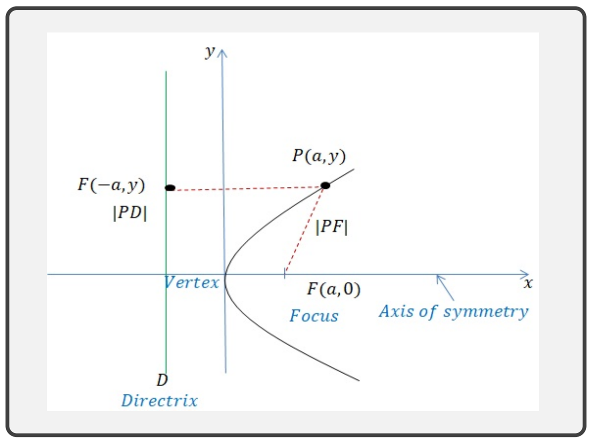


Figure 1.1: An illustrative graph of of the parabola.

The result is the equation of a parabola with vertex at the origin, that opens to the right. Similarly, we can extract the other equations of the parabola. In each case, $a > 0$ which represents the distance from the vertex to the focus. The **axis of symmetry** of the parabola is a line that passes through the vertex and is perpendicular to the directrix.

1.1.1 Vertical Parabolas

When a parabola opens right or left, it has a vertical axis of symmetry. In this case, the parabola is called a vertical parabola. We study the special and general cases of the vertical parabolas. In the special case, we assume that the vertex of the parabola is at the origin. In the general case, we assume that the vertex is at $V(h,k)$.

(A) Vertical Parabolas with the Vertex at the Origin.

The equation of the vertical parabola with the vertex at the origin is $x^2 = \pm 4ay$, where $a > 0$.

(1) The equation $x^2 = 4ay$ has the following properties:

- The vertex of the parabola is $V(0,0)$.
- The parabola opens upwards.
- The axis of symmetry of the parabola is y-axis.
- The focus of the parabola is $F(0,a)$.
- The directrix of the parabola is $y = -a$.

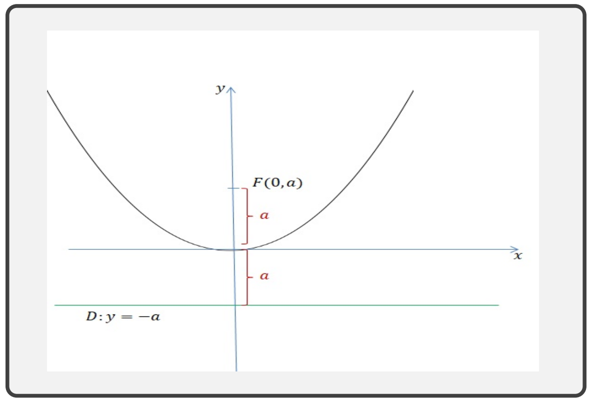


Figure 1.2: The graph of the parabola $x^2 = 4ay$.

(2) The equation $x^2 = -4ay$ has the following properties:

- The vertex of the parabola is $V(0,0)$.
- The parabola opens downwards.
- The axis of symmetry of the parabola is y-axis.
- The focus of the parabola is $F(0,-a)$.
- The directrix of the parabola is $y = a$.

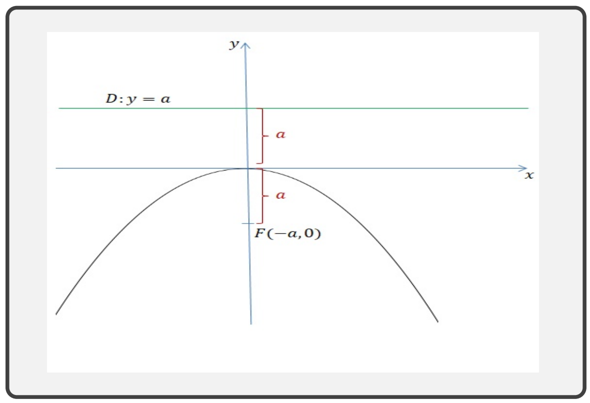


Figure 1.3: The graph of the parabola $x^2 = -4ay$.

(B) Vertical Parabolas with the Vertex at $V(h,k)$.

The equation of the vertical parabola with the vertex at $V(h,k)$ is $(x-h)^2 = \pm 4a(y-k)$, where $a > 0$. The previous form is the general formula of the vertical parabolas.

(1) The equation $(x-h)^2 = 4a(y-k)$ has the following properties:

- The vertex of the parabola is $V(h,k)$.
- The parabola opens upwards.
- The axis of symmetry of the parabola is parallel to y-axis.
- The focus of the parabola is $F(h,k+a)$.
- The directrix of the parabola is $y = k - a$.

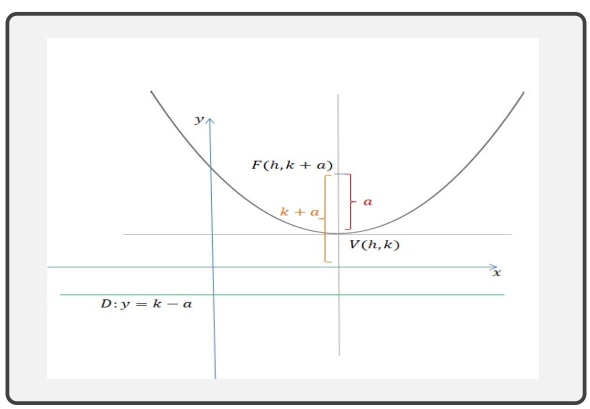


Figure 1.4: The graph of the parabola $(x-h)^2 = 4a(y-k)$ for $h,k > 0$.

(2) The equation $(x-h)^2 = -4a(y-k)$ has the following properties:

- The vertex of the parabola is $V(h,k)$.
- The parabola opens downwards.
- The axis of symmetry of the parabola is parallel to y-axis.
- The focus of the parabola is $F(h,k-a)$.
- The directrix of the parabola is $y = k+a$.

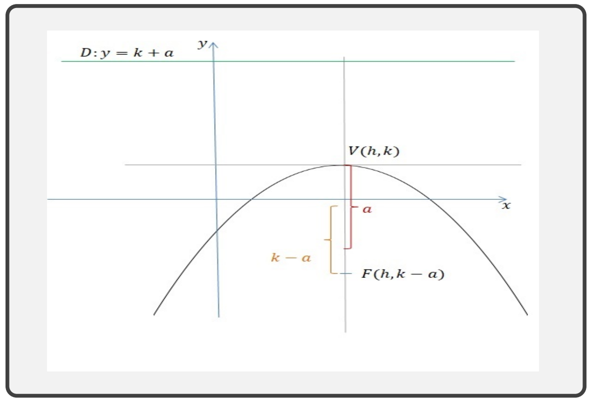


Figure 1.5: The graph of the parabola $(x-h)^2 = -4a(y-k)$ for $h,k > 0$.

■ **Example 1.1** Find the focus and the directrix of the parabola $x^2 = 4y$, and sketch its graph.

Solution:
The equation $x^2 = 4y$ takes the form $x^2 = 4ay$ with $a = 1$.

Therefore, the parabola has the following properties:

- The vertex of the parabola is $V(0,0)$.
- The parabola opens upwards.
- The axis of symmetry of the parabola is y-axis.
- The focus of the parabola is $F(0,1)$.
- The directrix of the parabola is $y = -1$.

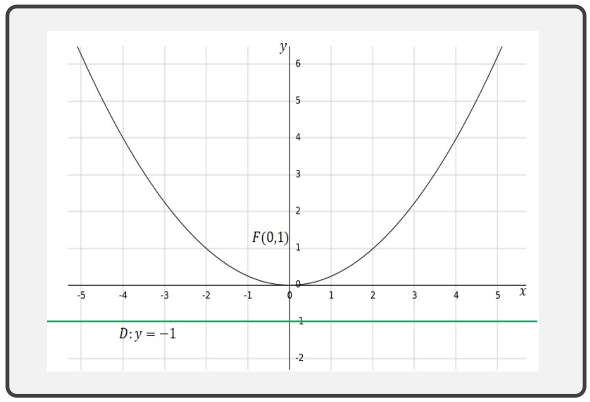


Figure 1.6: The parabola $x^2 = 4y$.

■ **Example 1.2** Find the focus and the directrix of the parabola $(x+1)^2 = -4(y-1)$, and sketch its graph.

Solution: The equation $(x+1)^2 = -4(y-1)$ takes the form

$$(x-h)^2 = -4a(y-k) .$$

This implies $a = 1$, $h = -1$ and $k = 1$. The parabola has the following properties:

- The vertex of the parabola is $V(-1,1)$.
- The parabola opens downwards.
- The axis of symmetry of the parabola is parallel to y-axis.
- The focus of the parabola is $F(-1,-1)$.
- The directrix of the parabola is $y = 2$.

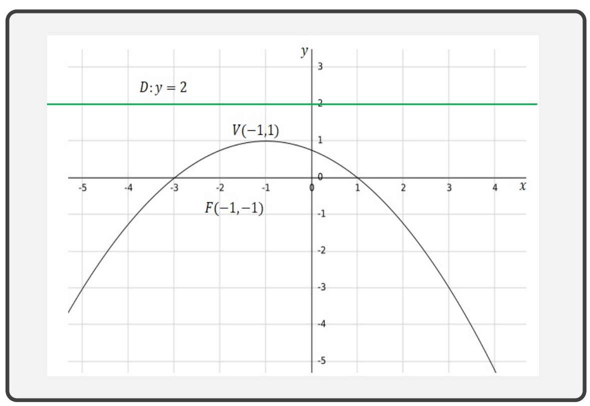


Figure 1.7: The graph of the parabola $(x+1)^2 = -4(y-1)$.

■ **Example 1.3** Find the equation of the parabola with vertex $(2,1)$ and focus $F(2,3)$. Then, sketch the graph.

Solution:
Since the vertex and focus are in the same line $x = 2$, then the axis of symmetry of the parabola is parallel to the y-axis. Also, from the y-coordinate of the vertex and focus, the parabola opens upwards. Thus, the equation of the parabola takes the form

$$(x-h)^2 = 4a(y-k) \ .$$

From the vertex and focus, we have

$$V(h,k) = (2,1) \Rightarrow h = 2 \text{ and } k = 1$$

$$F(h,k+a) = (2,3) \Rightarrow a = 2$$

By substituting the values of a, h and k, the equation of the parabola becomes $(x-2)^2 = 8(y-1)$.

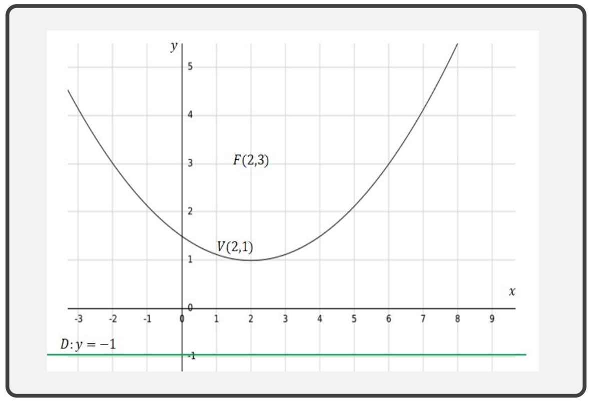


Figure 1.8: The graph of the parabola $(x-2)^2 = 8(y-1)$.

1.1.2 Horizontal Parabolas

When a parabola opens upwards or downwards, it has a horizontal axis of symmetry. In this case, the parabola is called a horizontal parabola. We consider the two cases: the vertex at the origin and the vertex at $V(h,k)$.

(A) Horizontal Parabolas with the Vertex at the Origin. The equation of the horizontal parabola with the vertex at the origin is $y^2 = \pm 4ax$, where $a > 0$.

(1) The equation $y^2 = 4ax$ has the following properties:

- The vertex of the parabola is $V(0,0)$.
- The parabola opens to the right.
- The axis of symmetry of the parabola is x-axis.
- The focus of the parabola is $F(a,0)$.
- The directrix of the parabola is $x = -a$.

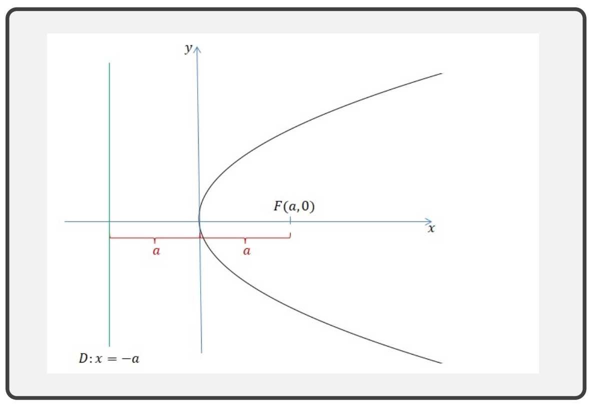


Figure 1.9: The graph of the parabola $y^2 = 4ax$.

(2) The equation $y^2 = -4ax$ has the following properties:

- The vertex of the parabola is $V(0,0)$.
- The parabola opens to the left.
- The axis of symmetry of the parabola is x-axis.
- The focus of the parabola is $F(-a,0)$.
- The directrix of the parabola is $x = a$.

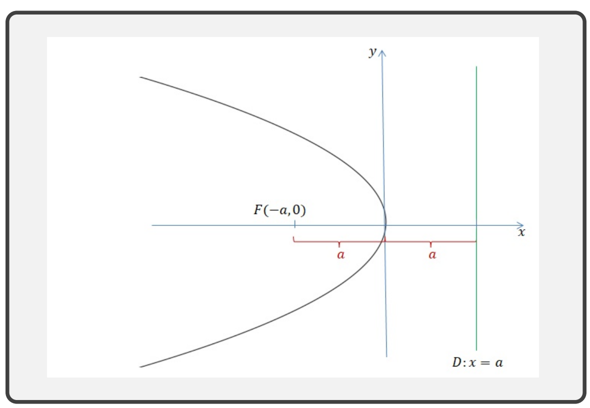


Figure 1.10: The graph of the parabola $y^2 = -4ax$.

(B) Horizontal Parabolas with the Vertex at $V(h,k)$. The general equation of the horizontal parabola with the vertex at $V(h,k)$ is $(y-k)^2 = \pm 4a(x-h)$, where $a > 0$.

(1) The equation $(y-k)^2 = 4a(x-h)$ has the following properties:

- The vertex of the parabola is $V(h,k)$.
- The parabola opens to the right.
- The axis of symmetry of the parabola is parallel to x-axis.
- The focus of the parabola is $F(h+a,k)$.
- The directrix of the parabola is $x = h-a$.

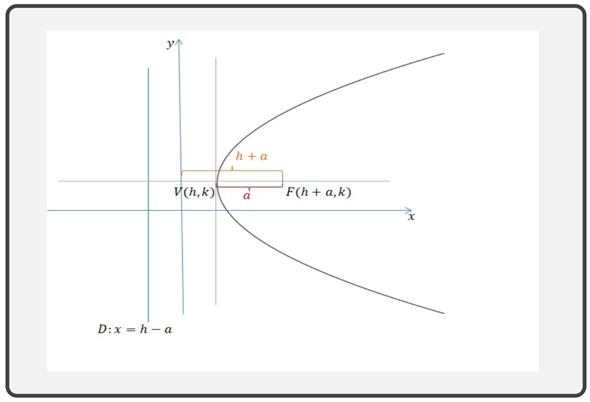


Figure 1.11: The graph of the parabola $(y-k)^2 = 4a(x-h)$ for $h,k > 0$.

(2) The equation $(y-k)^2 = -4a(x-h)$ has the following properties:

- The vertex of the parabola is $V(h,k)$.
- The parabola opens to the left.
- The axis of symmetry of the parabola is parallel to x-axis.
- The focus of the parabola is $F(h-a,k)$.
- The directrix of the parabola is $x = h+a$.

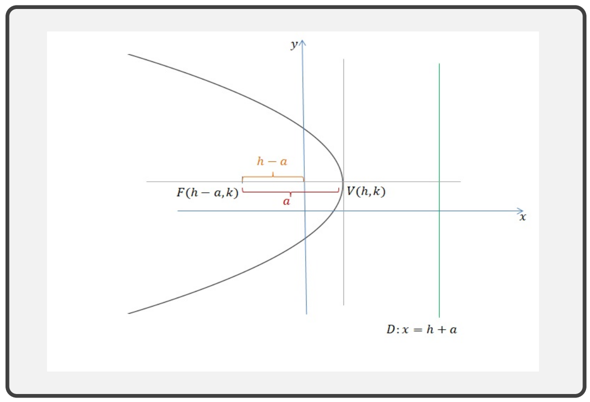


Figure 1.12: The graph of the parabola $(y-k)^2 = -4a(x-h)$.

■ **Example 1.4** Find the focus and the directrix of the parabola $y^2 = -8x$, and sketch its graph.

Solution:

The equation $y^2 = -8x$ takes the form $y^2 = -4ax$ with $a = 2$.

The parabola has the following properties:

- The vertex of the parabola is $V(0,0)$.
- The parabola opens to the left.
- The axis of symmetry of the parabola is x-axis.
- The focus of the parabola is $F(-2,0)$.
- The directrix of the parabola is $x = 2$.

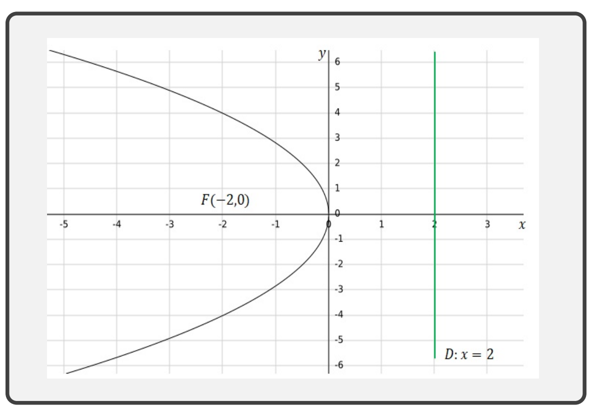


Figure 1.13: The parabola $y^2 = -8x$.

■ **Example 1.5** Find the equation of the parabola with focus $(3,0)$ and directrix $x = -3$. Then, sketch the graph.

Solution:

Since the focus of the parabola is $F(3,0) = F(a,0)$, the axis of symmetry of the parabola is x-axis and it opens to the right.

Thus, the equation of the parabola takes the form $y^2 = 4ax$.

Since $a = 3$, then the equation of the parabola is $y^2 = 12x$. The directrix of the parabola is $x = -3$.

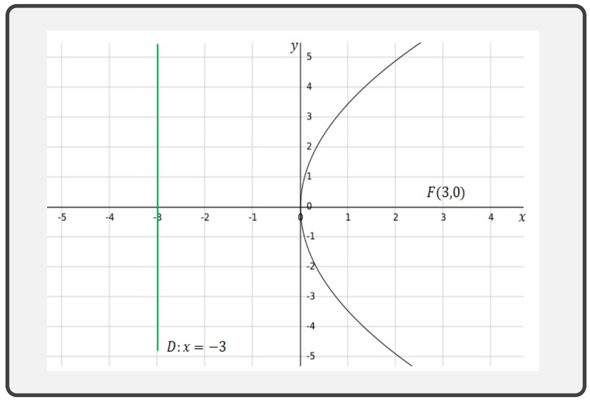


Figure 1.14: The parabola $y^2 = 12x$.

■ **Example 1.6** Find the focus and the directrix of the parabola $2y^2 - 4y + 8x + 10 = 0$, and sketch its graph.

Solution:

Since the quadrature is on the $y-$term, then the parabola takes the form $(y-k)^2 = \pm 4a(x-h)$.

$$\begin{aligned}
2y^2 - 4y + 8x + 10 &= 0, && \text{divide all terms by 2} \\
y^2 - 2y + 4x + 5 &= 0, \\
y^2 - 2y &= -4x - 5, && \text{isolate } y\text{-terms} \\
\underbrace{(y^2 - 2y + 1)}_{\text{completing square}} &= -4x - 4 \\
(y-1)^2 &= -4(x+1) && (y-k)^2 = -4a(x-h)
\end{aligned}$$

The parabola has the following properties:

- The vertex of the parabola is $V(-1,1)$.
- The parabola opens to the left.
- The axis of symmetry of the parabola is parallel to x-axis.
- The focus of the parabola is $F(-2,1)$.
- The directrix of the parabola is $x = 0$.

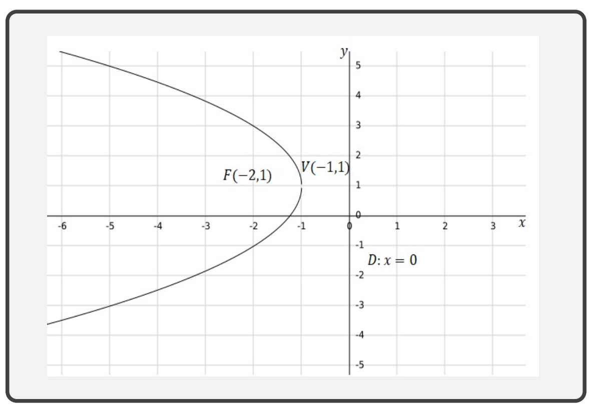


Figure 1.15: The graph of the parabola $(y-1)^2 = -4(x+1)$.

1.2 Ellipse

Definition 1.2 An ellipse is a set of all points in a plane such that the sum of the distances from each point to two fixed points (called foci) is constant.

- Each of the two fixed points mentioned in the previous definition is called a focus. The line containing the foci intersects the ellipse at points called vertices.

- The line segment between the vertices is called the major axis, and its midpoint is the center of the ellipse.

- A line perpendicular to the major axis through the center intersects the ellipse is called the minor axis and its endpoints called co-vertices.

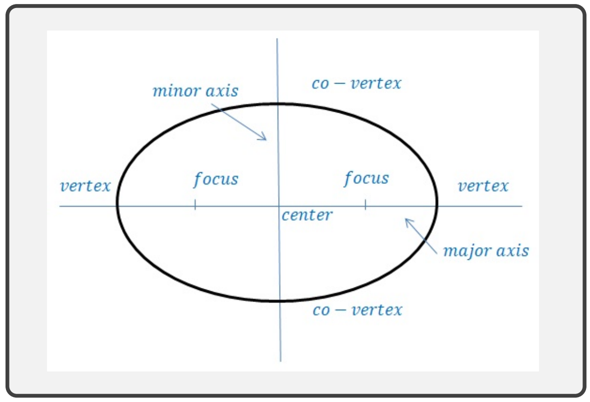


Figure 1.16: An illustrative graph of of the ellipse.

Let c_2 be a circle with midpoint F_2 and radius $2a$. From Figure 1.17, the distance of the point P to the circle c_2 equals the distance to the focus F_1. Therefore, if the point $P = W_1(0,b)$, then $|PF_1| = |Pc_2| = a$. From Pythagoras' theorem, we have $a^2 = b^2 + c^2$.

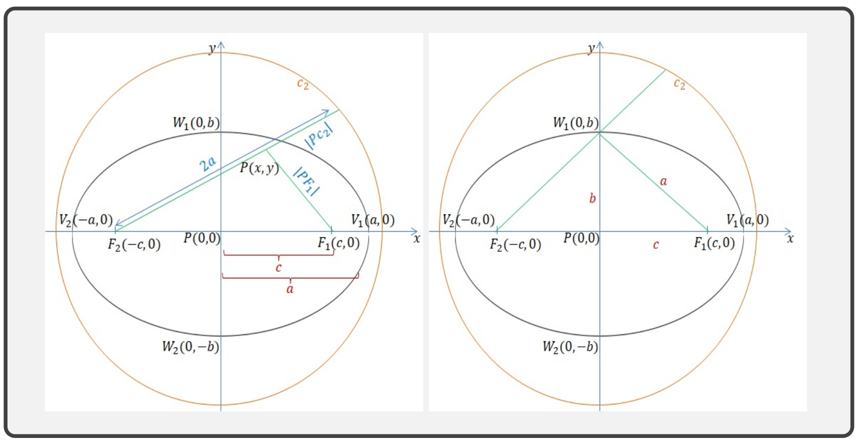


Figure 1.17

From Definition 1.2, we have

$|PF_2| + |PF_1| = 2a$

$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$

$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$

$\sqrt{(x-c)^2+y^2} = -\left(a - \frac{c}{a}x\right)$

$(x-c)^2+y^2 = a^2 - \frac{c^2}{a^2}x^2 - 2cx$

$x^2 - 2cx + c^2 + y^2 = a^2 - \frac{c^2}{a^2}x^2 - 2cx$ isolate x-terms any y-terms

$\left(\frac{a^2-c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$ divide both sides by $a^2 - c^2$

$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $b^2 = a^2 - c^2$

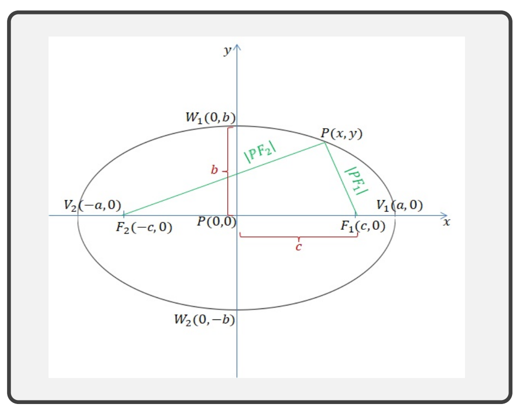


Figure 1.18

1.2.1 Ellipses with the Center at the Origin

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(A) If $a > b$, the ellipse has the following properties:

- The center of the ellipse is $P(0,0)$.
- The vertices of the ellipse are $V_1(a,0)$, $V_2(-a,0)$.
- The foci of the ellipse are $F_1(c,0)$, $F_2(-c,0)$, where
$$c = \sqrt{a^2 - b^2}\ .$$
- The major axis of the ellipse is x-axis with length $2a$.
- The minor axis endpoints (co-vertices) are $W_1(0,b)$, $W_2(0,-b)$.
- The minor axis of the ellipse is y-axis with length $2b$.

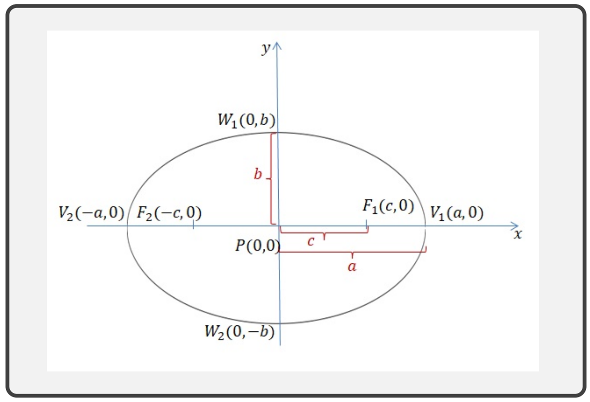


Figure 1.19: The graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.

(B) If $b > a$, the ellipse has the following properties:

- The center of the ellipse is $P(0,0)$.
- The vertices of the ellipse are $V_1(0,b)$, $V_2(0,-b)$.
- The foci of the ellipse are $F_1(0,c)$, $F_2(0,-c)$, where
$$c = \sqrt{b^2 - a^2}\ .$$
- The major axis of the ellipse is x-axis with length $2b$.
- The minor axis endpoints (co-vertices) are $W_1(a,0)$, $W_2(-a,0)$.
- The minor axis of the ellipse is y-axis with length $2a$.

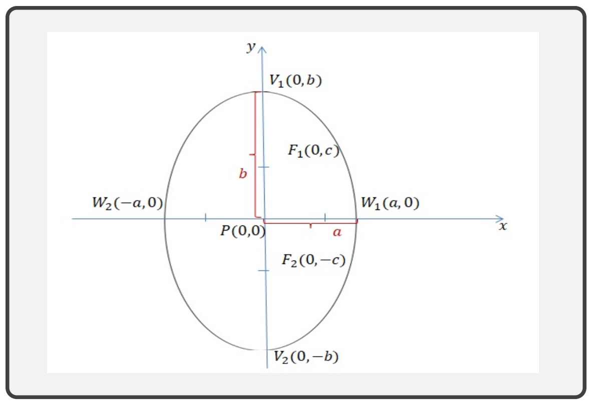


Figure 1.20: The graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b > a$.

■ **Example 1.7** Identify the features of the ellipse and sketch its graph.

(1) $9x^2 + 25y^2 = 225$ **(2)** $16x^2 + 9y^2 = 144$

Solution:

1. By dividing both sides by 225, we have $\frac{x^2}{25} + \frac{y^2}{9} = 1$. The result takes the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = 5$ and $b = 3$. Since $a > b$, then $c = \sqrt{25-9} = \sqrt{16} = 4$.

The ellipse has the following properties:

- The center of the ellipse is $P(0,0)$.
- The vertices of the ellipse are $V_1(5,0)$, $V_2(-5,0)$.
- The foci of the ellipse are $F_1(4,0)$, $F_2(-4,0)$
- The major axis of the ellipse is x-axis with length 10.
- The minor axis endpoints (co-vertices) are $W_1(0,3)$, $W_2(0,-3)$.
- The minor axis of the ellipse is y-axis with length 6.

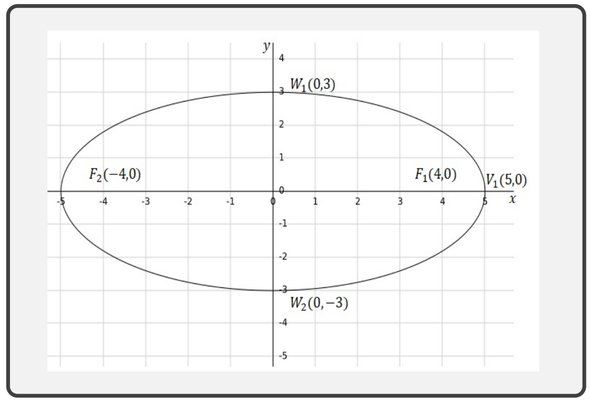


Figure 1.21: The graph of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, where $a>b$.

2. By dividing both sides by 144, we have $\frac{x^2}{9}+\frac{y^2}{16}=1$. The result takes the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a=3$ and $b=4$. Since $b>a$, then $c=\sqrt{16-9}=\sqrt{7}$.

The ellipse has the following properties:

- The center of the ellipse is $P(0,0)$.
- The Vertices of the ellipse are $V_1(0,4)$, $V_2(0,-4)$.
- The foci of the ellipse are $F_1(0,\sqrt{7})$, $F_2(0,-\sqrt{7})$
- The of the ellipse is x-axis with length 8.
- The minor axis endpoints (co-vertices) are $W_1(3,0)$, $W_2(-3,0)$.
- The minor axis of the ellipse is y-axis with length 6.

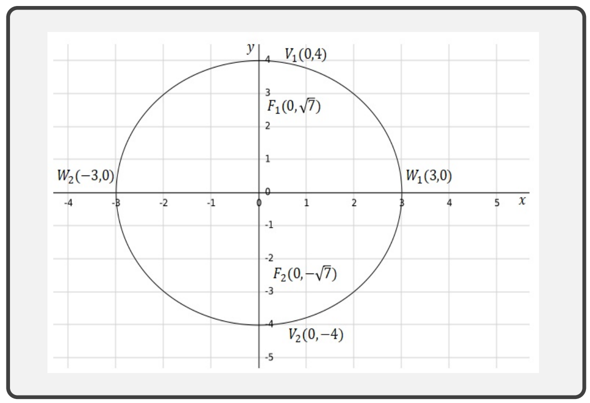


Figure 1.22: The graph of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$, where $b>a$.

■ **Example 1.8** Find an equation of an ellipse if the center is at the origin and

(1) Major axis on x-axis
Major axis length = 14
Minor axis length = 10

(2) Major axis on y-axis
Minor axis length = 14
Distance of foci from center = $10\sqrt{2}$

Solution:

(1) Since the major axis is on x-axis, then the equation of the ellipse takes the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$. Also, the major axis length is $2a=14$ and this implies $a=7$. The minor axis length is $2b=10$, so $b=5$. From this, the equation of the ellipse becomes $\frac{x^2}{49}+\frac{y^2}{25}=1$.

(2) Since the major axis is on y-axis, then the equation of the ellipse takes the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b>a$. From the minor axis length $2a=14$, we have $a=7$. Also, the distance of foci from the center is $c=10\sqrt{2}$. Since $c^2=b^2-a^2$, then $b^2=249$. By substituting the values of a and b, we have $\frac{x^2}{49}+\frac{y^2}{249}=1$.

1.2.2 Ellipses with the Center Not at the Origin

The equation of an ellipse of the form is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

(A) If $a>b$, the ellipse has the following properties:

- The center of the ellipse is $P(h,k)$.
- The Vertices of the ellipse are $V_1(h+a,k)$, $V_2(h-a,k)$.
- The foci of the ellipse are $F_1(h+c,k)$, $F_2(h-c,k)$, where
$$c=\sqrt{a^2-b^2}\,.$$
- The major axis of the ellipse is x-axis with length $2a$.
- The minor axis endpoints are $W_1(h,k+b)$, $W_2(h,k-b)$.
- The minor axis of the ellipse is y-axis with length $2b$.

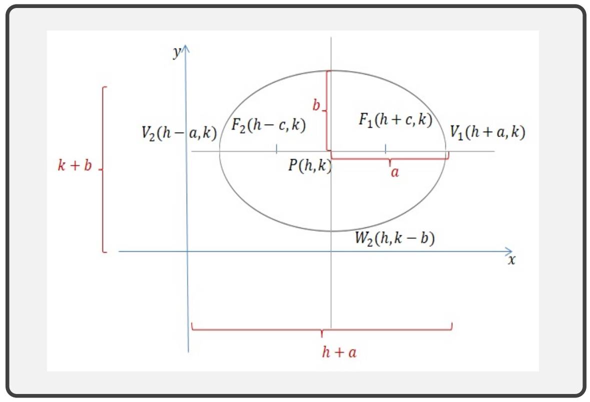


Figure 1.23: The graph of the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>b$.

(B) If $a<b$, the ellipse has the following properties:

- The center of the ellipse is $P(h,k)$.
- The Vertices of the ellipse are $V_1(h,k+b)$, $V_2(h,k-b)$.
- The foci of the ellipse are $F_1(h,k+c)$, $F_2(h,k-c)$, where
$$c=\sqrt{b^2-a^2}\,.$$
- The major axis of the ellipse is y-axis with length $2b$.
- The minor axis endpoints are $W_1(h+a,k)$, $W_2(h-a,k)$.
- The minor axis of the ellipse is x-axis with length $2a$.

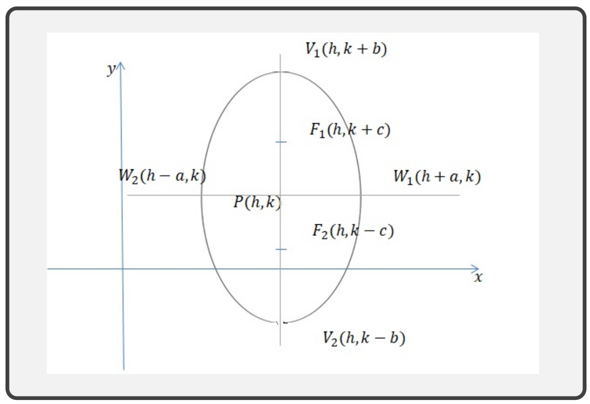


Figure 1.24: The graph of the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $b>a$.

■ **Example 1.9** Find the equation of the ellipse with foci at $(-3,1)$, $(5,1)$ and one of its vertice is $(7,1)$, then sketch its graph.

Solution:

Since the $y-$term in the foci is constant, the equation of the ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $a>b$.

From the given foci, we have

$$F_1(h+c,k)=(5,1)\Rightarrow h+c=5,\ k=1$$
$$F_2(h-c,k)=(-3,1)\Rightarrow h-c=-3,\ k=1$$

By doing some calculation, we obtain $h=1$ and $c=4$.

Illustration:

$$h+c=5\rightarrow 1$$
$$h-c=-3\rightarrow 2$$

Also, from the given vertex, we have $V_1(h+a,k)=(7,1)$ and by substituting the value of h, we obtain $a=6$.

From the formula $c^2=a^2-b^2$, we have $b^2=36-16=20$, so $b=2\sqrt{5}$. Thus, the equation of the ellipse is

$$\frac{(x-1)^2}{36}+\frac{(y-1)^2}{20}=1\,.$$

The ellipse has the following properties:

- The center of the ellipse is $P(1,1)$.
- The Vertices of the ellipse are $V_1(7,1)$, $V_2(-5,1)$.
- The foci of the ellipse are $F_1(5,1)$, $F_2(-3,1)$.
- The major axis of the ellipse is x-axis with length 12.
- The endpoints of the minor axis are $W_1(1,1+4\sqrt{5})$ and $W_2(1,1-4\sqrt{5})$.
- The minor axis of the ellipse is y-axis with length $8\sqrt{5}$.

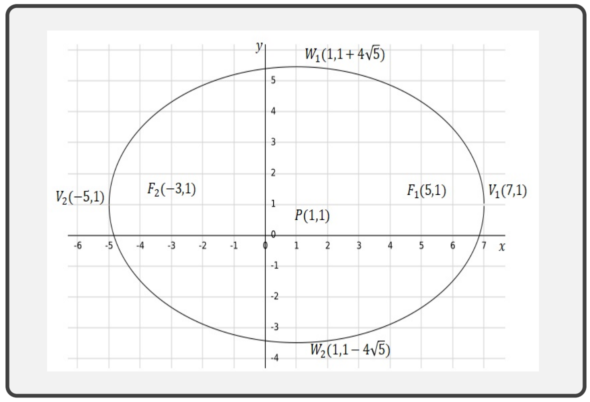


Figure 1.25: The graph of the ellipse $\frac{(x-1)^2}{36}+\frac{(y-1)^2}{20}=1$.

■ **Example 1.10** Find the equation of the ellipse with foci at $(2,5)$, $(2,-3)$ and the length of its minor axis equals 6, then and sketch its graph.

Solution:

Since the $x-$term in the foci is constant, the equation of the ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $b>a$.

From the given foci, we have

$$F_1(h,k+c)=(2,5)\Rightarrow h=2,\ k+c=5$$

$$F_2(h,k-c)=(2,-3)\Rightarrow h=2,\ k-c=-3$$

By doing some calculation, we obtain $k=1$ and $c=4$.

Illustration:

$$k+c=5\rightarrow 1$$
$$k-c=-3\rightarrow 2$$

Also, the length of its minor axis equals $2a=6$, hence $a=3$. From the formula $c^2=b^2-a^2$, we have $b^2=16+9=25$, so $b=5$. Thus, the equation of the ellipse is

$$\frac{(x-2)^2}{9}+\frac{(y-1)^2}{25}=1\ .$$

The ellipse has the following properties:

- The center of the ellipse is $P(2,1)$.
- The Vertices of the ellipse are $V_1(2,6)$, $V_2(2,-4)$.
- The foci of the ellipse are $F_1(2,5)$, $F_2(2,-3)$.
- The major axis of the ellipse is y-axis with length 10.
- The minor axis endpoints are $W_1(5,1)$, $W_2(-1,1)$.
- The minor axis of the ellipse is x-axis with length 6.

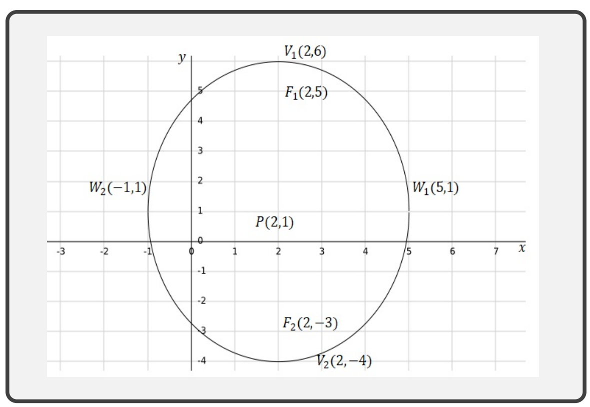


Figure 1.26: The graph of the ellipse $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{25}=1$.

■ **Example 1.11** Identify the features of the ellipse $4x^2+2y^2-8x-8y-20=0$, then sketch its graph.

Solution:

$$
\begin{aligned}
4x^2+2y^2-8x-8y-20&=0\\
2x^2+y^2-4x-4y-10&=0\\
2x^2-4x+y^2-4y&=10 \qquad \text{isolate } x \text{ any } y \text{ terms}\\
2(x^2-2x+1)+(y^2-4x+4)&=10+2+4 \qquad \text{completing square: } (a\pm b)^2=a^2\pm 2ab+b^2\\
2(x-1)^2+(y-2)^2&=16\\
\frac{(x-1)^2}{8}+\frac{(y-2)^2}{16}&=1 \qquad \text{divide by } 16\ .
\end{aligned}
$$

The result takes the standard form

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1\ ,$$

where

$$h=1,\ k=2,\ a=2\sqrt{2},\ \text{ and }\ b=4,\ \text{ then }\ c=\sqrt{16-8}=2\sqrt{2}\ .$$

The ellipse has the following properties:

- The center of the ellipse is $P(1,2)$.
- The Vertices of the ellipse are $V_1(1,6)$, $V_2(1,-2)$.
- The foci of the ellipse are $F_1(1,2+2\sqrt{2})$, $F_2(1,2-2\sqrt{2})$.
- The major axis of the ellipse is y-axis with length 8.
- The endpoints of the minor axis are $W_1(1+2\sqrt{2},2)$ and $W_2(1-2\sqrt{2},2)$.
- The minor axis of the ellipse is x-axis with length $4\sqrt{2}$.

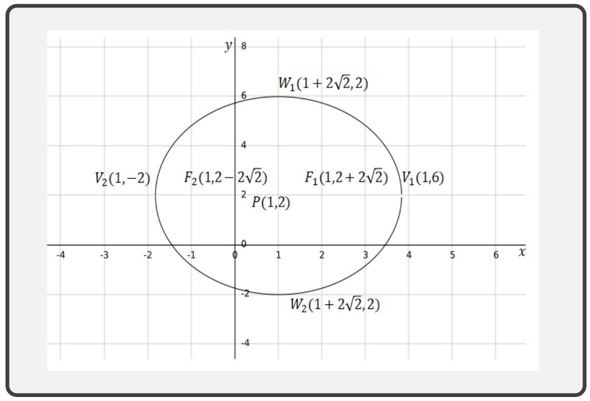


Figure 1.27: The graph of the ellipse $\frac{(x-1)^2}{8}+\frac{(y-2)^2}{16}=1$.

1.3 Hyperbola

Definition 1.3 A hyperbola is the set of all points in a plane such that the absolute value of the difference of the distances of each point from two fixed points (called foci) is constant.

- Each fixed point mentioned in the previous definition is called a focus.
- The point midway between the foci is called the center. The line containing the foci is the **transverse axis**.
- The graph of the hyperbola is made up of two parts called branches. Each branch intersects the transverse axis at a point called the vertex.

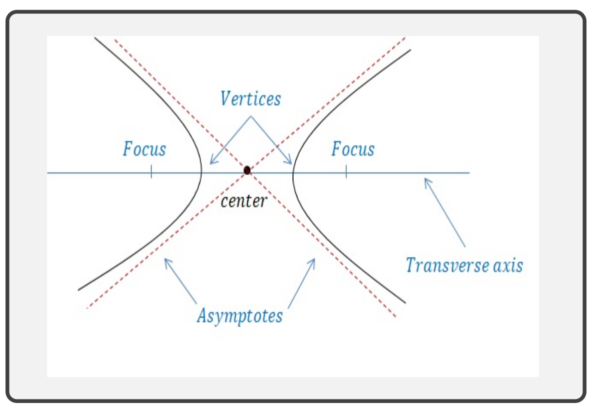


Figure 1.28: An illustrative graph of the hyperbola.

Let c_2 be the circle with midpoint F_2 and radius $2a$. The distance of a point P of the right branch to the circle c_2 equals the distance to the focus F_1: $|PF_1| = |Pc_2|$.

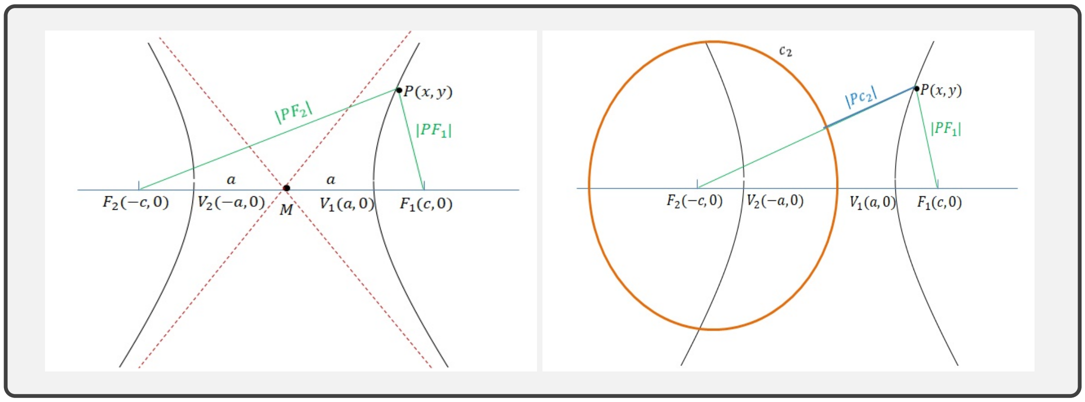


Figure 1.29

From Definition 1.3, we have

$$||PF_1| - |PF_2|| = 2a$$

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = 2a$$

$$x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2) \qquad \text{Rearranging and completing the square}$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1 \qquad \text{dividing both sides by } c^2-a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad b^2 = c^2 - a^2\ .$$

1.3.1 Hyperbola with the Center at the Origin

(A) The equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The hyperbola has the following properties:

- The center of the ellipse is $P(0,0)$.
- The vertices of the ellipse are $V_1(a,0)$, $V_2(-a,0)$.
- The foci of the ellipse are $F_1(c,0)$, $F_2(-c,0)$, where
$$c = \sqrt{a^2+b^2}\ .$$
- The transverse axis is x-axis with length $2a$.
- The asymptotes are $y = \pm\frac{b}{a}x$.

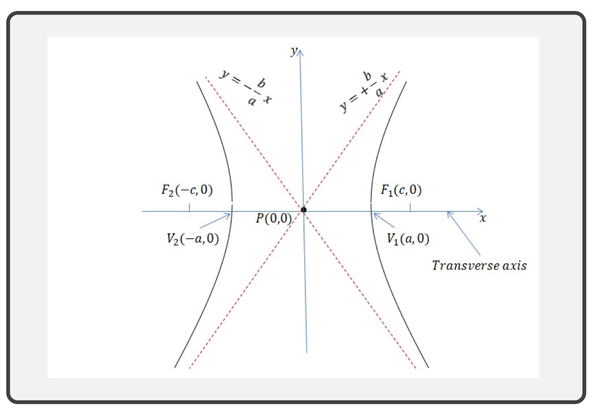


Figure 1.30: The graph of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(B) The equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

The hyperbola has the following properties:

- The center of the ellipse is $P(0,0)$.
- The vertices of the ellipse are $V_1(0,b)$, $V_2(0,-b)$.
- The foci of the ellipse are $F_1(0,c)$, $F_2(0,-c)$, where
$$c = \sqrt{a^2+b^2}\ .$$
- The transverse axis is x-axis with length $2b$.
- The asymptotes are $y = \pm\frac{b}{a}x$.

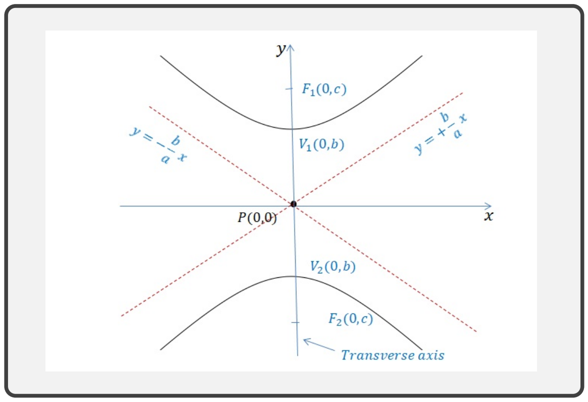


Figure 1.31: The graph of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

■ **Example 1.12** Identify the features of the hyperbola and sketch its graph.

(1) $4x^2 - 16y^2 = 64$

(2) $4y^2 - 9x^2 = 36$

Solution:

(1) By dividing both sides by 64, we have $\frac{x^2}{16} - \frac{y^2}{4} = 1$. The result takes the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since $a = 4$ and $b = 2$, then $c = \sqrt{16+4} = 2\sqrt{5}$.

The hyperbola has the following properties:

- The center of the ellipse is $P(0,0)$
- The vertices of the ellipse are $V_1(4,0)$, $V_2(-4,0)$.
- The foci of the ellipse are $F_1(2\sqrt{5},0)$, $F_2(-2\sqrt{5},0)$.
- The transverse axis is x-axis with length 8.
- The asymptotes are $y = \pm\frac{1}{2}x$.

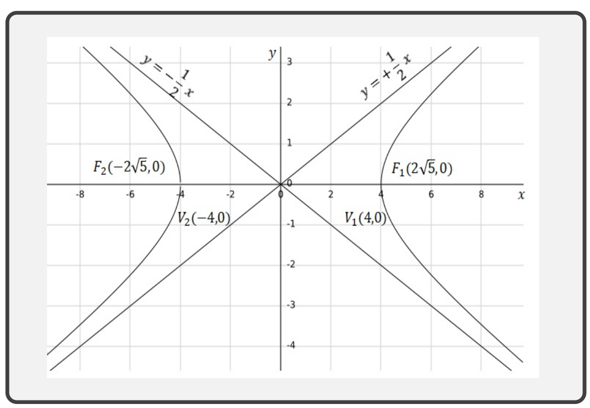


Figure 1.32: The graph of the hyperbola $\frac{x^2}{16} - \frac{y^2}{4} = 1$.

(2) Divide both sides by 36 to have $\frac{y^2}{9} - \frac{x^2}{4} = 1$. The result takes the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. Since $a = 2$ and $b = 3$, then $c = \sqrt{4+9} = \sqrt{13}$.

The hyperbola has the following properties:

- The center of the ellipse is $P(0,0)$
- The vertices of the ellipse are $V_1(0,3)$, $V_2(0,-3)$.
- The foci of the ellipse are $F_1(0,\sqrt{13})$, $F_2(0,-\sqrt{13})$.
- The transverse axis is x-axis with length 6.
- The asymptotes are $y=\pm\frac{3}{2}x$.

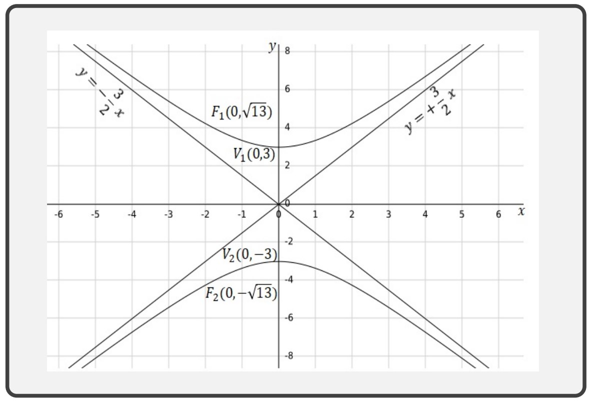


Figure 1.33: The graph of the hyperbola $\frac{y^2}{9}-\frac{x^2}{4}=1$.

■ **Example 1.13** Find an equation of the hyperbola if its vertices are $V_1(3,0)$ and $V_2(-3,0)$, and one of its foci $(4,0)$, then sketch its graph.

Solution: Since the $y-$term in the vertices is constant, the equation of the hyperbola takes the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Also, $V_1(a,0)=V_1(3,0)$ implies $a=3$ and $F_1(c,0)=F_1(4,0)$ implies $c=4$.

From the formula $c^2=a^2+b^2$, we have $b=\sqrt{16-9}=\sqrt{7}$.
Thus, the equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{7}=1$.
The hyperbola has the following properties:

- The center of the ellipse is $P(0,0)$
- The vertices of the ellipse are $V_1(3,0)$, $V_2(-3,0)$.
- The foci of the ellipse are $F_1(4,0)$ and $F_2(-4,0)$.
- The transverse axis is x-axis with length 6.
- The asymptotes are $y=\pm\frac{\sqrt{7}}{3}x$.

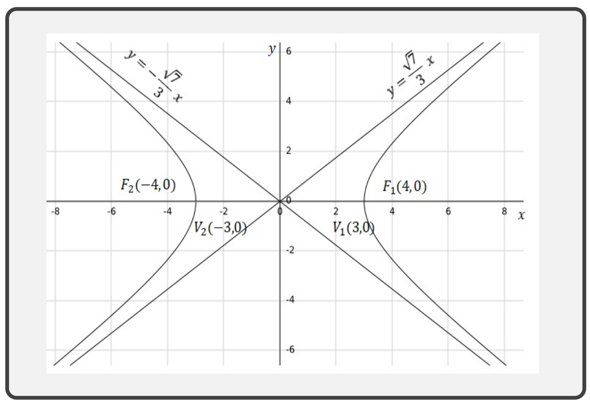


Figure 1.34: The graph of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

1.3.2 Hyperbola with the Center Not at the Origin

(A) The equation of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

The hyperbola has the following properties:

- The center of the ellipse is $P(h,k)$
- The vertices of the ellipse are $V_1(h+a,k)$, $V_2(h-a,k)$.
- The foci of the ellipse are $F_1(h+c,k)$, $F_2(h-c,k)$, where
$$c=\sqrt{a^2+b^2}\,.$$
- The transverse axis is x-axis with length $2a$.
- The asymptotes are $(y-k)=\pm\frac{b}{a}(x-h)$.

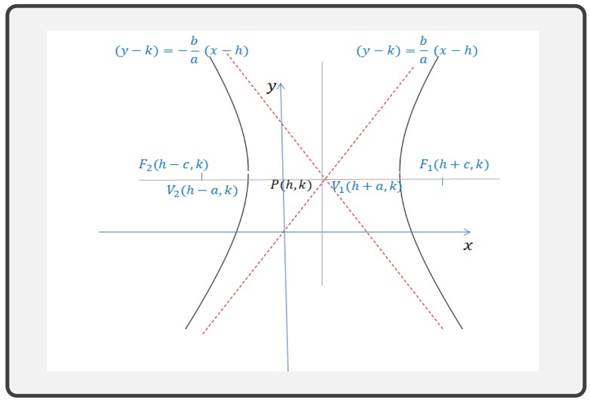


Figure 1.35: The graph of the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

(B) The equation of the hyperbola is $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$.

The hyperbola has the following properties:

- The center of the ellipse is $P(h,k)$
- The vertices of the ellipse are $V_1(h,k+b)$, $V_2(h,k-b)$.
- The foci of the ellipse are $F_1(h,k+c)$, $F_2(h,k-c)$.
- The transverse axis is x-axis with length $2b$.
- The asymptotes are $(y-k)=\pm\frac{b}{a}(x-h)$.

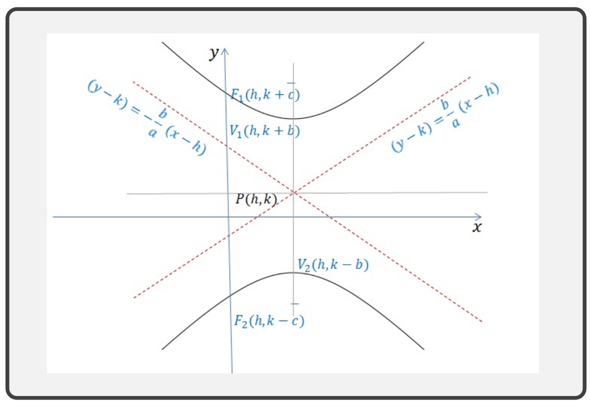


Figure 1.36: The graph of the hyperbola $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$.

■ **Example 1.14** Find the equation of the hyperbola with foci at $(-2,2)$, $(6,2)$ and one of its vertices is $(5,2)$, then sketch its graph.

Solution:

Since the $y-$term in the foci is constant, then the equation of the hyperbola takes the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.
From the given foci, we have

$$F_1(h+c,k)=(6,2)\Rightarrow h+c=6, k=2$$
$$F_2(h-c,k)=(-2,2)\Rightarrow h-c=-2, k=2$$

By doing some calculation, we obtain $h=2$ and $c=4$.

Illustration:

$$h+c=6\rightarrow 1$$
$$h-c=-2\rightarrow 2$$

Also, from the given vertex $V_1(h+a,k)=(5,2)$, we have $h+a=5$. By substituting the value of h, we obtain $a=3$. From the formula $c^2=a^2+b^2$, we find $b^2=16-9=7$ and this implies $b=\sqrt{7}$. Thus, the equation of the hyperbola is $\frac{(x-2)^2}{9}+\frac{(y-2)^2}{7}=1$.

The hyperbola has the following properties:

- The center of the ellipse is $P(2,2)$
- The vertices of the ellipse are $V_1(5,2)$, $V_2(-1,2)$.
- The foci of the ellipse are $F_1(6,2)$, $F_2(-4,2)$.
- The transverse axis is x-axis with length 6.
- The asymptotes are $(y-2) = \pm\frac{\sqrt{7}}{3}(x-2)$.

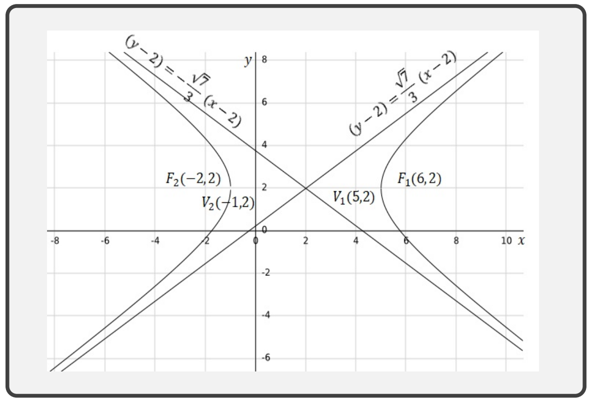


Figure 1.37: The graph of the hyperbola $\frac{(x-2)^2}{9} + \frac{(y-2)^2}{7} = 1$.

■ **Example 1.15** Find the equation of the hyperbola with foci at $(-1,-6)$, $(-1,4)$ and the length of its transverse axis is 8, and sketch its graph.

Solution:

Since the $x-$term in the foci is constant, the equation of the hyperbola takes the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.

From the foci, we have

$$F_1(h,k+c) = (-1,4) \Rightarrow h = -1, k+c = 4$$

$$F_2(h,k-c) = (-1,-6) \Rightarrow h = -1, k-c = -6$$

By doing some calculation, we obtain $k = -1$ and $c = 5$.

Illustration:

$$k + c = 4 \rightarrow 1$$

$$k - c = -6 \rightarrow 2$$

Also, the length of the transverse axis is $2b = 8$ and this implies $b = 4$. From the formula $c^2 = a^2 + b^2$, we have $a^2 = 25 - 16 = 9$, so $a = 3$.

Thus, the equation of the hyperbola is

$$\frac{(y+1)^2}{16} - \frac{(x+1)^2}{9} = 1 \ .$$

The hyperbola has the following properties:

- The center of the ellipse is $P(-1,-1)$
- The vertices of the ellipse are $V_1(-1,3)$, $V_2(-1,-4)$.
- The foci of the ellipse are $F_1(-1,4)$, $F_2(-1,-6)$.
- The transverse axis is x-axis with length 8.
- The asymptotes are $(y+1) = \pm\frac{4}{3}(x+1)$.

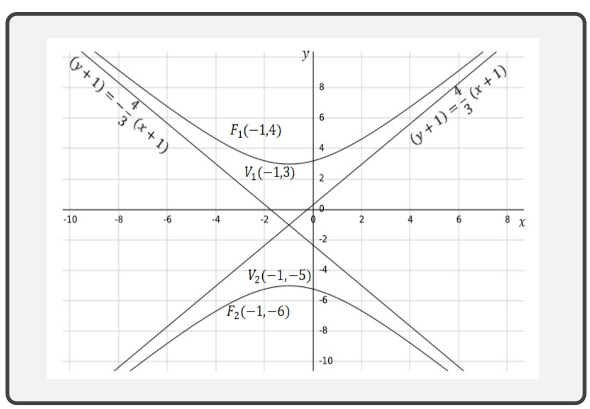


Figure 1.38: The graph of the hyperbola $\frac{(y+1)^2}{16} + \frac{(x+1)^2}{9} = 1$.

■ Example 1.16 Identify the features of the hyperbola $2y^2-4x^2-4y-8x-34=0$. Then, sketch its graph.

Solution:

$$\begin{aligned}
2y^2-4x^2-4y-8x-34&=0,\\
2y^2-4y-4x^2-8x&=34\\
2(y^2-2y)-4(x^2-2x)&=34 && \text{Rearranging } x\text{-terms and } y\text{-terms}\\
2(y^2-2y+1)-4(x^2-2x+1)&=34+2+4 && \text{completing the square}\\
2(y-1)^2-4(x+1)^2&=40\\
\frac{(y-1)^2}{20}-\frac{(x+1)^2}{10}&=1 && \text{dividing both sides by 40}
\end{aligned}$$

.

From the standard form $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$, we have $h=-1,\ k=1,\ a=\sqrt{10}$, and $b=2\sqrt{5}$. Thus, from the formula $c^2=a^2+b^2$, we have $c=\sqrt{30}$.

The hyperbola has the following properties:

- The center of the ellipse is $P(-1,1)$
- The vertices of the ellipse are
$$V_1(-1,1+2\sqrt{5}),\ V_2(-1,1-2\sqrt{5})\ .$$
- The foci of the ellipse are
$$F_1(-1,1+\sqrt{30}),\ F_2(-1,1-\sqrt{30})\ .$$
- The transverse axis is x-axis with length $4\sqrt{5}$.
- The asymptotes are $(y-1)=\pm\sqrt{2}(x+1)$.

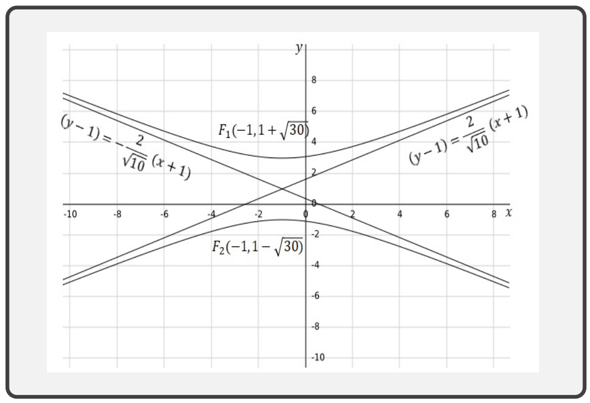


Figure 1.39: The graph of the hyperbola $\frac{(y-1)^2}{20}-\frac{(x+1)^2}{10}=1$.

Exercises

1 - 13 Write an equation of the parabola with the given elements, then sketch the graph.

1 Vertex at $(-4,2)$ and focus at $(-\frac{7}{2},2)$.

2 Vertex at $(2,6)$, passes through $(-1,4)$ and opens to the left.

3 Vertex at $(-1,1)$ and y-intercept of $(0,2)$.

4 Focus at $(2,4)$ and directrix is $y=-2$.

5 Vertex at $(0,-2)$, passes through $(-2,0)$ and opens to the left.

6 Vertex at $(5,3)$, passes through $(3,-1)$ and opens downwards.

7 Vertex at $(1,2)$ and focus at $(1,3)$.

8 Vertex at $(4,-3)$, passes through $(10,-6)$ and opens downwards.

9 Vertex at $(-3,-5)$, passes through $(-5,-6)$ and opens downwards.

10 Focus at $(4,1)$ and directrix is the y-axis.

11 Vertex at $(4,-5)$, passes through $(6,1)$ and opens upwards.

12 Vertex at $(-8,2)$, passes through $(2,-3)$ and opens downwards.

13 Focus at $(3,6)$ and the directrix $y=2$.

14 - 21 Write an equation of the ellipse with the given elements, then sketch the graph.

14 Center at the origin and major axis on x-axis and its length equals 8 and minor axis length equals 6.

15 One of its vertices $(3,0)$, one of its foci at $(2,0)$.

16 Center at $(2,2)$, one of its vertices $(4,2)$, and one of its foci $(2+\sqrt{3},2)$.

17 Center at $(1,-1)$, one of its vertices $(4,-1)$, and one of its foci $(1+\sqrt{5},-1)$.

18 Center at $(-2,3)$, major axis is parallel to y-axis, and its length equals 8 and minor axis length equals 4.

19 Vertices $(2,3)$ and $(2,-2)$, and minor axis is parallel to x-axis and its length equals 2.

20 Vertices $(-1,-1)$ and $(-1,9)$, and minor axis is parallel to x-axis with length 8.

21 Foci $(10,-2),(4,-2)$ and one of its vertices $(12,-2)$.

22 - 31 Write an equation of the hyperbola with the given elements, then sketch the graph.

22 Vertices $(0,-2)$ and $(0,2)$ and one of its foci $(0,\sqrt{13})$.

23 Vertices $(0,-6)$ and $(0,6)$, and one of its foci $(0,-8)$.

24 Vertices $(1,1)$ and $(11,1)$, and one of its foci $(12,1)$.

25 One of its vertices $(-4,0)$ and the asymptotes $y=\pm x$.

26 One of its vertices $(1,0)$ and the asymptotes $y=\pm 2x$.

27 One of its vertices $(0,5)$ and the asymptotes $y=\pm\frac{5}{3}x$.

28 One of its vertices $(0,-\frac{7}{2})$ and the asymptotes $y=\pm\frac{1}{2}x$.

29 Center at $(3,5)$, one of its vertices $(3,11)$ and one of foci $(3,5+2\sqrt{10})$.

30 Center at $(4,2)$, one of its vertices $(9,2)$ and one of foci $(4+\sqrt{26},2)$.

31 Foci $(4,-2)$ and $(10,-2)$, and one of its vertices $(8,-2)$.

32 - 75 ■ Determine the elements of the conic section and sketch its graph.

32 $(x-1)^2=8(y+1)$

33 $y=-(x-2)^2-2$

34 $y=-\frac{3}{4}(x+2)^2+3$

35 $y=\frac{1}{2}(x+2)^2-5$

36 $y=-4(x-1)^2+1$

37 $y=4x^2+24x+25$

38 $y=4(x-5)^2-7$

39 $y=-5(x+4)^2+9$

40 $y=x^2-8x+7$

41 $4x^2+10y^2=100$

42 $x^2+9y^2=36$

43 $\frac{x^2}{100}+\frac{y^2}{49}=1$

44 $\frac{x^2}{5}+\frac{y^2}{7}=1$

45 $\frac{x^2}{49}+\frac{y^2}{36}=1$

46 $\frac{(x+3)^2}{16}+\frac{(y-2)^2}{9}=1$

47 $\frac{x^2}{36}+\frac{y^2}{81}=1$

48 $\frac{x^2}{15}+\frac{y^2}{30}=1$

49 $\frac{x^2}{55}+\frac{y^2}{27}=1$

50 $\frac{x^2}{64}+\frac{y^2}{10}=1$

51 $25(x-3)^2+10(y+2)^2=100$

52 $\frac{x^2}{9}+\frac{(y-2)^2}{25}=1$

53 $49x^2+4y^2=196$

54 $y=-2x^2-28x-89$

55 $y=x^2+6x+5$

56 $y=2(x-4)^2-3$

57 $y=-2x^2-16x-35$

58 $x-5=(y-3)^2$

59 $x=-2y^2-4y-5$

60 $x^2+5y^2+6x-40y+84=0$

61 $x^2+2y+2x=2$

62 $\frac{x^2}{25}-\frac{y^2}{9}=1$

63 $\frac{x^2}{16}-\frac{y^2}{9}=1$

64 $\frac{y^2}{49}-\frac{x^2}{25}=1$

65 $\frac{x^2}{4}-\frac{y^2}{49}=1$

66 $\frac{x^2}{25}-\frac{y^2}{81}=1$

67 $\frac{y^2}{64}-\frac{x^2}{25}=1$

68 $\frac{x^2}{36}-\frac{y^2}{20}=1$

69 $\frac{(x+3)^2}{16}-\frac{(y-2)^2}{9}=1$

70 $\frac{(x-2)^2}{25}-\frac{y^2}{16}=1$

71 $\frac{(y-5)^2}{64}-\frac{(x-6)^2}{25}=1$

72 $\frac{(x+4)^2}{81}-\frac{(y-5)^2}{55}=1$

73 $-4x^2+10y^2=100$

74 $10y^2+49x^2=490$

75 $y^2-5x^2+6y-40x-76=0$

Chapter 2

MATRICES AND DETERMINANTS

2.1 Matrices

2.1.1 Definitions and Notations

Definition 2.1 A matrix A of order $m \times n$ is a set of numbers or expressions arranged in a rectangular array of m rows and n columns.

A matrix is a rectangular table of form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix}.$$

Notes :

(1) The horizontal arrays of a matrix are called its rows and the vertical arrays are called its columns.

(2) a_{ij} represents the element of the matrix A that lies in row i and column j.

(3) The matrix A can also be written as $A = [a_{ij}]_{m\times n}$.

■ **Example 2.1** Find the order of each matrix, then find the given elements.

(1) A = $\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}$, a_{11} and a_{22}

(2) B = $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \end{bmatrix}$, a_{12}, a_{21} and a_{23}

Solution:

1. The matrix A is of order 2×2. The elements $a_{11} = 2$ and $a_{22} = 0$.
2. The matrix B is of order 2×3. The elements $a_{12} = 3$, $a_{21} = 2$ and $a_{23} = 0$.

Definition 2.2 Two matrices $A = (a_{ij}]$ and $B = [b_{ij}]$ having the same order $m \times n$ are equal if $a_{ij} = b_{ij}$ for each $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

■ **Example 2.2** Find the value of x if the matrices $A = B$.

A = $\begin{bmatrix} 1 & 2 \\ -1 & 4x-1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 2 \\ -1 & 11 \end{bmatrix}$

Solution:

Since the matrices $A = B$, then from Definition 2.2, we have $4x - 1 = 11$. This implies $x = 3$.

2.1.2 Special Types of Matrices

1. **Row vector.** A row vector of order n is a matrix of order $1 \times n$ written as $A = [a_1\ a_2\ ...\ a_n]$. For example, $A = [2\ \ 7\ \ 0\ \ -1\ \ 9]$ is a row vector of order 5.

2. **Column vector.** A column vector of order n is a matrix of order $n \times 1$ written as A = $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. For example, A = $\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$ is a column vector of order 3.

3. **Null matrix.** The matrix $A = [a_{ij}]_{m \times n}$ is called a null matrix if $a_{ij} = 0$ for all i and j i.e.
$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$
For example, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null matrix of order 2×3.

4. **Square matrix.** If the number of rows equals the number of columns $(m = n)$, then the matrix is called a square matrix of order n. In a square matrix $A = [a_{ij}]$, the set of elements of the form a_{ii} is called the diagonal of the matrix. For example, the diagonal of the following square matrix is highlighted in red $\begin{bmatrix} 2 & -7 & 3 \\ 1 & 0 & 9 \\ -1 & 6 & 8 \end{bmatrix}$.

5. **Upper triangular matrix.** The square matrix $A = [a_{ij}]$ of order n is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$:
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$
For example, $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ is an upper triangular matrix of order 3.

6. **Lower triangular matrix.** The square matrix $A = [a_{ij}]$ of order n is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$:
$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

For example, $\begin{bmatrix} 4 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 3 & 5 \end{bmatrix}$ is a lower triangular matrix of order 3.

7. **Diagonal matrix.** The square matrix $A = [a_{ij}]$ of order n is called a diagonal matrix if $a_{ij} = 0$ for all ij:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

This can be written as $diag(a_{11}, a_{22}, ..., a_{nn})$.
For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix of order 3. Note that a square matrix that is both upper and lower triangular is called a diagonal matrix.

8. **Identity matrix.** The square matrix $I_n = [a_{ij}]$ of order n is called an identity matrix if $a_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$.
An identity matrix of order n can be represents by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3.

2.1.3 Operations on Matrices

(1) Addition and subtraction of matrices :

Definition 2.3 Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of order $m \times n$. Then,

1. $A + B = C$ with $c_{ij} = a_{ij} + b_{ij}$.
2. $A - B = C$ with $c_{ij} = a_{ij} - b_{ij}$.

From Definition 2.3, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of order $m \times n$, then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

Also,

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{bmatrix}.$$

■ **Example 2.3** If A = $\begin{bmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 0 & 8 \\ 1 & 4 & -1 \\ 10 & 11 & -2 \end{bmatrix}$, find $A+B$ and $A-B$.

Solution:

A+B = $\begin{bmatrix} 1+5 & 3+0 & 2+8 \\ 5+1 & -4+4 & 6+(-1) \\ 0+10 & 9+11 & 2+(-2) \end{bmatrix}$ = $\begin{bmatrix} 6 & 3 & 10 \\ 6 & 0 & 5 \\ 10 & 20 & 0 \end{bmatrix}$.

A-B = $\begin{bmatrix} 1-5 & 3-0 & 2-8 \\ 5-1 & -4-4 & 6-(-1) \\ 0-10 & 9-11 & 2-(-2) \end{bmatrix}$ = $\begin{bmatrix} -4 & 3 & -6 \\ 4 & -8 & 7 \\ -10 & -2 & 4 \end{bmatrix}$.

(2) Multiplying a matrix by a scalar:

Definition 2.4 Let $A = [a_{ij}]$ be a matrix of order $m \times n$. Then, for any $k \in \mathbb{R}$, kA is a matrix $C = [c_{ij}]$ with $c_{ij} = ka_{ij}$.

From Definition 2.4, if $A = [a_{ij}]$ is a matrix of order $m \times n$ and $k \in \mathbb{R}$ then $k\,A = [k\,aij]$.

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

■ **Example 2.4** If A = $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 9 & 2 \end{bmatrix}$, find $3A$.

Solution:

$$3A = \begin{bmatrix} 3\times 1 & 3\times 3 & 3\times 2 \\ 3\times 0 & 3\times 9 & 3\times 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 6 \\ 0 & 27 & 6 \end{bmatrix}$$

■ **Example 2.5** If A = $\begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}$, find $-2A+3B$.

Solution:

$$-2A+3B = -2\begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ 0 & 24 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 4 & 16 \end{bmatrix}.$$

Theorem 2.5 Let A, B and C be matrices of order $m \times n$, and let $k, \ell \in \mathbb{R}$. Then

1. The addition of matrices is commutative: $A + B = B + A$.
2. The addition of matrices is associative: $(A+B)+C = A+(B+C)$.
3. The null matrix is the identity matrix of addition: $A + 0 = A$.
4. $(k+\ell)A = kA + \ell A$.
5. $k(\ell A) = (k\ell)A$.

Definition 2.6 Let $A = [a_{ij}]$ be a matrix of order $m \times n$. There exists a matrix B such that $A + B = 0$. This matrix B is called the additive inverse of A and it is denoted by $-A = (-1)A$.

(3) Multiplication of matrices:

Definition 2.7 Let $A = [a_{ij}]$ be a matrix of order $m \times n$ and $B = [b_{ij}]$ be a matrix of order $n \times p$. The multiplication AB is a matrix $C = [c_{ij}]$ of order $m \times p$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj} \ .$$

Note: the multiplication AB is defined if and only if the number of columns of A equals the number of rows of B; otherwise, we say the multiplication is undefined.

■ **Example 2.6** If A = $\begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}$, find AB.

Solution:

$$AB = \begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 6 \times 10 & 1 \times 3 + 6 \times 8 \\ -2 \times 2 + 4 \times 0 & -2 \times 3 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 2 & 51 \\ -4 & 26 \end{bmatrix}.$$

■ **Example 2.7** If A = $\begin{bmatrix} 1 & 6 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 7 \end{bmatrix}$, find AB.

Solution:

$$AB = \begin{bmatrix} 1 & 6 & 2 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 29 & 26 \\ -8 & 11 & 15 \end{bmatrix}.$$

A special case of multiplication of matrices is multiplying a row vector by a column vector. Let $A = [a_1\ a_2\ \dots\ a_n]$ be a row vector of order n and B = $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column vector of order n. Then the multiplication AB is a matrix $C = [c]$ of order 1, where

$$c = \sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \ .$$

■ **Example 2.8** If A = $\begin{bmatrix} 2 & 1 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix}$, find AB

Solution:

$AB = [2 \times 8 + 1 \times (-3) + 4 \times 5] = [33]$.

■ **Example 2.9** If A = $\begin{bmatrix} 2 & 1 & 4 & -6 \end{bmatrix}$ and B = $\begin{bmatrix} 1 \\ 3 \\ -4 \\ 7 \end{bmatrix}$, find AB

Solution:

$AB = [2 \times 1 + 1 \times 3 + 4 \times (-4) + (-6) \times 7] = [-53]$.

■ **Example 2.10** If A = $\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$, Compute (if possible) 1. AB **2.** BC.

Solution:

(1) $AB = A = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = [-1]$.

(2) The multiplication BC is not possible since the matrix B of order 3×1 and the matrix C of order 3×2.

Theorem 2.8 Let A be a matrix of order $m \times n$, B be a matrix of order $n \times p$ and C be a matrix of order $p \times q$. Then,

1. The multiplication of matrices is not commutative: $AB \neq BA$.
2. The multiplication of matrices is associative: $(AB)C = A(BC)$.
3. The matrix I_n is the identity matrix of multiplication: $AI_n = A$.
4. For any $k \in \mathbb{R}$, $(kA)B = k(AB) = A(kB)$.

Theorem 2.9 Let A and B be any two matrices of order $m \times n$. The multiplication of matrices is distributive:

1. $(A+B)C = AC+BC$, where C is a matrix of order $n \times p$.
2. $C(A+B) = CA+CB$, where C is a matrix of order $p \times m$.

■ **Example 2.11** If A = $\begin{bmatrix} 4 & 3 & 9 \\ -1 & 2 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ Compute (if possible) **1.** AB **2.** BA.

Solution:

(1) The multiplication BC is not possible since the matrix B of order 3×1 and the matrix C of order 3×2.

(2) $BA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 9 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 13 \\ -1 & 2 & 0 \end{bmatrix}$.

■ **Example 2.12** If A = $\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 7 & 5 \\ 1 & 0 \end{bmatrix}$, compute (if possible) **1.** AB **2.** BA.

Solution:

(1) $AB = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ -5 & -5 \end{bmatrix}$.

(2) $BA = \begin{bmatrix} 7 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 38 \\ 3 & 4 \end{bmatrix}$.

From this example, we find $AB \neq BA$ and this means the multiplication of matrices is not commutative.

■ **Example 2.13** If A = $\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$, find AI_2

Solution: $AI_n = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$.

Definition 2.10 Let $A = [a_{ij}]$ be a matrix of order $m \times n$. Then, the transpose of A is $A^t = [a_{ji}]_{n \times m}$.

Theorem 2.11 Let A and B be any two matrices of order $m \times n$ and $k \in \mathbb{R}$.

1. $(A^t)^t = A$.
2. $(A+B)^t = A^t + B^t$.
3. $(kA)^t = kA^t$.
4. $(AB)^t = B^t A^t$.

Remark 2.12

1. The transpose of a row vector is a column vector and vice-versa.
2. The transpose of a lower triangular matrix is an upper triangular matrix and vice-versa.

Example 2.14 If A = $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ and C = $\begin{bmatrix} 5 & 3 \\ 1 & 4 \\ -1 & 2 \end{bmatrix}$, Compute

(1) $(A^t)^t$

(2) $(A+B)^t$

(3) $(3A)^t$

(4) $(AC)^t$

Solution:

(1) $(A^t) = \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$, so $(A^t)^t = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$.

(2) $(A+B) = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 1 & 5 & 2 \end{bmatrix}$. From this, $(A+B)^t = \begin{bmatrix} 5 & 1 \\ 0 & 5 \\ -2 & 2 \end{bmatrix}$.

(3) $(3A)^t = 3A^t = 3 \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ -3 & 15 \\ 0 & 3 \end{bmatrix}$.

(4) $(AC)^t = C^t A^t = \begin{bmatrix} 5 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 5 & 50 \end{bmatrix}$.

2.2 Determinants of Matrices

Let A be a square matrix. Then, the determinant of A is denoted by $det(A)$ or $|A|$.

Definition 2.13 Let $A = [a_{ij}]$ be a square matrix of order n. Then, the determinant of A can be defined as follows:

$$det(A) = \begin{cases} a & : A = [a] \\ \sum_{j=1}^{n}(-1)^{i+j}a_{ij}A_{ij}(i = 1,...,n) & : \text{otherwise}, \end{cases}$$

where A_{ij} is $det(A)$ after removing the row i and clomun j.

2.2.1 The determinant of an 2×2 Matrix

Let A be a square matrix of order 2 as follows:

A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

■ **Example 2.15** Find the determinant of the matrix.

(1) A = $\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$ **(2)** B = $\begin{bmatrix} 4 & -1 \\ 2 & 9 \end{bmatrix}$

Solution:

1. $det(A) = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 1 \times 7 - 3 \times 5 = 7 - 15 = -8.$

2. $det(B) = \begin{vmatrix} 4 & -1 \\ 2 & 9 \end{vmatrix} = 4 \times 9 - (-1) \times 2 = 36 + 2 = 38.$

2.2.2 The determinant of an $n \times n$ Matrix

Before starting evaluating the determinant of an $n \times n$ matrix, we first need to define the minor and cofactor of that matrix. The **minor** M_{ij} is the determinant of the matrix obtained by eliminating the i^{th} row and the j^{th} column of A.

■ **Example 2.16** If A = $\begin{bmatrix} 1 & 3 & 1 \\ -2 & -1 & 2 \\ 2 & 4 & 5 \end{bmatrix}$, find the minors M_{11}, M_{12} and M_{13}.

Solution:

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 5 \end{vmatrix} = -1 \times 5 - 2 \times 4 = -13.$$

$$M_{12} = \begin{vmatrix} -2 & 2 \\ 2 & 5 \end{vmatrix} = -2 \times 5 - 2 \times 2 = -14.$$

$$M_{13} = \begin{vmatrix} -2 & -1 \\ 2 & 4 \end{vmatrix} = -2 \times 4 - (-1) \times 2 = -6.$$

The **cofactor** C_{ij} of the matrix A is defined as follows:

$$C_{ij} = (-1)^{i+j} M_{ij} \ .$$

Note that the cofactor C_{ij} depends on the minor M_{ij}.

■ **Example 2.17** In the previous example, calculate the corresponding cofactors of the minors M_{11}, M_{12} and M_{13}.

Solution:

$C_{11} = (-1)^{(1+1)} M_{11} = (1)(-13) = -13.$

$C_{12} = (-1)^{(1+2)} M_{12} = (-1)(-14) = 14.$

$C_{13} = (-1)^{(1+3)} M_{13} = (1)(-6) = -6.$

(1) The determinant of an 3×3 Matrix

The determinant of a matrix A is obtained as follows:

- Choose a row or a column of A (we usually choose a row).
- Multiply each of the elements a_{ij} of the row (or column) by its corresponding cofactor C_{ij}.

Let A be a square matrix of order 3 as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \ .$$

To calculate the determinant, choose the first row of A and multiply each of its elements by the corresponding cofactor:

$$\begin{aligned} det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}(-1)^{(1+1)}M_{11} + a_{12}(-1)^{(1+2)}M_{12} + a_{13}(-1)^{(1+3)}M_{13} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \ . \end{aligned}$$

■ **Example 2.18** Find the determinant of the matrix.

(1) $A = \begin{bmatrix} 1 & 6 & 3 \\ 5 & -1 & 4 \\ -2 & 9 & 7 \end{bmatrix}$ **(2)** $B = \begin{bmatrix} 4 & 1 & 5 \\ 2 & 1 & -2 \\ 1 & 8 & 7 \end{bmatrix}$

Solution:

$$(\mathbf{1})\quad det(A) = \begin{vmatrix} 1 & 6 & 3 \\ 5 & -1 & 4 \\ -2 & 9 & 7 \end{vmatrix} = 1C_{11} + 6C_{12} + 3C_{13}$$
$$= 1(-1)^{(1+1)}M_{11} + 6(-1)^{(1+2)}M_{12} + 3(-1)^{(1+3)}M_{13}$$
$$= 1\begin{vmatrix} -1 & 4 \\ 9 & 7 \end{vmatrix} - 6\begin{vmatrix} 5 & 4 \\ -2 & 7 \end{vmatrix} + 3\begin{vmatrix} 5 & -1 \\ -2 & 9 \end{vmatrix}$$
$$= 1(-7-36) - 6(35+8) + 3(45-2) = -42 - 258 + 129 = -171\ .$$

$$(\mathbf{2})\quad det(B) = \begin{vmatrix} 4 & 1 & 5 \\ 2 & 1 & -2 \\ 1 & 8 & 7 \end{vmatrix} = 4C_{11} + 1C_{12} + 5C_{13}$$
$$= 4(-1)^{(1+1)}M_{11} + 1(-1)^{(1+2)}M_{12} + 5(-1)^{(1+3)}M_{13}$$
$$= 4\begin{vmatrix} 1 & -2 \\ 8 & 7 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ 1 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 1 & 8 \end{vmatrix}$$
$$= 4(7+16) - 1(14+2) + 5(16-1) = 92 - 16 + 75 = 151\ .$$

(2) The determinant of an 4×4 Matrix

Let A be a square matrix of order 4 as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Then

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$
$$= a_{11}(-1)^{(1+1)}M_{11} + a_{12}(-1)^{(1+2)}M_{12} + a_{13}(-1)^{(1+3)}M_{13} + a_{14}(-1)^{(1+4)}M_{14}$$

where

$$\mathrm{M}_{11} = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, M_{12} = \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix},$$

$$\mathrm{M}_{13} = \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}, M_{14} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}.$$

■ **Example 2.19** Find the determinant of the matrix.

$$(\mathbf{1})\ \ \mathrm{A} = \begin{bmatrix} 1 & 6 & 3 & 2 \\ 5 & -1 & 4 & 1 \\ -2 & 9 & 7 & 3 \\ 7 & 1 & 3 & -6 \end{bmatrix} \qquad (\mathbf{2})\ \ \mathrm{B} = \begin{bmatrix} 4 & 1 & 5 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 3 & 9 \\ 1 & 7 & 4 & 6 \end{bmatrix}$$

Solution:

(1) $det(A_1) = \begin{vmatrix} -1 & 4 & 1 \\ 9 & 7 & 3 \\ 1 & 3 & -6 \end{vmatrix} = -1\begin{vmatrix} 7 & 3 \\ 3 & -6 \end{vmatrix} - 4\begin{vmatrix} 9 & 3 \\ 1 & -6 \end{vmatrix} + 1\begin{vmatrix} 9 & 7 \\ 1 & 3 \end{vmatrix}$

$= -1(-42-9) - 4(-54-3) + 1(27-7) = 51 + 228 + 20 = 299 .$

$det(A_2) = \begin{vmatrix} 5 & 4 & 1 \\ -2 & 7 & 3 \\ 7 & 3 & -6 \end{vmatrix} = 5\begin{vmatrix} 7 & 3 \\ 3 & -6 \end{vmatrix} - 4\begin{vmatrix} -2 & 3 \\ 7 & -6 \end{vmatrix} + 1\begin{vmatrix} -2 & 7 \\ 7 & 3 \end{vmatrix}$

$= 5(-42-9) - 4(12-21) + 1(-6-49) = -255 + 36 - 55 = -274 .$

$det(A_3) = \begin{vmatrix} 5 & -1 & 1 \\ -2 & 9 & 3 \\ 7 & 1 & -6 \end{vmatrix} = 5\begin{vmatrix} 9 & 3 \\ 1 & -6 \end{vmatrix} + 1\begin{vmatrix} -2 & 3 \\ 7 & -6 \end{vmatrix} + 1\begin{vmatrix} -2 & 9 \\ 7 & 1 \end{vmatrix}$

$= 5(-54-3) + 1(12-21) + 1(-2-63) = -285 - 9 - 65 = -359 .$

$det(A_4) = \begin{vmatrix} 5 & -1 & 4 \\ -2 & 9 & 7 \\ 7 & 1 & 3 \end{vmatrix} = 5\begin{vmatrix} 9 & 7 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} -2 & 7 \\ 7 & 3 \end{vmatrix} + 4\begin{vmatrix} -2 & 9 \\ 7 & 1 \end{vmatrix}$

$= 5(27-7) + 1(-6-49) + 4(-2-63) = 100 - 55 - 260 = -215 .$

Thus,

$det(A) = 1\,det(A_1) - 6\,det(A_2) + 3\,det(A_3) - 2\,det(A_4) = 1(299) - 6(-274) + 3(-359) - 2(-215) = 1296 .$

(2) $det(B_1) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & 9 \\ 7 & 4 & 6 \end{vmatrix} = 1\begin{vmatrix} 3 & 9 \\ 4 & 6 \end{vmatrix} - 0\begin{vmatrix} 1 & 9 \\ 7 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 7 & 4 \end{vmatrix}$

$= 1(18-36) - 0 + 1(4-21) = -18 - 17 = -35 .$

$det(B_2) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 9 \\ 1 & 4 & 6 \end{vmatrix} = 2\begin{vmatrix} 3 & 9 \\ 4 & 6 \end{vmatrix} - 0\begin{vmatrix} 0 & 9 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$

$= 2(18-36) - 0 + 1(0-3) = -36 - 3 = -39 .$

$det(B_3) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 9 \\ 1 & 7 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 9 \\ 7 & 6 \end{vmatrix} - 1\begin{vmatrix} 0 & 9 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 7 \end{vmatrix}$

$= 2(6-63) - 1(0-9) + 1(0-1) = -114 + 9 - 1 = -106 .$

$det(B_4) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 7 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 7 & 4 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 7 \end{vmatrix}$

$= 2(4-21) - 1(0-3) + 0 = -35 + 3 = -32 .$

Thus,

$det(B) = 4\,det(B_1) - 1\,det(B_2) + 5\,det(B_3) - 2\,det(B_4) = 4(-35) - 1(-39) + 5(-106) - 2(-32) = -773 .$

Theorem 2.14

1. If A is a square matrix having a zero row (or a zero column), then $det(A) = 0$.
2. If A is a square matrix having two equal rows (or two equal columns), then $det(A) = 0$.
3. If A is a square matrix having a row which is a multiple of another row (or a column which is a multiple of another column), then $det(A) = 0$.
4. If A is a diagonal matrix or an upper triangular matrix or a lower triangular matrix, then $det(A)$ is the product of the elements of the main diagonal.
5. The determinant of the null matrix is 0 and the determinant of the identity matrix is 1.
6. If B is obtained from A by multiplying a row (or column) by λ, then $det(B) = \lambda\, det(A)$.
7. If B is obtained from A by interchanging two rows (or two columns), then $det(B) = -det(A)$.
8. If B is obtained from A by multiplying a row by a non-zero constant and adding the result to another row (or multiplying a column by a non-zero constant and adding the result to another column), then $det(B) = det(A)$.

■ **Example 2.20** Find the determinant the matrix.

(1) $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 3 & 4 & 7 \end{bmatrix}$ **(3)** $C = \begin{bmatrix} 1 & 2 & -2 \\ 4 & 7 & 5 \\ 3 & 6 & -6 \end{bmatrix}$

(2) $B = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 5 & 6 \\ 3 & 4 & 3 \end{bmatrix}$ **(4)** $D = \begin{bmatrix} 3 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:

(1) The matrix A contains a zero row, so from (item **1**) in Theorem 2.14, we have $det(A) = 0$.

(2) The matrix A contains two equal columns, so from (item **2**) in Theorem 2.14, we have $det(A) = 0$.

(3) The third row in matrix A is a multiple of the first row by 2, so from (item **3**) in Theorem 2.14, we have $det(A) = 0$.

(4) The matrix A is an upper triangular matrix, so from (item **4**) in Theorem 2.14, we have $det(A) = -15$.

■ **Example 2.21** Find the determinant the matrix.

(1) $A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$ **(3)** $C = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 1 \\ 0 & -3 & 2 \end{bmatrix}$

(2) $B = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$ **(4)** $D = \begin{bmatrix} 1 & 3 & 1 \\ 6 & 8 & 2 \\ 0 & -3 & 2 \end{bmatrix}$

Solution:

(1) $det(A) = 1(4-3) - 3(8-0) + 1(-12-0) = -35$.

(2) The matrix B is obtained from A (in item 1) by multiplying the first row by 2, then $det(B) = 2det(A) = -70$.

(3) The matrix C is obtained from A by interchanging the first and second rows, then $det(C) = -det(A) = 35$.

(4) The matrix C is obtained from A by multiplying the first row by 2 and adding the result to the second row. Therefore, $det(D) = det(A) = -35$.

Theorem 2.15

1. Let A and B be square matrices of order n. Then $det(AB) = det(A)det(B)$.

2. Let A be a square matrix. Then $det(A) = det(A^t)$.

■ **Example 2.22** If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$, find **(1)** $det(AB)$ **(2)** $det(A^t)$.

Solution:

First, we compute $det(A)$ and $det(B)$.

$det(A) = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10$ and $det(B) = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = -2$.

(1) From Theorem 2.15, $det(AB) = det(A)det(B) = -10 \times (-2) = 20$.

(2) From Theorem 2.15, $det(A^t) = det(A) = -10$.

Exercises

1 - 10 ■ If A = $\begin{bmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 0 \\ 1 & 4 \\ 10 & 11 \end{bmatrix}$ and C = $\begin{bmatrix} -2 & 0 \\ 0 & 7 \\ 5 & 3 \end{bmatrix}$, compute the following (if possible):

1 $B + C$

2 $2B + 3C$

3 $C - B$

4 $A - C$

5 AB

6 BA

7 A^t

8 $(3A)^t$

9 $det(A)$

10 $det(2A)$

11 - 20 ■ If A = $\begin{bmatrix} 4 & -1 \\ 1 & 5 \\ 2 & 7 \end{bmatrix}$, B = $\begin{bmatrix} -2 & 1 \\ 3 & 6 \\ 1 & 4 \end{bmatrix}$ and C = $\begin{bmatrix} 1 & 2 & 1 \\ 9 & 5 & 3 \end{bmatrix}$, compute the following (if possible):

11 $A+B$

12 $5A$

13 $-3A+2B$

14 $A-C$

15 AC

16 BC

17 AB

18 B^t

19 $(2A)^t$

20 $(A^t)^t$

21 - 24 ■ If $det(B)=2$ and $det(A)=-3$, find the following:

21 $7det(A)$

22 $det(AB)$

23 $det(A^t)$

24 $det((AB)^t)$

25 - 32 ■ Find the determinant.

25 $\begin{vmatrix} 1 & -2 \\ 2 & 7 \end{vmatrix}$

26 $\begin{vmatrix} 3 & 1 & 3 \\ 7 & 2 & 9 \\ 1 & 7 & 4 \end{vmatrix}$

27 $\begin{vmatrix} 4 & -2 & 3 \\ 3 & 7 & 2 \\ 6 & 9 & 5 \end{vmatrix}$

28 $\begin{vmatrix} 4 & -2 & 3 & 1 \\ 3 & 7 & 2 & 2 \\ 6 & 9 & 5 & -1 \\ 1 & 0 & 5 & 1 \end{vmatrix}$

29 $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$

30 $\begin{vmatrix} 2 & -2 & 2 \\ -3 & 10 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

31 $\begin{vmatrix} 1 & -2 & 3 \\ 4 & 0 & 1 \\ 2 & 7 & 0 \end{vmatrix}$

32 $\begin{vmatrix} 1 & 5 & 3 & 6 \\ 1 & 0 & 1 & 2 \\ 2 & 7 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{vmatrix}$

Chapter 3

SYSTEMS OF LINEAR EQUATIONS

3.1 Linear Systems

Definition 3.1 A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \cdots \quad + \cdots \quad + \cdots \quad + \cdots \quad + \cdots &= \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + + a_{mn}x_n &= b_m \end{aligned} \tag{3.1}$$

where $a_{ij}, b_j \in \mathbb{R}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

The above system of linear equations can be written as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

A is called the coefficients matrix

X is called the column vector of the variables (or column vector of the unknowns)

B is called the column vector of constants (or column vector of the resultants)

A special case of the linear system of equations is a system of two different variables x_1 and x_2:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

The above system of linear equations can be written as $AX = B$ where A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and B = $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

3.2 Solution of Linear Equations Systems

A solution of the linear system $Ax = B$ is a column vector Y with entries $y_1, y_2, ..., y_n$ such that the linear system (3.1) is satisfied if we replace y_i with x_i i.e., $AY = B$ holds where $Y^t = [y_1, y_2, ..., y_n]$. Note that for the linear system of equations $AX = 0$, the column vector $X^t = [0, 0, ..., 0]$ is always solution and it is called the trivial solution.

In this chapter, we present three methods to solve the system of linear equations (3.1): **Cramer's method**, **Gauss elimination method**, and **Gauss-Jordan method**.

3.2.1 Cramer's Method

Theorem 3.2 Let $AX = B$ be a linear system with n equations in n variables. The system has a solution if $det(A) \neq 0$.

Theorem 3.3 Let $AX = B$ be a linear system with n equations in n variables. If $det(A) \neq 0$, then the unique solution to this system is

$$x_i = \frac{det(A_i)}{det(A)} \quad \text{for every} i = 1, 2, ..., n\ ,$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector of constants B.

The matrix A_1 is formed by replacing the first column of A by the column vector of constants B:

$$A_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The matrix A_2 is formed by replacing the second column of A by the column vector of constants B:

$$A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}.$$

By continuing doing so, the matrix A_n is formed by replacing the last column of A by the column vector of constants B:

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{bmatrix}.$$

Example 3.1 Solve the linear system by Cramer's rule.

(1) $2x+3y=7$
$-x+y=4$

(2) $2x+y+z=3$
$4x+y-z=-2$
$2x-2y+z=6$

(3) $x_1+2x_2=1$
$2x_1+x_2=-1$

(4) $x+y+z=12$
$x-y=2$
$x-z=4$

Solution:

(1) $A=\begin{bmatrix}2&3\\-1&1\end{bmatrix}\Rightarrow det(A)=5.$

$A_1=\begin{bmatrix}7&3\\4&1\end{bmatrix}\Rightarrow det(A_1)=-5.$

$A_2=\begin{bmatrix}2&7\\-1&4\end{bmatrix}\Rightarrow det(A_2)=15.$

Hence,

$$x_1=\frac{det(A_1)}{det(A)}=\frac{-5}{5}=-1 \text{ and } x_2=\frac{det(A_2)}{det(A)}=\frac{15}{5}=3$$

The column vector of variables is $X=\begin{bmatrix}-1\\3\end{bmatrix}$.

(2) $A=\begin{bmatrix}2&1&1\\4&1&-1\\2&-2&1\end{bmatrix}\Rightarrow det(A)=-18.$

$A_1=\begin{bmatrix}3&1&1\\-2&1&-1\\6&-2&1\end{bmatrix}\Rightarrow det(A_1)=-9.$

$A_2=\begin{bmatrix}2&3&1\\4&-2&-1\\2&6&1\end{bmatrix}\Rightarrow det(A_2)=18.$

$A_3=\begin{bmatrix}2&1&3\\4&1&-2\\2&-2&6\end{bmatrix}\Rightarrow det(A_3)=-54.$

Hence,

$$x=\frac{det(A_1)}{det(A)}=\frac{-9}{-18}=\frac{1}{2},\ y=\frac{det(A_2)}{det(A)}=\frac{18}{-18}=-1 \ and\ z=\frac{det(A_3)}{det(A)}=\frac{-54}{-18}=3\ .$$

The column vector of variables is $X=\begin{bmatrix}\frac{1}{2}\\-1\\3\end{bmatrix}$.

(3) $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}\Rightarrow det(A)=-3.$

$A_1=\begin{bmatrix}1&2\\-1&1\end{bmatrix}\Rightarrow det(A_1)=3.$

$A_2=\begin{bmatrix}1&1\\2&-1\end{bmatrix}\Rightarrow det(A_2)=-3.$

From this,

$$x_1 = \frac{det(A_1)}{det(A)} = \frac{3}{-3} = -1 \ and \ x_2 = \frac{det(A_2)}{det(A)} = \frac{-3}{-3} = 1 \ .$$

The column vector of variables is $X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(4) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow det(A) = 3.$

$A_1 = \begin{bmatrix} 12 & 1 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \Rightarrow det(A_1) = 18.$

$A_2 = \begin{bmatrix} 1 & 12 & 1 \\ 1 & 2 & 0 \\ 1 & 4 & -1 \end{bmatrix} \Rightarrow det(A_2) = 12.$

$A_3 = \begin{bmatrix} 1 & 1 & 12 \\ 1 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix} \Rightarrow det(A_3) = 6.$

Therefore,

$$x = \frac{det(A_1)}{det(A)} = \frac{18}{3} = 6, \ y = \frac{det(A_2)}{det(A)} = \frac{12}{3} = 4 \ and \ z = \frac{det(A_3)}{det(A)} = \frac{6}{3} = 2 \ .$$

The column vector of variables is $X = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$.

3.3 Gauss Elimination Method

Definition 3.4 The following operations are called elementary operations:

1. Interchange of two equations.
2. Multiply a non-zero constant throughout an equation.
3. Replace an equation by itself plus a constant multiple of another equation.

Definition 3.5 Two linear systems are said to be equivalent if one can be obtained from the other by a finite number of elementary operations.

The elementary operations help us in getting a linear system from the system (3.1), which is easily solvable.

Theorem 3.6 Let $CX = D$ be the linear system obtained from the linear system $AX = B$ by a finite number of elementary operations. Then, the linear systems $AX = B$ and $CX = D$ have the same set of solutions.

Elementary Row Operations

Rewriting the linear system of equations in a matrix form simplifies the way in which we solve it. The operations on the corresponding matrix are exactly the same operations on the original system of equations as follows:

1. **Multiply some row with $\alpha \neq 0$.** We can multiply any equation by a non-zero real number. The corresponding matrix operation consists of multiplying a row of the matrix by that number.
2. **Add some multiple of a row to another row.** We can replace any equation by the original equation plus a number multiple of another equation. Equivalently, we can replace any row of the corresponding matrix by that row plus a multiple of another row.
3. **Switch two rows.** We can interchange two equations. In the matrix, we interchange the corresponding rows.

The previous operations can be summarized in the following table:

Elementary operations on linear systems	Elementary Row Operations
■ Multiply i^{th} equation by λ ■ Multiply i^{st} equation by λ and add the result to j^{th} equation ■ Replace i^{th} equation by j^{th} equation	Multiply i^{th} row R_i by λ: $\xrightarrow{\lambda R_i}$ Multiply i^{th} row R_i by λ and add the result to j^{th} row R_j: $\xrightarrow{\lambda R_i + R_j}$ Replace i^{th} row R_i by j^{th} row R_j: $\xrightarrow{R_i \leftrightarrow R_j}$

Table 3.1: The elementary Row Operations

For example, for a linear system with two equations

$$\begin{aligned} x + y &= 11 \to 1 \\ 2x + y &= 25 \to 2 \end{aligned}$$

Multiply the 1^{st} row by -2 and add the result to 2^{nd} row

$$\begin{aligned} -2x - 2y &= -22 \\ 2x + y &= 25 \end{aligned}$$

This can be represented by the following elementary row operation:

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 11 \\ 2 & 1 & 25 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|c} -2 & -2 & -22 \\ 2 & 1 & 25 \end{array}\right]$$

Definition 3.7 Two matrices are said to be row-equivalent if one can be obtained from the other by a finite number of elementary row operations.

■ **Example 3.2** The three matrices given below are row equivalent.

$$\begin{bmatrix} 2 & -3 & 1 \\ -1 & 5 & 2 \\ 1 & -2 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 5 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} -2 & 10 & 4 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix}.$$

Definition 3.8 Gaussian elimination is a method of solving a linear system $AX = B$ by constructing the augmented matrix $[A|B]$ and transforming the matrix A to an upper triangular matrix $[C|D]$.

The Method:

1. Construct the augmented matrix $[A|B]$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right].$$

2. Use the elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with a leading coefficient of each row equals 1:

$$\left[\begin{array}{cccccc|c} 1 & c_{12} & c_{13} & c_{14} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{33} & c_{24} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & c_{(n-1)n} & d_{n-1} \\ 0 & 0 & 0 & \cdots & 1 & 1 & d_n \end{array}\right].$$

3. From the last augmented matrix, we have $x_n = d_n$ and the rest of the unknowns can be calculated by backward substitution.

■ **Example 3.3** Solve the linear system by Gauss elimination method.

(1) $3x_1 + x_2 = 9$
$x_1 + 2x_2 = 8$

(2) $x - 2y + z = 4$
$-x + 2y + z = -2$
$4x - 3y - z = -4$

(3) $x + y + z = 2$
$x - y + 2z = 0$
$2x + z = 2$

(4) $x + 2y + 3z = 14$
$2x + y + 2z = 10$
$3x + 4y - 3z = 2$

Solution: For each system, construct the augmented matrix $[A|B]$. Then, use elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with leading coefficient of each row equals 1.

(1) $[A|B] = \left[\begin{array}{cc|c} 3 & 1 & 9 \\ 1 & 2 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow{-3\,R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -13 \end{array}\right] \xrightarrow{-\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & \frac{13}{5} \end{array}\right].$

Thus, $x_2 = \frac{13}{5}$ and $x_1 + 2x_2 = 8$. By substituting the value of x_2, we obtain $x_1 = \frac{14}{5}$. Therefore, the column vector of variables is $X = \begin{bmatrix} \frac{14}{5} \\ \frac{13}{5} \end{bmatrix}$.

(2) $[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow[-4R_1+R_3]{1R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 5 & -5 & -20 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -20 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{\frac{1}{5}R_2}$
$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

Hence, $z = 1$, $y - z = -4$ and $x - 2y + z = 4$. By doing some substitution, we obtain $y = -3$ and $x = -3$. The column vector of variables is $X = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$.

(3) $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-2R_1+R_3]{-1R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right]$
$\xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$.

Thus, $z = 0$, $y - \frac{1}{2}z = 1$ and $x + y + z = 2$. By substituting the value of z and then y, we have $y = 1$ and $x = 1$. The column vector of variables is $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(4) $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 2 & 1 & 2 & 10 \\ 3 & 4 & -3 & 2 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -3 & -4 & -18 \\ 0 & -2 & -12 & -40 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -3 & -4 & -18 \\ 0 & 1 & 6 & 20 \end{array}\right]$
$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 6 & 20 \\ 0 & -3 & -4 & -18 \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 14 & 42 \end{array}\right] \xrightarrow{\frac{1}{14}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

Hence, we have $z = 3$, $y + 6z = 20$ and $x + 2y + 3z = 14$. By substituting the value of z and then y, we have $y = 2$ and $x = 1$. The column vector of variables is $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

3.4 Gauss-Jordan Method

Definition 3.9 Gauss-Jordan elimination is a method of solving a linear system $AX = B$ by constructing the augmented matrix $[A|B]$ and transforming the matrix A to an identity matrix $[I_n|D]$.

The Method:

1. Construct the augmented matrix $[A|B]$.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right].$$

2. Use the elementary row operations on the augmented matrix $[A|B]$ to transform the matrix A to the identity matrix I_n.

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & d_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & d_n \end{array}\right].$$

3. From the last augmented matrix, $x_i = d_i$ for every $i = 1, 2, , n$.

Example 3.4 Solve the linear system by Gauss-Jordan elimination method.

(1) $x + y = 2$
$2x + y = 1$

(2) $x + y + z = 2$
$x - y + 2z = 0$
$2x + z = 2$

(3) $x - 2y + 2z = 5$
$5x + 3y + 6z = 57$
$x + 2y + 2z = 21$

Solution: For each linear system, construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

(1) $$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{1R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right].$$

Thus, $x = -1$ and $y = 3$. The column vector of variables is $X = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(2) $$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-2R_1+R_3]{-1R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-1R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence, $x = 1$, $y = 1$ and $z = 0$. The column vector of variables is $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(3) $$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 5 & 3 & 6 & 57 \\ 1 & 2 & 2 & 21 \end{array}\right] \xrightarrow[-1R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 4 & 0 & 16 \end{array}\right] \xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 1 & 0 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 13 & -4 & 32 \end{array}\right] \xrightarrow{-13R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -4 & -20 \end{array}\right] \xrightarrow{-\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 13 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right].$$

Hence, $x = 3$, $y = 4$ and $z = 5$. The column vector of variables is $X = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

Exercises

1 - 8 ■ Use Cramer's rule to solve the system of linear equations

1 $x+y+z=18$
$x-y+z=6$
$x+y-z=4$

2 $x_1+x_2-2x_3=1$
$2x_1-3x_2+x_3=-8$
$3x_1+x_2+4x_3=7$

3 $x+2y-z=-1$
$x-3y+2z=3$
$x+y-z=2$

4 $2x_1+2x_2-x_3=1$
$2x_1-x_2+2x_3=1$
$x_1+x_2-4x_3=5$

5 $x+y-z=1$
$x-y+2z=2$
$4x+y-z=2$

6 $x_1+3x_2-x_3=2$
$x_1-x_2+5x_3=3$
$3x_1+x_2-x_3=1$

7 $x+y-z=-3$
$x-6y+5z=1$
$x+4y+z=1$

8 $2x-4y+3z=10$
$3x+y-2z=6$
$x+3y-z=20$

9 - 16 ■ Use Gauss elimination method to solve the system

9 $x+y+z=18$
$x-y+z=6$
$x+y-z=4$

10 $x_1+3x_2-x_3=2$
$x_1-x_2+5x_3=3$
$3x_1+x_2-x_3=1$

11 $x-y-z=1$
$x-6y+5z=4$
$2x+y+z=6$

12 $x_1-x_2-x_3=4$
$-2x_1-x_2+x_3=2$
$x_1+x_2+3x_3=7$

13 $x_1+4x_2-x_3=1$
$x_1-2x_2+7x_3=9$
$x_1+2x_2+x_3=3$

14 $x+2y+z=11$
$x+y-z=20$
$x-y+z=5$

15 $x_1+2x_2+x_3=15$
$x_1+3x_2+x_3=5$
$-3x_1-x_2+2x_3=1$

16 $x+y+z=12$
$x-y=2$
$x-z=4$

17 - 24 ■ Use Gauss-Jordan method to solve the system

17 $x-3y+z=21$
$4x+2y+z=14$
$3x+3y+z=7$

18 $x_1+3x_2-x_3=2$
$x_1-x_2+5x_3=3$
$3x_1+x_2-x_3=1$

19 $2x+2y+6z=8$
$x+2y-z=1$
$x+y-3z=1$

20 $x_1+x_2-2x_3=1$
$2x_1-3x_2+x_3=-8$
$3x_1+x_2+4x_3=7$

21 $2x+y+3z=5$
$-5x-3y+z=13$
$x+y+2z=7$

22 $x_1+x_2+x_3=1$
$-x_1+x_2+x_3=3$
$2x_1+x_2-3x_3=5$

23 $x_1+6x_2+3x_3=4$
$2x_1+x_2+2x_3=1$
$3x_1+x_2+x_3=5$

24 $2x-4y+3z=10$
$3x+y-2z=6$
$x+3y-z=20$

Chapter 4

INTEGRATION

4.1 Antiderivatives and Indefinite Integrals

We begin with a definition of antiderivatives and indefinite integrals. Then, we provide basic integration rules. First, we answer the question: given a function f, can we find a function F whose derivative F' is f?

4.1.1 Antiderivatives

Definition 4.1 A function F is called an antiderivative of f on an interval I if

$$F^{'}(x) = f(x) \text{ for every } x \in I.$$

■ **Example 4.1**

(1) Let $F(x) = x^2 + 3x + 1$ and $f(x) = 2x + 3$.
Since $F^{'}(x) = f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.

(2) Let $G(x) = \sin\, x + x$ and $g(x) = \cos\, x + 1$.
Since $G^{'}(x) = g(x)$, then the function $G(x)$ is an antiderivative of $g(x)$.

If $F(x)$ is an antiderivative of $f(x)$, then every function $F(x) + c$ is also antiderivative of $f(x)$, where c is a constant. For different values of c, we have different antiderivatives, but these integrals are very similar geometrically. The upcoming theorem states that any antiderivative $G(x)$, which is different from $F(x)$ can be expressed as $F(x) + c$ where c is an arbitrary constant.

Theorem 4.2 Functions with same derivatives differ by a constant.

■ **Example 4.2** If $f(x) = 2x$, the functions

$F(x) = x^2 + 2,$

$G(x) = x^2 - \frac{1}{2},$

$H(x) = x^2 - \sqrt[3]{2},$

are antiderivatives of the function f. Therefore, $F(x) = x^2 + c$ is a general form of the antiderivatives of the function $f(x) = 2x$.

■ **Example 4.3** Find the general form of the antiderivatives of $f(x) = 6x^5$.

Solution: To find an antiderivative of a given function, we search for a function whose derivative is $f(x) = 6x^5$. By inspection, we have $F(x) = x^6$ since $F'(x) = 6x^5 = f(x)$. Therefore, the function $F(x) = x^6 + c$ is the general antiderivative of f.

4.1.2 Indefinite Integrals

From Theorem 4.2, if the function $F(x) + c$ is an antiderivative of $f(x)$, then there exist no antiderivatives in different forms for the function $f(x)$. This leads us to define indefinite integrals. We introduce a symbol, namely, $\int f(x)\,dx$ which will represent the antiderivatives and it is read as the indefinite integral of f with respect to x.

Definition 4.3 Let f be a continuous function on an interval I. The indefinite integral of f is the general antiderivative of f on I:

$$\int f(x)\,dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Now, by using the previous definition, the general antiderivatives in Example 4.1 are

(1) $\int (2x+3)\,dx = x^2 + 3x + c.$

(2) $\int (\cos\, x + 1)\,dx = \sin\, x + x + c.$

4.2 Properties of Indefinite Integrals

Theorem 4.4 Assume f and g have antiderivatives on an interval I, then

1. $\dfrac{d}{dx}\int f(x)\,dx = f(x).$
2. $\int \dfrac{d}{dx}(F(x))\,dx = F(x) + c.$
3. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$
4. $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant.

Note:

1. The properties **3.** and **4.** can be generalized to a finite number of functions $f_1, f_2, ..., f_n$ and real numbers, $k_1, k_2, ..., k_n$:

$$\int \left(k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)\right)\, dx = k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx + ... + +k_n \int f_n(x)\, dx\,.$$

2. From now onwards, we shall write only one constant of integration in the final answer.

4.2.1 Integration as an Inverse Process of Differentiation

We are given the derivative of a function and asked to find its antiderivatives. They are evidently obtained directly from the corresponding formula of differentiation. That is by knowing the formulas for the derivatives, we can write the corresponding formulas for the integrals.

In the following, we write a list of integrals by using the corresponding formula of the differential calculus.

■ **Rule 1:** Power of x.

$$\boxed{\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \implies \int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1}$$

In words, to integrate the function x^n, we add 1 to the power (i.e., $n+1$) and divide the function by $n+1$. If $n = 0$, we have a special case

$$\int 1\, dx = x + c.$$

Recall, if $c \in \mathbb{R}$, then $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = x^n$. Thus, antiderivatives are not unique.

Note that **Rule 1** cannot be applied for $n = -1$. For this value, the formula gives $\int x^{-1}\, dx = \frac{x^0}{0} = \infty$.

■ **Example 4.4** Evaluate the integral.

(1) $\int (x+1)\, dx$

(2) $\int (4x^3 + 2x^2 + 1)\, dx$

(3) $\int (x^2 - \frac{1}{x^3})\, dx$

Solution:

(1) $\int (x+1)\, dx = \frac{x^2}{2} + x + c\,.$

(2) $\int (4x^3 + 2x^2 + 1)\, dx = \frac{4x^4}{4} + \frac{2}{3}x^3 + x + c = x^4 + \frac{2}{3}x^3 + x + c\,.$

(3) $\int (x^2 - \frac{1}{x^3})\, dx = \int x^2\, dx - \int x^{-3}\, dx = \frac{x^3}{3} + \frac{x^{-2}}{2} + c\,.$

Rule 2: Trigonometric Functions.

- $\frac{d}{dx}(\sin\ x) = \cos\ x \Rightarrow \int \cos\ dx = \sin\ x + c$
- $\frac{d}{dx}(\cos\ x) = -\sin\ x \Rightarrow \int -\sin\ x\, dx = \cos\ x + c$
- $\frac{d}{dx}(\tan\ x) = \sec^2\ x \Rightarrow \int \sec^2\ x\, dx = \tan\ x + c$
- $\frac{d}{dx}(\cot\ x) = -\csc\ x \Rightarrow \int -\csc^2\ x\, dx = \cot\ x + c$
- $\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x \Rightarrow \int \sec\ x\ \tan\ x\, dx = \sec\ x + c$
- $\frac{d}{dx}(\csc\ x) = -\csc\ x\ \cot\ x \Rightarrow \int -\csc\ x\ \cot\ x\, dx = \csc\ x + c$

Example 4.5 Evaluate the integral.

(1) $\int (\sin x - \csc x\ \cot x)\ dx$

(2) $\int \left(\frac{1}{\sec x} + \cos x\right)\ dx$

(3) $\int \left(\frac{\tan x}{\cos x} - x^2\right)\ dx$

Solution:

(1) $\int (\cos x - \csc x\ \cot x)\ dx = \int \cos x\ dx - \int \csc x\ \cot x\ dx = \sin x + \csc x + c$.

(2) $\int \left(\frac{1}{\sec x} - \sin x\right)\ dx = \int \cos x\ dx - \int \sin x\ dx = \sin x + \cos x + c$.

(3) $\int \left(\frac{\tan x}{\cos x} - x^2\right)\ dx = \int \tan x\ \sec x\ dx - \int x^2\ dx = \sec x - \frac{x^3}{3} + c$.

Rule 3: Natural Logarithmic and Exponential Functions.

- $\frac{d}{dx}(\ln\ x) = \frac{1}{x} \Longrightarrow \int \frac{1}{x}\ dx = \ln x + c$
- $\frac{d}{dx}(e^x) = e^x \Longrightarrow \int e^x\ dx = e^x + c$

Example 4.6 Evaluate the integral.

(1) $\int \frac{3}{x}\ dx$

(2) $\int (x^3 + x^{-1} + e^x)\ dx$

(3) $\int \left(\frac{1}{3e^{-x}} + \frac{1}{x^2}\right)\ dx$

Solution:

(1) $\int \frac{3}{x}\ dx = 3\int \frac{1}{x}\ dx = 3\ln|x| + c$.

(2) $\int \left(x^3 + x^{-1} + e^x\right) dx = \int x^3\, dx + \int \frac{1}{x}\, dx + \int e^x\, dx = \frac{x^4}{4} + \ln|x| + e^x + c\,.$

(3) $\int \left(\frac{1}{3e^{-x}} + \frac{1}{x^2}\right) dx = \frac{1}{3}\int e^x\, dx + \int x^{-2}\, dx = \frac{1}{3}e^x - x^{-1} + c\,.$

■ **Rule 4:** Inverse Trigonometric Functions.

- $\frac{d}{dx}\left(\sin^{-1}\right) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\cos^{-1}\right) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\tan^{-1}\right) = \frac{1}{1+x^2}$
- $\frac{d}{dx}\left(\cot^{-1}\right) = -\frac{1}{1+x^2}$
- $\frac{d}{dx}\left(\sec^{-1}\right) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\left(\csc^{-1}\right) = -\frac{1}{x\sqrt{x^2-1}}$

In general, we have

- $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} big(\frac{x}{a} big) + c$
- $\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1} big(\frac{x}{a} big) + c$
- $\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1} big(\frac{x}{a} big) +$
c

■ **Example 4.7** Evaluate the integral.

(1) $\int \frac{6}{4+x^2}\, dx$

(2) $\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x}}\right) dx$

(3) $\int \left(5x + \frac{1}{x\sqrt{x^2-5}}\right) dx$

Solution:

(1) $\int \frac{6}{4+x^2}\, dx = 6\int \frac{1}{4+x^2}\, dx = 6\frac{1}{2}\tan^{-1}(\frac{x}{2}) + c\,.$

(2) $\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x}}\right) dx = \int \frac{1}{\sqrt{1-x^2}}\, dx + \int x^{-\frac{1}{2}}\, dx = \sin^{-1} x + 2\sqrt{x} + c\,.$

(3) $\int \left(5x + \frac{1}{x\sqrt{x^2-5}}\right) dx = \int 5x\, dx + \int \frac{1}{x\sqrt{x^2-5}}\, dx = \frac{5x^2}{2} + \frac{1}{\sqrt{5}}\ \sec^{-1}(\frac{x}{\sqrt{5}}) + c\,.$

■ **Example 4.8** Solve the differential equation $f'(x) = x^2$ subject to the initial condition $f(0) = 1$.

Solution:

$$\int f'(x)\,dx = \int x^2\,dx$$
$$\Rightarrow f(x) = \frac{1}{3}x^3 + c\,.$$

To determine c, we have to apply the condition $f(0) = 1$. Hence, $f(0) = \frac{1}{3}(0)^3 + c = 1$ and this implies $c = 1$. The solution of the differential equation is $f(x) = \frac{1}{3}x^3 + 1$.

Notes:

- We can always check our answers by differentiating the results.

- In the previous examples, we use x as a variable of the integration. However, for this role, we can use any variable such as y, z, t, etc . That is, instead of $f(x)\,dx$, we can integrate $f(y)\,dy$ or $f(t)\,dt$.

Example 4.9 Evaluate the integral.

(1) $\int (y^2 + y + 1)\,dy$

(2) $\int (\cos\,t + \sec^2\,t)\,dt$

Solution:

(1) $\int (y^2 + y + 1)\,dy = \frac{y^3}{3} + \frac{y^2}{2} + y + c\,.$

(2) $\int (\cos\,t + \sec^2\,t)\,dt = \sin\,t + \tan\,t + c\,.$

4.3 Definite Integrals

4.3.1 Summation Notation

Summation (or sigma notation) is a simple form used to give a concise expression for a sum of values.

Definition 4.5 Let $\{a_1, a_2, ..., a_n\}$ be a set of numbers. The symbol $\sum_{k=1}^{n} a_k$ represents their sum:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

Example 4.10 Evaluate the sum.

(1) $\sum_{i=1}^{3} i^3$ **(2)** $\sum_{j=1}^{4} (j^2 + 1)$ **(3)** $\sum_{k=1}^{3} (k+1)k^2$

Solution:

(1) $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36.$

(2) $\sum_{j=1}^{4}(j^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1) = 2+5+10+17 = 34.$

(3) $\sum_{k=1}^{3}(k+1)k^2 = (1+1)(1)^2+(2+1)(2)^2+(3+1)(3)^2 = 2+12+36 = 50.$

Theorem 4.6 Let $\{a_1,a_2,...,a_n\}$ and $\{b_1,b_2,...,b_n\}$ be sets of real numbers. If n is any positive integer, then

1. $\sum_{k=1}^{n} c = \underbrace{c+c+...+c}_{\text{n-times}} = nc$ for any $c \in \mathbb{R}$.
2. $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$.
3. $\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

■ **Example 4.11** Evaluate the sum.

(1) $\sum_{k=1}^{10} 15$ **(2)** $\sum_{k=1}^{4}(k^2+2k)$ **(3)** $\sum_{k=1}^{3} 3(k+1)$

Solution:

(1) $\sum_{k=1}^{10} 15 = (10)(15) = 150.$

(2) $\sum_{k=1}^{4}(k^2+2k) = \sum_{k=1}^{4} k^2 + 2\sum_{k=1}^{4} k = (1^2+2^2+3^2+4^2)+2(1+2+3+4) = 30+20 = 50.$

(3) $\sum_{k=1}^{3} 3(k+1) = 3\sum_{k=1}^{3}(k+1) = 3(2+3+4) = 27.$

4.3.2 Riemann Sum

A Riemann sum is a mathematical form used in this book to approximate the area of a region underneath the graph of a function. Before start-up this issue, we provide some basic definitions.

Definition 4.7 A set $P = \{x_0, x_1, x_2, ..., x_n\}$ is called a partition of a closed interval $[a,b]$ if for any positive integer n,

$$a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b.$$

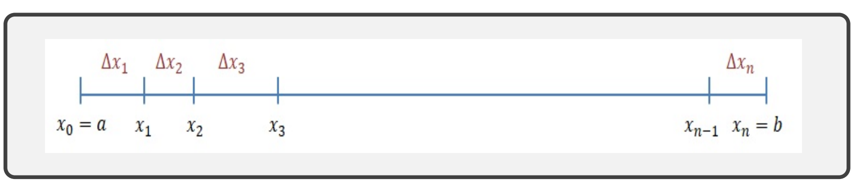


Figure 4.1: A partition of the interval $[a,b]$.

Notes:

■ The division of the interval $[a,b]$ by the partition P generates n subintervals: $[x_0,x_1],[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$.

■ The length of each subinterval $[x_{i-1},x_i]$ is $\Delta x_i = x_i - x_{i-1}$.

Remark 4.8

1. The partition P of the interval $[a,b]$ is regular if $\Delta x_0 = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \Delta x$.
2. For any positive integer n, if the partition P is regular, then

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = x_0 + i\,\Delta x.$$

To see this, let P be a regular partition of the interval $[a,b]$. Since $x_0 = a$ and $x_n = b$, then

$$\begin{aligned}
&x_1 = x_0 + \Delta x\ ,\\
&x_2 = x_1 + \Delta x = (x_0 + \Delta x) + \Delta x = x_0 + 2\Delta x\ ,\\
&x_3 = x_2 + \Delta x = (x_0 + 2\Delta x) + \Delta x = x_0 + 3\Delta x.
\end{aligned}$$

By continuing doing so, we have $x_i = x_0 + i\,\Delta x$.

Figure 4.2: A regular partition of the interval $[a,b]$.

■ **Example 4.12** Define a regular partition P that divides the interval $[1,3]$ into 4 subintervals.

Solution:

Since P is a regular partition of $[1,3]$ with $n = 4$, then

$$\Delta x = \frac{3-1}{4} = \frac{1}{2} \text{ and } x_i = 1 + i\,\frac{1}{2}.$$

Therefore,

$x_0 = 1$
$x_1 = 1 + \frac{1}{2} = 1\frac{1}{2}$
$x_2 = 1 + 2(\frac{1}{2}) = 2$
$x_3 = 1 + 3(\frac{1}{2}) = 2\frac{1}{2}$
$x_4 = 1 + 4(\frac{1}{2}) = 3$

The regular partition is $P = \{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3\}$.

Definition 4.9 The norm of the partition P is the largest length among $\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n$ i.e.,

$$||\ P\ || = max\{\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n\}.$$

In the following, we will see how to compute definite integrals. The definite integral has a unique value and it is denoted by $\int_a^b f(x)\,dx$. We should distinguish between definite and indefinite integrals. A definite integral is a number, whereas the indefinite integral is a function.

Definition 4.10 Let f be a continuous function on the interval $[a,b]$. For any partition P of the interval $[a,b]$ and any choice of x_i^* in the i^{th} subinterval, the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{||P||\to 0} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x_i}_{\text{Riemann sum}}$$

if the limit exists. The numbers a and b are called the limits of the integration.

■ **Example 4.13** Evaluate the integral $\displaystyle\int_2^4 (x+2)\,dx$.

Solution:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of the interval $[2,4]$, then $\Delta x = \frac{4-2}{n} = \frac{2}{n}$ and $x_i = x_0 + i\Delta x$. Also, let the mark x^* be the right endpoint of each subinterval, so $x_i^* = x_i = 2 + \frac{2i}{n}$ and then $f(x_i^*) = \frac{2}{n}(2n+i)$.

The Riemann sum of f for P is

$$R_p = \sum_i f(x_i^*)\Delta x_i = \frac{4}{n^2}\sum_i (2n+i) = \frac{4}{n^2}\left(2n^2 + \frac{n(n+1)}{2}\right) = 8 + \frac{2(n+1)}{n}.$$

From Definition 4.10, $\displaystyle\int_2^4 (x+2)\,dx = \lim_{n\to\infty} R_p = 8 + \lim_{n\to\infty} \frac{2n(n+1)}{n^2} = 8+2 = 10.$

4.3.3 Properties of Definite Integrals

Theorem 4.11

1. $\displaystyle\int_a^b c\,dx = c(b-a)$.

2. $\displaystyle\int_a^a f(x)\,dx = 0$ if $f(a)$ exists.

Theorem 4.12

1. If f and g are integrable on $[a,b]$, then $f+g$ and $f-g$ are integrable on $[a,b]$ and

$$\int_a^b \big(f(x) \pm g(x)\big)\ dx = \int_a^b f(x) \pm \int_a^b g(x)\ dx.$$

2. If f is integrable on $[a,b]$ and $k \in \mathbb{R}$, then $k\ f$ is integrable on $[a,b]$ and

$$\int_a^b k\ f(x)\ dx = k\ \int_a^b f(x)\ dx.$$

Theorem 4.13

1. If f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

2. If f is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\ dx \geq 0.$$

Theorem 4.14 If f is integrable on the intervals $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

Theorem 4.15 If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

■ **Example 4.14** Evaluate the integral.

(1) $\displaystyle\int_0^2 5\ dx$ **(2)** $\displaystyle\int_2^2 \sqrt{x^2+4}\ dx$

Solution:

(1) $\displaystyle\int_0^2 3\ dx = 3(2-0) = 6.$

(2) $\displaystyle\int_2^2 (x^2+4)\ dx = 0.$

■ **Example 4.15** If $\displaystyle\int_a^b f(x)\ dx = 7$ and $\displaystyle\int_a^b g(x)\ dx = 4$, then find $\displaystyle\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\ dx.$

Solution:

$$\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\,dx = 3\int_a^b f(x)\,dx - \frac{1}{2}\int_a^b g(x)\,dx \qquad \text{Theorem 4.12}$$
$$= 3(4) - \frac{1}{2}(2) = 11\ .$$

■ **Example 4.16** Prove that $\int_0^2 (x^3+x^2+2)\,dx \geq \int_0^2 (x^2+1)\,dx$ without evaluating the integrals.

Solution: Let $f(x) = x^3+x^2+2$ and $g(x) = x^2+1$. We can find that $f(x)-g(x) = x^3+1 > 0$ for all $x \in [0,2]$. This implies that $f(x) > g(x)$ and from Theorem 4.13, we have

$$\int_0^2 (x^3+x^2+2)\,dx \geq \int_0^2 (x^2+1)\,dx.$$

4.3.4 The Fundamental Theorem of Calculus

Theorem 4.16 Suppose that f is continuous on the closed interval $[a,b]$.

1. If $F(x) = \int_a^x f(t)\,dt$ for every $x \in [a,b]$, then $F(x)$ is an antiderivative of f on $[a,b]$.
2. If $F(x)$ is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

From the previous theorem, a definite integral $\int_a^b f(x)\,dx$ is evaluated by two steps:

Step 1: Find the indefinite integral $\int f(x)\,dx = F(x)$ and no need to put a constant c.

Step 2: Evaluate the antiderivative F at upper and lower limits by substituting $x = b$ and $x = a$ then calculate $F(b) - F(a)$.

■ **Example 4.17** Evaluate the integral.

(1) $\int_{-1}^2 (2x+1)\,dx$

(2) $\int_0^3 (x^2+1)\,dx$

(3) $\int_1^2 \frac{1}{\sqrt{x^3}}\,dx$

(4) $\int_0^{\frac{\pi}{2}} (\sin\,x+1)\,dx$

(5) $\int_{\frac{\pi}{4}}^{\pi} (\sec^2\,x-4)\,dx$

(6) $\int_0^{\frac{\pi}{3}} (\sec\,x\,\tan\,x+x)\,dx$

Solution:

(1) $\int_{-1}^2 (2x+1)\,dx = \Big[x^2+x\Big]_{-1}^2 = \big(4+2\big) - \big((-1)^2+(-1)\big) = 6-0 = 6.$

(2) $\int_0^3 (x^2+1)\,dx = \Big[\frac{x^3}{3}+x\Big]_0^3 = (\frac{27}{3}+3) - 0 = 12.$

(3) $\displaystyle\int_1^2 \frac{1}{\sqrt{x^3}}\, dx = \left[\frac{-2}{\sqrt{x}}\right]_1^2 = \frac{-2}{\sqrt{2}} - (-2) = \frac{-2+2\sqrt{2}}{\sqrt{2}} = -\sqrt{2}+2.$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} (\sin\, x+1)\, dx = \Big[-\cos\, x+x\Big]_0^{\frac{\pi}{2}} = (-\cos\, \tfrac{\pi}{2}+\tfrac{\pi}{2}) - (-\cos\, 0+0) = \tfrac{\pi}{2}+1.$

(5) $\displaystyle\int_{\frac{\pi}{4}}^{\pi} (\sec^2\, x-4)\, dx = \Big[\tan\, x-4x\Big]_{\frac{\pi}{4}}^{\pi} = (\tan\, \pi-4\pi) - (\tan\, \tfrac{\pi}{4}-4\tfrac{\pi}{4}) = -4\pi-(1-\pi) = -3\pi-1.$

(6) $\displaystyle\int_0^{\frac{\pi}{3}} (\sec\, x\, \tan\, x+x)\, dx = \Big[\sec\, x+\tfrac{x^2}{2}\Big]_0^{\frac{\pi}{3}} = (\sec\, \tfrac{\pi}{3}+\tfrac{(\frac{\pi}{3})^2}{2}) - (\sec\, 0+\tfrac{0}{2}) = 2+\tfrac{\pi^2}{18}-1 = 1+\tfrac{\pi^2}{18}.$

4.4 Techniques of Integration

In the preceding section, we have shown the importance of the basic integrals and the integral properties in evaluating many integrals. However, those rules are elementary for simple functions, in a sense that we need new techniques that will enable us to evaluate more complex functions. In this section, we present three techniques of integration: integration by substitution, integration by parts, integration of rational functions.

4.4.1 Integration By Substitution

The integration by substitution (known as u-substitution) is one of the important methods for evaluating the integral. The method is based on changing the variable of the integration to obtain a simple integral.

Theorem 4.17 Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g. If F is an antiderivative of the function f on J, then

$$\int f(g(x))g^{'}(x)\, dx = F(g(x))+c, \;\; \forall x \in I.$$

The task here is to recognize whether an integrand has the form $f(g(x))g^{'}(x)$. The following examples simplify the idea of the integration by substitution.

■ **Example 4.18** Evaluate the integral $\displaystyle\int 3x^2\, (x^3+1)^5\, dx.$

Solution: We can use the previous theorem as follows:

if $f(x) = x^5$ and $g(x) = x^3+1$, then $f\big(g(x)\big) = (x^3+1)^5$. Since $g^{'}(x) = 3x^2$, then from Theorem 4.17, we have

$$\int 3x^2(x^3+1)^5\, dx = \frac{(x^3+1)^6}{6}+c.$$

Note that in the previous example, we can end with the same solution by using the following steps of the substitution method.

■ **Steps of the integration by substitution:**

Step 1: Choose a new variable u. Observe the integrand $f(x)$ and choose an inside function u as a function of x. Then check if $f(x)$ can be decomposed into

$$f(x) = (\text{function of } u).\text{constant multiple of } u'(x)$$

Step 2: Determine the value of du.

Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral in terms of u.

Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x.

In Example 4.18, let $u = x^3 + 1$, this implies $du = 3x^2\,dx$. Now apply the substitution by substituting all x-terms into u-terms:

$$\int \underbrace{(x^3+1)}_{u}{}^5\ \underbrace{3x^2}_{du}\,dx = \int u^5\,du = \frac{u^6}{6} + c\,.$$

By returning the evaluation to the initial variable x, we have

$$\int 3x^2(x^3+1)^5\,dx = \frac{(x^3+1)^6}{6} + c\,.$$

■ **Example 4.19** Evaluate the integral $\displaystyle\int \tan\,x\,dx$.

Solution: Write $\tan\,x = \frac{\sin\,x}{\cos\,x}$. Set the substitution $u = \cos\,x$, this implies $du = -\sin\,x\,dx$ and $-du = \sin\,x\,dx$.

Hence,

$$\int \tan\,x\,dx = \int \frac{\sin\,x}{\cos\,x}\,dx = \int \frac{1}{\cos\,x}.\sin\,x\,dx = -\int \frac{1}{u}\,du = -\ln|u| + c\,.$$

By returning the evaluation to the initial variable x, we have

$$\int \tan\,x\,dx = -\ln|\cos\,x| + c\,.$$

Rules:

- $\displaystyle\int \tan\,x\,dx = -\ln|\cos\,x| + c = \ln|\sec\,x| + c$
- $\displaystyle\int \cot\,x\,dx = \ln|\sin\,x| + c = -\ln|\csc\,x| + c$

■ **Example 4.20** Evaluate the integral $\displaystyle\int \frac{\cos\,\sqrt{x}}{\sqrt{x}}\,dx$.

Solution: By using Theorem 4.17, we have $f(x) = \cos\,x$ and $g(x) = \sqrt{x}$, then $f\big(g(x)\big) = \cos\,\sqrt{x}$. Since $g'(x) = 1/(2\sqrt{x})$, then

$$\int \frac{\cos\,\sqrt{x}}{\sqrt{x}}\,dx = 2\int \frac{\cos\,\sqrt{x}}{2\sqrt{x}}\,dx = 2\sin\,\sqrt{x} + c\,.$$

By using the steps of the substitution method, we let $u = \sqrt{x}$, this implies $du = \frac{1}{2\sqrt{x}}\,dx$. By substitution, we obtain

$$2\int \cos\,u\,du = 2\sin\,u + c = 2\sin\,\sqrt{x} + c.$$

■ **Example 4.21** Evaluate the integral $\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$.

Solution: Let $u = x^3-3x+1$, then $du = 3(x^2-1)\,dx$. By substitution, we have

$$\frac{1}{3}\int u^{-6}\,du = \frac{1}{3}\,\frac{1}{-5\,u^5}+c = \frac{-1}{15(x^3-3x+1)^5}+c.$$

■ **Example 4.22** Evaluate the integral $\int \sin^3 x\,\cos x\,dx$.

Solution:

Let $u = \sin x$, then $du = \cos x\ dx$. By substitution, we have

$$\int u^3\,du = \frac{u^4}{4}+c = \frac{\sin^4 x}{4}+c.$$

Note:
Any power of a trigonometric function or an inverse trigonometric function can be integrated by **Rule 1** on page 54 when accompanied by its differential.

■ **Example 4.23** Evaluate the integral $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$.

Solution: Let $u = \sin^{-1} x$, then $du = \frac{1}{\sqrt{1-x^2}}\,dx$. By substitution, we have

$$\int u\,du = \frac{u^2}{2}+c = \frac{(\sin^{-1} x)^2}{2}+c.$$

Corollary 4.18 If $\int f(x)\,dx = F(x)+c$, then for any $a \neq 0$,

$$\int f(ax \pm b)\,dx = \frac{1}{a}F(ax \pm b)+c.$$

■ **Example 4.24** Evaluate the integral.

(1) $\int \sqrt{3x-2}\,dx$

(2) $\int \sec^2\,(5x+4)\,dx$

Solution: From Corollary 4.18, we have

(1) $\int \sqrt{3x-2}\,dx = \frac{1}{3}\frac{(3x-2)^{3/2}}{3/2}+c = \frac{(3x-2)^{3/2}}{3}+c.$

(2) $\int \sec^2\,(5x+4)\,dx = \frac{1}{5}\tan\,(5x+4)+c.$

Note that when using the substitution method to evaluate the definite integral $\int_a^b f(x)\,dx$, we have two options:

Option 1: Change the limits of integration to the new variable. In this case, we evaluate the integral without returning to the original variable.

Option 2: Leave the limits in terms of the original variable. In this case, we evaluate the integral and return the result to the original variable. After that, we substitute $x = b$ and $x = a$ into the antiderivative.

Example 4.25 Evaluate the integral $\int_0^1 2x\sqrt{x^2+1}\,dx$.

Solution:

Option 1: Let $u = x^2 + 1$, this implies $du = 2x\,dx$. Change the limits $u(0) = 1$ and $u(1) = 2$. By substitution, we have

$$\int_1^2 u^{1/2}\,du = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^2 = \frac{2}{3}\left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{3}\left(2\sqrt{2} - 1\right).$$

Option 2: Let $u = x^2 + 1$, then $du = 2x\,dx$. By substitution, we have $\int u^{1/2}\,du = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(x^2+1)^{\frac{3}{2}}$. Thus,

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \frac{2}{3}\left[(x^2+1)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left(2\sqrt{2} - 1\right).$$

Example 4.26 Evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$.

Solution: Let $u = \cos^2 x$, this implies $du = -\sin x\,dx$. Change the limits $u(0) = 1$ and $u(\frac{\pi}{2}) = 0$. By substitution, we have

$$-\int_1^0 \frac{1}{1+u^2\,x}\,du = -\left[\tan^{-1} u\right]_1^0 = -\left(\tan^{-1}(0) - \tan^{-1}(1)\right) = -(0 - \frac{\pi}{4}) = \frac{\pi}{4}.$$

4.4.2 Integration by Parts

In this section we will learn another important technique of integration, called integration by parts. This technique depends on the product rule in differentiation, so it can be thought as the product formula for integration. In practice, the integrand is divided into two parts u and dv, then we find du by deriving u and v by integrating dv. This method transfers a product integral (the original integrand) into another product integral that can be evaluated.

Theorem 4.19 If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u\,dv = uv - \int v\,du.$$

Theorem 4.19 shows that the integration by parts transfers the integral $\int u\,dv$ into an easier integral $\int v\,du$. The question here is, what we choose as u and what we choose as $dv = v'\,dx$.

A Guideline for Choosing u and dv:

(1) Choose u a portion of the integrand that can be easily differentiated. We can choose u to be the function that comes first in this list:

(**a**) Inverse trigonometric function.
(**b**) Logarithmic function.
(**c**) Algebraic function.
(**d**) Exponential function.
(**e**) Trigonometric function.

(**2**) Choose dv the most complicated portion of the integrand that can be easily integrated.

This guideline is useful, but is not enough to solve all product integrals. Sometimes we need to analyze the integrand and examine the best way of using integration by parts.

■ **Example 4.27** Evaluate the integral $\int x\, e^x\, dx$.

Solution:
The integrand xe^x is a product of two functions x and e^x. Now we need to identify one function as u and the other one as dv such that the new product integral $\int v\, du$ is easier than the original integrand.

Let $I = \displaystyle\int x\, e^x\, dx$ and choose $u = x$, and $dv = e^x\, dx$. Then,

$$u = x \Rightarrow du = dx\,,$$
$$dv = e^x\, dx \Rightarrow v = \int e^x\, dx = e^x.$$

From Theorem 4.19, we have

$$I = x\, e^x - \int e^x\, dx = x\, e^x - e^x + c.$$

- We choose $u = x$ because it can be differentiated to a constant. Thus the new product integral will not involve a product anymore.
- Try to choose

$u = e^x$ and $dv = x\, dx$

You will obtain

$$I = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\, dx.$$

However, the integral $\int \frac{x^2}{2}e^x\, dx$ is more difficult than the original one $\int xe^x\, dx$.

■ **Example 4.28** Evaluate the integral $\int x \cos x\, dx$.

Solution: In the same manner as in the preceding example, let $I = \int x \cos x\, dx$. Set $u = x$ and $dv = \cos x\, dx$. Hence,

$$u = x \Rightarrow du = dx\,,$$
$$dv = \cos x\, dx \Rightarrow v = \int \cos x\, dx = \sin x.$$

Try to choose
$u = \cos x$ and $dv = x\, dx$
Do you have the same result?

From Theorem 4.19, we have

$$I = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c.$$

■ **Example 4.29** Evaluate the integral $\int \ln x\, dx$.

Solution: Let $I = \int \ln x \, dx$ and choose $u = \ln x$, and $dv = dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} \, dx \,,$$
$$dv = dx \Rightarrow v = \int 1 \, dx = x.$$

Remember:
If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}$$

Apply Theorem 4.19, to have $I = x\ln x - \int x\frac{1}{x} \, dx = x\ln x - \int 1 \, dx = x\ln x - x + c$.

■ **Example 4.30** Evaluate the integral $\int x^3 \, \ln x \, dx$.

Solution:

Let $I = \int x^3 \, \ln x \, dx$ and choose $u = \ln x$, and $dv = x^3 \, dx$.
Then,

$$u = \ln x \Rightarrow du = \frac{1}{x}dx \,,$$
$$dv = x^3 \, dx \Rightarrow v = \int x^3 \, dx = \frac{x^4}{4}.$$

From Theorem 4.19, we have

$$\begin{aligned} I &= \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \, \frac{1}{x} \, dx \\ &= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 \, dx \\ &= \frac{x^4}{4}\ln x - \frac{x^4}{16} + c \,. \end{aligned}$$

Rule:
To evaluate $\int x^n \, \ln x \, dx$, let
$u = \ln x \Rightarrow du = \frac{1}{x}dx$
$dv = x^n \, dx \Rightarrow v = \int x^n \, dx = \frac{x^{n+1}}{n+1}$

Hence,

$$\int x^n \, \ln x \, dx = \frac{x^{n+1}}{n+1} \, \ln x + \frac{x^{n+1}}{(n+1)^2} + c$$

Note:

1. When we consider the integration by parts, we want to obtain an easier integral. As we saw in Example 4.31, if we choose $u = e^x$ and $dv = x \, dx$, we have $\int \frac{x^2}{2} \, e^x \, dx$ which is more difficult than the original one.
2. When considering the integration by parts, we have to choose dv a function that can be integrated (see Example 4.31).

■ **Example 4.31** Evaluate the integral $\int \sin \, x \, \ln(\cos \, x) \, dx$.

Solution:

Let $I = \int \sin x \ln(\cos x)\, dx$ and choose $u = \ln(\cos x)$ for $\cos x > 0$, and $dv = \sin x\, dx$. Then,

$$u = \ln(\cos x) \Rightarrow du = \frac{-\sin x}{\cos x} dx = -\tan x\, dx,$$
$$dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x.$$

Hence,

$$\begin{aligned} I &= -\cos x \ln(\cos x) - \int \cos x \tan x\, dx \\ &= -\cos x \ln(\cos x) - \int \sin x\, dx \\ &= -\cos x \ln(\cos x) + \cos x + c\,. \end{aligned}$$

■ **Example 4.32** Evaluate the integral $\int_0^1 \tan^{-1} x\, dx$.

Solution:

Let $I = \int \tan^{-1} x\, dx$ and choose $u = \tan^{-1} x$, and $dv = dx$. Hence,

$$u = \tan^{-1} x \Rightarrow du = \frac{1}{x^2+1} dx\,,$$
$$dv = dx \Rightarrow v = \int 1\, dx = x.$$

Any inverse trigonometric function such that the differential is not accompanied can be integrated in the same manner of Example 4.32.

By applying Theorem 4.19, we obtain

$$I = x \tan^{-1} x - \underbrace{\frac{1}{2}\int \frac{2x}{x^2+1}\, dx}_{\text{apply substitution } u=x^2+1} = x \tan^{-1} x - \frac{1}{2}\ln(x^2+1) + c.$$

Therefore,

$$\int_0^1 \tan^{-1} x\, dx = \left[x \tan^{-1} x - \frac{1}{2}\ln(x^2+1)\right]_0^1 = (\tan^{-1}(1) - \frac{1}{2}\ln 2) - (0 - \frac{1}{2}\ln 1) = \frac{\pi}{4} - \ln\sqrt{2}.$$

Remark 4.20 Sometimes we need to use the integration by parts twice as in Examples 4.33 and 4.34.

■ **Example 4.33** Evaluate the integral $\int x^2 e^x\, dx$.

Solution:

Let $I = \int x^2 e^x \, dx$ and choose $u = x^2$, and $dv = e^x \, dx$.
Then,

$$u = x^2 \Rightarrow du = 2x \, dx \,,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

In successive application of the integration by parts, do not switch choices for u and dv.

This implies $I = x^2 e^x - 2\int xe^x \, dx$.

We use the integration by parts again for the integral $\int xe^x \, dx$. Let $J = \int xe^x \, dx$.

Let $u = x$ and $dv = e^x \, dx$. Hence,

$$u = x \Rightarrow du = \; dx \,,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

Therefore, $J = xe^x - \int e^x \, dx = xe^x - e^x + c$. By substituting the result of J into I, we have

$$I = x^2 e^x - 2(xe^x - e^x) + c = e^x(x^2 - 2x + 2) + c.$$

■ **Example 4.34** Evaluate the integral $\int e^x \cos\, x \, dx$.

Solution: Let $I = \int e^x \cos\, x \, dx$ and choose $u = e^x$, and $dv = \cos\, x \, dx$.
Then,

$$u = e^x \Rightarrow du = e^x \, dx \,,$$
$$dv = \cos\, x \, dx \Rightarrow v = \int \cos\, x \, dx = \sin\, x.$$

In Example 4.34, try to choose

$u = \cos x$ and $dv = e^x$

Do you have the same result?

Hence, $I = e^x \sin\, x - \int e^x \sin\, x \, dx$.

The integral $\int e^x \sin\, x \, dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin\, x \, dx$. Let $u = e^x$ and $dv = \sin\, x \, dx$. Then,

$$u = e^x \Rightarrow du = e^x \, dx \,,$$
$$dv = \sin\, x \, dx \Rightarrow v = \int \sin\, x \, dx = -\cos\, x.$$

Hence, $J = -e^x \cos\, x + \int e^x \cos\, x \, dx$. By substituting the result of J into I, we have

$$\begin{aligned} I &= e^x \sin\, x - J \\ &= e^x \sin\, x + e^x \cos\, x - \int e^x \cos\, x \, dx \\ \Rightarrow I &= e^x \sin\, x + e^x \cos\, x - I. \end{aligned}$$

This implies

$$2I = e^x \sin\, x + e^x \cos\, x \Rightarrow I = \frac{1}{2}(e^x \sin\, x + e^x \cos\, x) \Rightarrow \int e^x \cos\, x\, dx = \frac{e^x}{2}(\sin\, x + \cos\, x) + c.$$

4.4.3 Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$. A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x^4 - x$. The degree of a polynomial $f(x)$ is the highest power occurring in the polynomial, for example the degree of $f(x)$ is 3 and the degree of $g(x)$ is 4.

Steps of the integration of the rational functions:

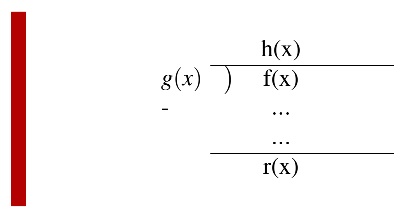
The practical steps to evaluate integrals of the rational functions can be summarized as follows:

➤ Step 1: If the degree of $f(x)$ is equal or greater than the degree of $g(x)$, we do polynomial long-division; otherwise we move to step 2.

By doing the long-division, we reduce the fraction to a mixed quantity.

$$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)}\ ,$$

where $h(x)$ is the quotient and $r(x)$ is the remainder. The degree of the numerator of the new fraction will be less than the degree of the denominator.



➤ Step 2: Factor the denominator $g(x)$ into irreducible polynomials where the factors are either linear or irreducible quadratic polynomials.

➤ Step 3: Find the partial fraction decomposition. This step depends on the result of step 2 where the fraction $\frac{f(x)}{g(x)}$ or $\frac{r(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x) = P_1(x) + P_2(x) + P_3(x) + \ldots + P_n(x)\ ,$$

each $P_k(x) = \frac{A_k}{(ax+b)^n}, n \in \mathbb{N}$ or $P_k(x) = \frac{A_k x + B_k}{(ax^2+bx+c)^n}$ if $b^2 - 4ac < 0$. The constants A_k and B_k are real numbers and computed later. Note that the denominators of the fractions $P_k(x)$ are the factors of the original denominator obtained in step 2.

➤ Step 4: Integrate the result of step 3:

$$\int q(x)\, dx = \int P_1(x)\, dx + \int P_2(x)\, dx + \int P_3(x)\, dx + \ldots + \int P_n(x)\, dx\ .$$

Cases of factoring the denominator $g(x)$:

Case 1: The denominator $g(x)$ is a product of distinct linear factors.

If $g(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$, then the fraction $\frac{f(x)}{g(x)}$ can be written as a sum of partial fractions:

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + + \frac{A_3}{a_3x+b_3} + \dots + \frac{A_n}{a_nx+b_n}$$

■ **Case 2:** The denominator $g(x)$ has repeated linear factors of the form $(a_ix+b_i)^k$ where $k > 1$. Then,

$$\frac{f(x)}{g(x)} = \frac{A_1}{(a_ix+b_i)} + \frac{A_2}{(a_ix+b_i)^2} + \frac{A_3}{(a_ix+b_i)^3} + \dots + \frac{A_n}{(a_ix+b_i)^k}.$$

■ **Case 3:** The denominator $g(x)$ has factors which are irreducible quadratics of the form $a_ix^2 + b_ix + c_i$ where $b_i^2 - 4a_ic_i < 0$. In this case, we include terms of the form $\frac{A_ix+B_i}{a_ix^2+b_ix+c_i}$.

■ **Example 4.35** Evaluate the integral $\displaystyle\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:

Step 1: This step can be skipped since the degree of $f(x) = x+1$ is less than the degree of $g(x) = x^2-2x-8$.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 - 2x - 8 = (x+2)(x-4).$$

Here we have case 2 in factoring the denominator $g(x)$.

Step 3: Find the partial fraction decomposition

$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax-4A+Bx+2B}{(x+2)(x-4)}.$$

We need to find the constants A and B by equating the coefficients of like powers of x in the two sides of the equation:

$$x+1 = (A+B)x - 4A + 2B$$

Coefficients of the numerators:

coefficients of x: $\quad A + B = 1 \rightarrow (1)$

constants: $\quad -4A + 2B = 1 \rightarrow (2)$

By doing some calculation, we obtain $A = \frac{1}{6}$ and $B = \frac{5}{6}$. Thus,

$$\frac{x+1}{x^2-2x-8} = \frac{1/6}{x+2} + \frac{5/6}{x-4}.$$

Illustration

Multiply equation (1) by 4 and add the result to equation (2)

$$\begin{aligned} 4A + 4B &= 4 \\ -4A + 2B &= 1 \\ \hline 6B &= 5 \end{aligned}$$

Step 4: Integrate the result of step 3.

$$\int \frac{x+1}{x^2-2x-8}\,dx = \int \frac{1/6}{x+2}\,dx + \int \frac{5/6}{x-4}\,dx = \frac{1}{6}\ln|x+2| + \frac{5}{6}\ln|x-4| + c.$$

Example 4.36 Evaluate the integral $\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2}\,dx$.

Solution:

Step 1: Do the polynomial long-division.
Since the degree of the denominator $g(x)$ is less than the degree of the numerator $f(x)$, we do the polynomial long-division given on the right side. Then, we have

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2}.$$

$$\begin{array}{r|rrrr} & 2x & -10 & & \\ \hline x^2+3x+2 & 2x^3 & -4x^2 & -15x & +5 \\ & -(2x^3 & +6x^2 & +4x) & \\ \hline & & -10x^2 & -19x & +5 \\ & & -(-10x^2 & -30x & -20) \\ \hline & & & 11x & +25 \end{array}$$

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2+3x+2 = (x+1)(x+2).$$

According to the cases of factoring the denominator $g(x)$, we have case 2.

Step 3: Find the partial fraction decomposition

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}.$$

We need to find the constants A and B.

Coefficients of the numerators:

$$\begin{aligned} \text{coefficients of } x: &\quad A+B = 11 \to (1) \\ \text{constants:} &\quad 2A+B = 25 \to (2) \end{aligned}$$

By doing some calculation, we have $A = 14$ and $B = -3$.
Hence,

$$q(x) = (2x-10) + \frac{14}{x+1} + \frac{-3}{x+2}.$$

Illustration

$-2 \times (1) + (2)$

$$\begin{array}{r} -2A-2B = -22 \\ 2A+B = 25 \\ \hline -B = 3 \end{array}$$

Step 4: Integrate the result of step 3.

$$\begin{aligned} \int q(x)\,dx &= \int (2x-10)\,dx + \int \frac{14}{x+1}\,dx + \int \frac{-3}{x+2}\,dx \\ &= x^2 - 10x + 14\ln|x+1| - 3\ln|x+2| + c. \end{aligned}$$

Example 4.37 Evaluate the integral $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}dx$.

Solution:

Steps 1 and 2 can be skipped in this example. According to the cases of factoring the denominator $g(x)$, we have cases 1 and 2.

Step 3: Find the partial fraction decomposition.

Since the denominator $g(x)$ has repeated factors, then

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5} = \frac{A(x^2-4x-5)+B(x-5)+C(x^2+2x+1)}{(x+1)^2(x-5)}.$$

Coefficients of the numerators:

$$\begin{aligned} \text{coefficients of } x^2: &\quad A+C=2 \rightarrow (1) \\ \text{coefficients of } x: &\quad -4A+B+2C=-25 \rightarrow (2) \\ \text{constants:} &\quad -5A-5B+C=-33 \rightarrow (3) \end{aligned}$$

Illustration

$5\times(2)+(3)=$

$-25A+11C=-158 \rightarrow (4)$

$25\times(1)+(4)=$

$36C=-108 \Rightarrow C=-3$

By solving the system of equations, we have $A=5$, $B=1$ and $C=-3$.

Step 4: Integrate the result of step 3.

$$\begin{aligned} \int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx &= \int \frac{5}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{-3}{x-5}\,dx \\ &= 5\ln|x+1| + \int (x+1)^{-2}\,dx - 3\ln|x-5| \\ &= 5\ln|x+1| - \frac{1}{(x+1)} - 3\ln|x-5| + c. \end{aligned}$$

Example 4.38 Evaluate the integral $\int \frac{x+1}{x(x^2+1)}dx$.

Solution:

Steps 1 and 2 can be skipped in this example. Here we have cases 1 and 3 of factoring the denominator $g(x)$.

Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}.$$

Coefficients of the numerators:

$$\begin{aligned} \text{coefficients of } x^2: &\quad A+B=0 \rightarrow (1) \\ \text{coefficients of } x: &\quad C=1 \rightarrow (2) \\ \text{constants:} &\quad A=1 \rightarrow (3) \end{aligned}$$

We have $A=1$, $B=-1$ and $C=1$.

Step 4: Integrate the result of step 3.

$$\begin{aligned}\int \frac{x+1}{x(x^2+1)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+1}\,dx \\ &= \ln|\,x\,| - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= \ln|\,x\,| - \frac{1}{2}\ln(x^2+1) + \tan^{-1}\,x + c.\end{aligned}$$

Exercises

1 - 38 ■ Evaluate the integral.

1 $\int \sec^2(3x-5)\ dx$

2 $\int \dfrac{dx}{\sqrt{16-x^2}}$

3 $\int xe^{2x}\ dx$

4 $\int x\cos(2x)\ dx$

5 $\int \sin^{-1}x\ dx$

6 $\int \dfrac{dx}{x^2-x-2}$

7 $\int x(2x^2-3)^8\ dx$

8 $\int \dfrac{\cos\sqrt[3]{x}}{\sqrt[3]{x^2}}\ dx$

9 $\int_0^3 (2-x+x^2)\ dx$

10 $\int_{-1}^1 (x^2+3x+1)\ dx$

11 $\int_0^{\frac{\pi}{2}} \cos\ x\ dx$

12 $\int_0^{\frac{\pi}{4}} (\sin\ x+\cos\ x)\ dx$

13 $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\ x\ (\tan\ x+\sec\ x)\ dx$

14 $\int_{-2}^4 2\ dx$

15 $\int_0^5 (3-x)\ dx$

16 $\int_{-1}^4 (2x^2+x-1)\ dx$

17 $\int_2^2 (6x^2+3)\ dx$

18 $\int_0^1 (x^3-4x^4)\ dx$

19 $\int_{-1}^2 x\ \sqrt{x^2+1}\ dx$

20 $\int_0^5 |\ x-1\ |\ \ dx$

21 $\int x\ln\sqrt{x}\ dx$

22 $\int_1^3 (x^2+1)\ dx$

23 $\int_e^5 \dfrac{1}{x-2}\ dx$

24 $\int_3^6 \left(\dfrac{1}{x-2}+\dfrac{2}{x+1}\right)\ dx$

25 $\int_0^{\pi/2} (1+\sqrt{\cos x})^2 \sin x\ dx$

26 $\int_0^{10} (x^{\frac{3}{2}}+1)\ dx$

27 $\int_1^2 \dfrac{2}{\sqrt{x}}\ dx$

28 $\int_0^2 |\ x-1\ |\ \ dx$

29 $\int_{-1}^1 |\ 3x+1\ |\ \ dx$

30 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \dfrac{1}{\sin^2\ x}\ dx$

31 $\int_0^3 |\ 2x-3\ |\ \ dx$

32 $\int_1^3 (x-2)(x+3)\ dx$

33 $\int_0^{\pi} \sin\ x\ dx$

34 $\int_0^{\frac{\pi}{4}} \cos\ 2x\ dx$

35 $\int_0^{\pi} \sec\ x\ (\tan\ x-\sec\ x)\ dx$

36 $\int x\ \cos\ x^2\ dx$

37 $\int \dfrac{\csc^2\ \sqrt{x}}{\sqrt{x}}\ dx$

38 $\int \dfrac{\sec x+\tan x}{\cos x}\ dx$

39 - 52 ■ Evaluate the integral.

39 $\int \sqrt{x}\,\ln x\,dx$

40 $\int x\,\sec^2 x\,dx$

41 $\int x\,e^{-4x}\,dx$

42 $\int (\ln x)^2\,dx$

43 $\int \frac{1}{(x-3)(x-1)^2}\,dx$

44 $\int \frac{1}{x^2+6x+8}\,dx$

45 $\int \frac{x}{x^2-x-2}\,dx$

46 $\int \frac{1}{x^2+2x-3}\,dx$

47 $\int_3^7 \frac{x^2}{x^2-x-2}\,dx$

48 $\int \frac{3x^2-10}{x^2-4x+4}\,dx$

49 $\int \frac{x^2-9}{x-1}\,dx$

50 $\int \frac{2x^4-3x^3-10x^2+2x+11}{x^3-x^2-5x-3}\,dx$

51 $\int \frac{1}{1+e^x}\,dx$

52 $\int \frac{2x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

53 - 58 ■ If $\int_a^b f(x)\,dx = 2$, $\int_b^c f(x)\,dx = 2$ and $\int_a^b g(x)\,dx = 3$ where $c \in (a,b)$, evaluate the integral.

53 $\int_b^a f(x)\,dx$

54 $\int_a^c f(x)\,dx$

55 $\int_a^b \big(2f(x)+g(x)\big)\,dx$

56 $\int_b^a \big(5f(x)-3g(x)\big)\,dx$

57 $\int_a^b \big(\frac{1}{3}f(x)+7g(x)\big)\,dx$

58 $\int_a^a \big(4f(x)+g(x)\big)\,dx$

59 - 64 ■ Use the properties of the definite integrals to prove the inequality without evaluating the integrals.

59 $\int_0^1 x\,dx \geq \int_0^1 x^2\,dx$

60 $\int_0^3 \frac{x}{x^3+2}\,dx \geq \int_0^3 x\,dx$

61 $\int_1^4 (2x+2)\,dx \geq \int_1^4 (3x+1)\,dx$

62 $\int_0^3 (x^2-3x+4)\,dx \geq 0$

63 $\int_1^2 \sqrt{5-x}\,dx \geq \int_1^2 \sqrt{x+1}\,dx$

64 $2 < \int_{-1}^2 \sqrt{1+x^2}\,dx$

65 - 71 ■ Choose the correct answer.

65 The value of the integral $\int \frac{\sin\ x}{\sqrt{2+\cos\ x}}\, dx$ is equal to

(a) $-2\sqrt{2+\cos\ x}+c$
(b) $\sqrt{2+\cos\ x}+c$
(c) $-\sqrt{2+\cos\ x}+c$
(d) $2\sqrt{2+\cos\ x}+c$

66 The value of the integral $\int \frac{\sin\ (\tan\ x)}{\cos^2\ x}\, dx$ is equal to

(a) $\cos\ (\tan\ x)+c$
(b) $\sin\ (\tan\ x)+c$
(c) $-\cos\ (\tan\ x)+c$
(d) $-\sin\ (\tan\ x)+c$

67 The integral $\int x\sqrt{x^2+1}\, dx$ is equal to

(a) $\frac{1}{2}x^2\sqrt{x^2+1}+c$
(b) $\frac{2}{3}(x^2+1)^{\frac{3}{2}}+c$
(c) $-\frac{2}{3}(x^2+1)^{\frac{3}{2}}+c$
(d) $\frac{1}{3}(x^2+1)^{\frac{3}{2}}+c$

68 The integral $\int \frac{x}{\cos^2\ x^2}\, dx$ is equal to

(a) $\frac{1}{2}\tan\ x^2+c$
(b) $\tan\ x^2+c$
(c) $\frac{1}{2}\tan\ x+c$
(d) $-\frac{1}{\cos\ x^2}+c$

69 The value of the integral $\int \frac{\sec^2\ x}{\cot^2\ x}\, dx$ is equal to

(a) $\frac{1+\cos^2\ x}{3\cos^3\ x}+c$
(b) $\frac{1-3\cos^2\ x}{3\cos^3\ x}+c$
(c) $\frac{\cot^4\ x}{4}+c$
(d) $\frac{\tan^3\ x}{3}+c$

70 The value of the integral $\int \frac{\cos\ x}{\sqrt{4+\sin\ x}}\, dx$

(a) $\frac{1}{2}\sqrt{\sin\ x+4}+c$
(b) $\sqrt{\sin\ x+4}+c$
(c) $2\sqrt{\sin\ x+4}+c$
(d) $-2\sqrt{\sin\ x+4}+c$

71 The value of the integral $\int_{-1}^{1} 2\mid x\mid^3\ dx$

(a) 2 (b) 1 (c) 0 (d) -1

Chapter 5

APPLICATIONS OF INTEGRATION

5.1 Areas

In this section, we shed a light on the following cases:

- The region between a curve, the x-axis from $x = a$ to $x = b$.
- The region between a curve, the y-axis from $y = c$ to $y = d$.
- The region between two curves.

5.1.1 Area Bounded by a Curve and x-axis

Consider the region between the graph of the function $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ as shown in Figure 5.1. Now the problem is to find the area of the shaded region.

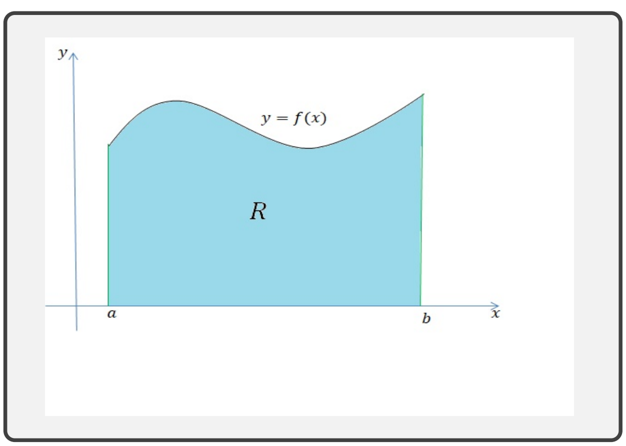


Figure 5.1: A region R bounded by the graph of $y = f(x)$ and the ordinates $x = a$ and $x = b$.

In order to solve this problem, we divide the interval $[a,b]$ into n subintervals and choose x_i^* in the i^{th} subinterval.

As shown in Figure 5.2, the amount $f(x_1^*)\Delta x_1$ is the area of the rectangle A_1, $f(x_2^*)\Delta x_2$ is the area of the rectangle A_2 and so on.

The sum of these areas approximates the area of the whole region under the graph of the function f from $x = a$ to $x = b$, where as the number of the subintervals increases $n \to \infty$ ($||P|| \to 0$), the estimation becomes better.
From Definition 4.10, we have

$$\int_a^b f(x)\,dx = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where P is a partition of $[a,b]$.
Thus, if $y = f(x)$ is continuous and $f(x) \geq 0$ on $[a,b]$, the definite integral $\displaystyle\int_a^b f(x)\,dx$ is exactly the area of the region under the graph of $y = f(x)$ from a to b:

$$\boxed{A = \int_a^b f(x)\,dx}$$

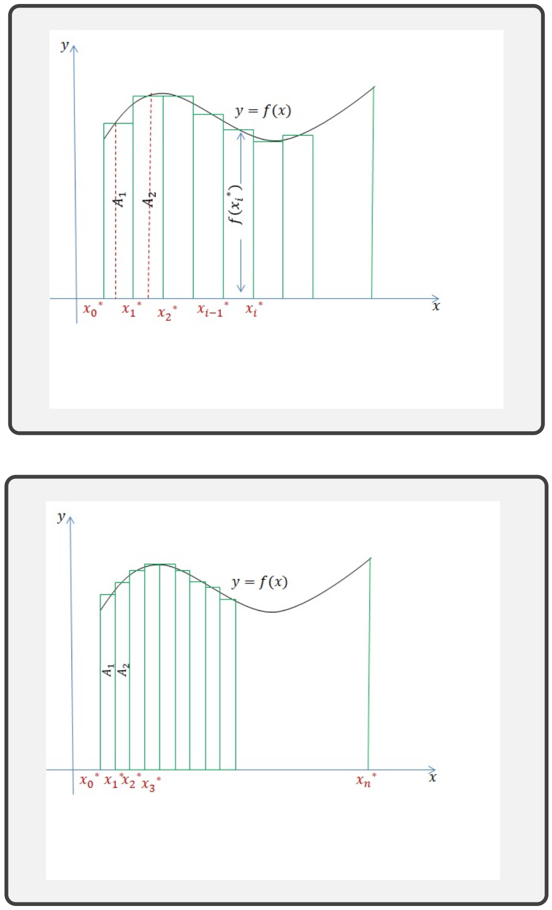


Figure 5.2: A region R bounded by the graph of $y = f(x)$ and the ordinates $x = a$ and $x = b$.

■ **Example 5.1** Express the area of the shaded region as a definite integral then find the area.

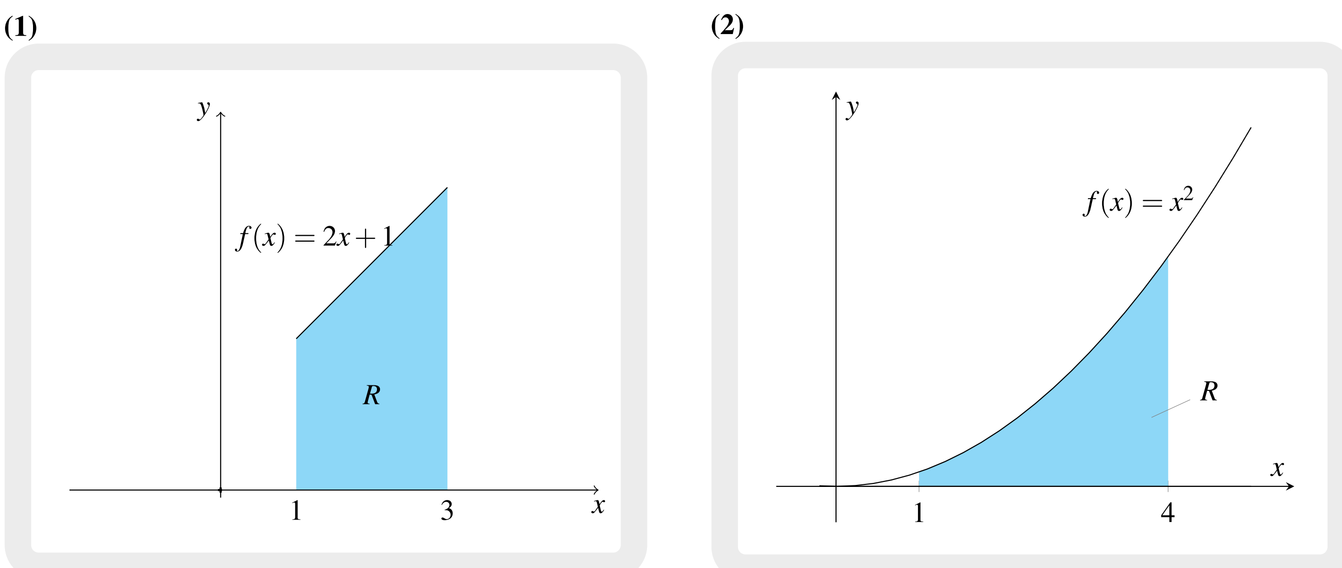


Figure 5.3

Solution:

(1) $A = \int_1^3 (2x+1)\,dx = \left[x^2+x\right]_1^3 = \left[(3^2+3)-(1^2+1)\right] = 12-2 = 10.$

(2) $A = \int_1^4 x^2\,dx = \frac{1}{3}\left[x^3\right]_1^4 = \frac{1}{3}\left[64-1\right] = \frac{63}{3} = 21.$

■ **Example 5.2** Sketch the region bounded by the graph of $y=\sqrt{x}$ from $x=0$ to $x=3$, then find its area.

Solution:
The region bounded by the function $y=\sqrt{x}$ in the interval $[0,3]$ is shown in the figure.

The area of the region is

$$\begin{aligned} A &= \int_0^3 \sqrt{x}\,dx \\ &= \frac{2}{3}\left[\,x^{3/2}\,\right]_0^3 \\ &= 2\sqrt{3}. \end{aligned}$$

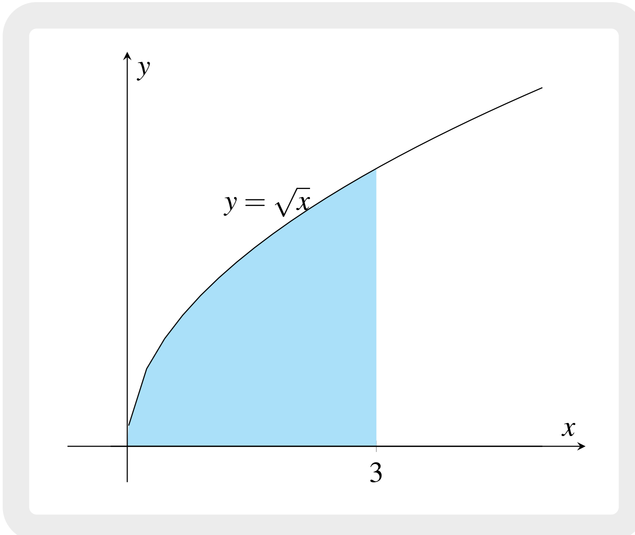


Figure 5.4

5.1.2 Area Bounded by a Curve and y-axis

Consider the region between the graph of the function $x=f(y)$, the y-axis and the ordinates $y=c$ and $y=d$ as shown in Figure 5.5. Now we want to find the area of the shaded region.

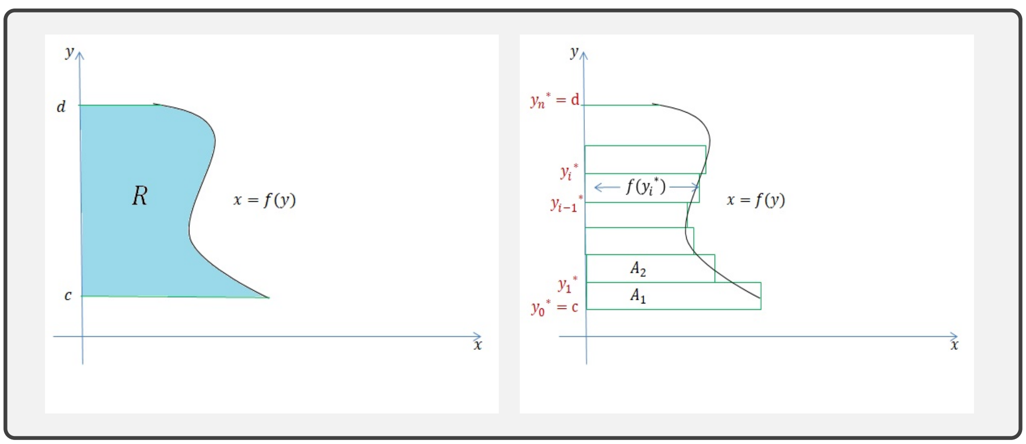


Figure 5.5: A region R bounded by the graph of $x=f(y)$ and the ordinates $y=c$ and $y=d$.

Divide the interval $[c,d]$ into n subintervals and choose y_i^* in the i^{th} subinterval. As shown in Figure **??**, the area of the rectangle A_1 is $f(y_1^*)\Delta y_1$, the area of the rectangle A_2 is $f(y_2^*)\Delta y_2$ and so on.

The sum of the areas of the rectangles approximates the area of the whole region under the graph of the function

$x = f(y)$ from $y = c$ to $y = d$ such that as the number of the subintervals increases $n \to \infty$ ($||P|| \to 0$), the estimation becomes better.

From Definition 4.10, we have

$$\int_c^d f(y)\,dx = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(y_i^*)\Delta y_i$$

where P is a partition of $[c,d]$. Thus, if $x = f(y)$ is continuous and $f(y) \geq 0$ on $[c,d]$, the definite integral $\int_c^d f(y)\,dy$ is exactly the area of the region under the graph of $x = f(y)$ from $y = c$ to $y = d$:

$$\boxed{A = \int_c^d f(y)\,dy}$$

■ **Example 5.3** Sketch the region bounded by the graph of $x = y + 1$ and y-axis over $[-1,0]$, then find its area.

Solution: Figure 5.6 shows the region that is bounded by the function $x = y + 1$ and y-axis over $[-1,0]$.

The area of the region is

$$\begin{aligned} A &= \int_{-1}^{0} (y+1)\,dy \\ &= \left[\frac{y^2}{2} + y\right]_{-1}^{0} \\ &= \left[0 - (\frac{1}{2} - 1)\right]_{-1}^{0} \\ &= \frac{1}{2}. \end{aligned}$$

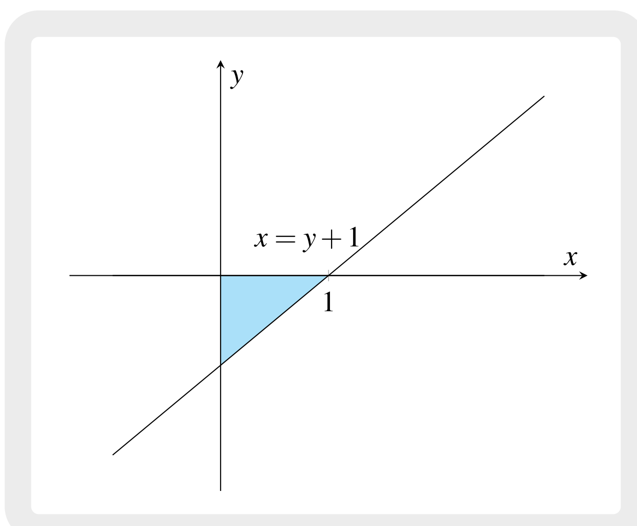


Figure 5.6

■ **Example 5.4** Sketch the region bounded by the graph of $x = \sqrt{y}$ from $y = 0$ to $y = 1$, then find its area.

Solution: The region bounded by the function $x = \sqrt{y}$ in the interval $[0,1]$ is shown in the figure.

The area of the region is

$$\begin{aligned} A &= \int_0^1 \sqrt{y}\, dy \\ &= \frac{2}{3}\left[\, y^{3/2} \,\right]_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

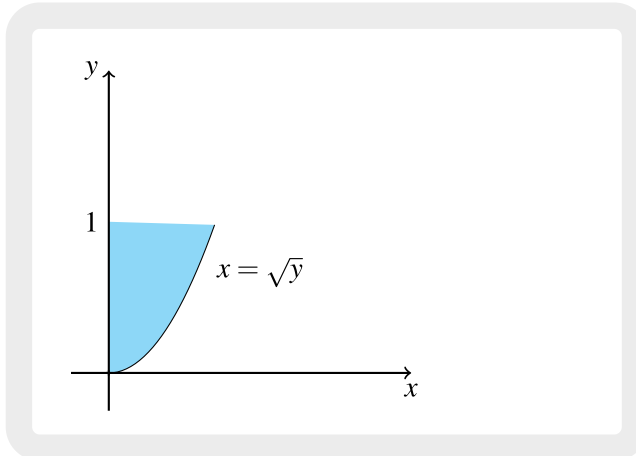


Figure 5.7

5.1.3 Area Between Two Curves

■ If $f(x)$ and $g(x)$ are continuous functions such that $f(x) \geq g(x)$ $\forall x \in [a,b]$, then the area A of the region bounded by the graphs of $f(x)$ (the upper boundary of R) and $g(x)$ (the lower boundary of R) from $x = a$ to $x = b$ is subtracting the area of the region under $g(x)$ from the area of the region under $f(x)$. This can be stated as follows:

$$\boxed{A = \int_a^b \big(f(x) - g(x)\big)\, dx}$$

■ If $f(y)$ and $g(y)$ are continuous functions such that $f(y) \geq g(y)$ $\forall y \in [c,d]$, then the area A of the region bounded by the graphs of $f(y)$ (the right boundary of R) and $g(y)$ (the left boundary of R) from $y = c$ to $y = d$ is subtracting the area of the region bounded by $g(y)$ from the area of the region bounded by $f(y)$. This can be stated as follows:

$$\boxed{A = \int_c^d \big(f(y) - g(y)\big)\, dy}$$

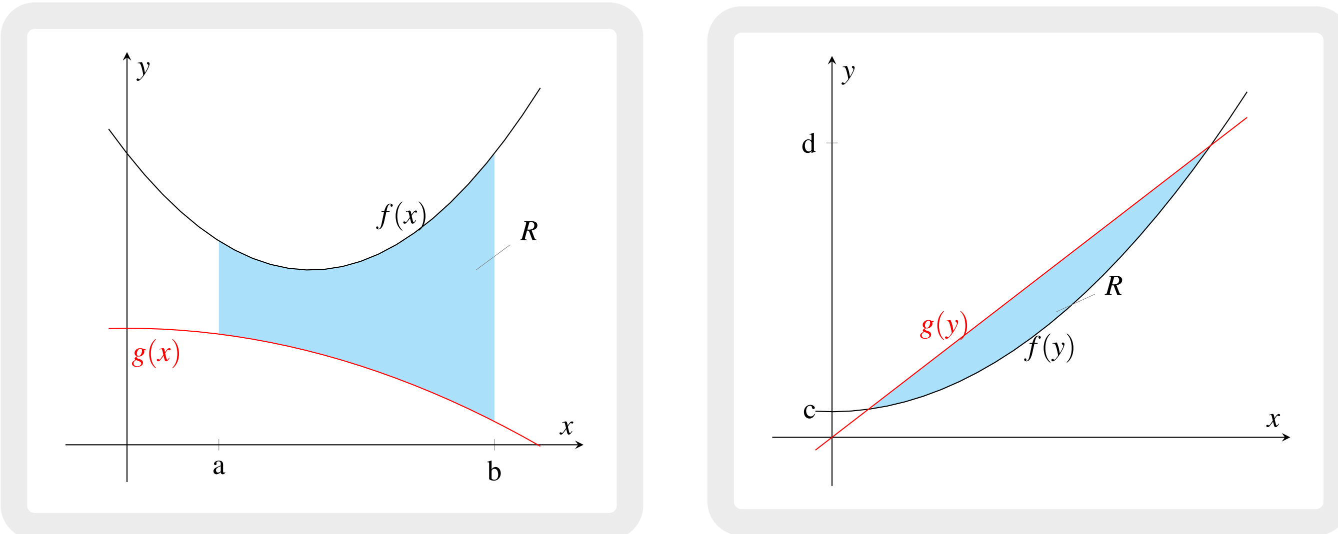
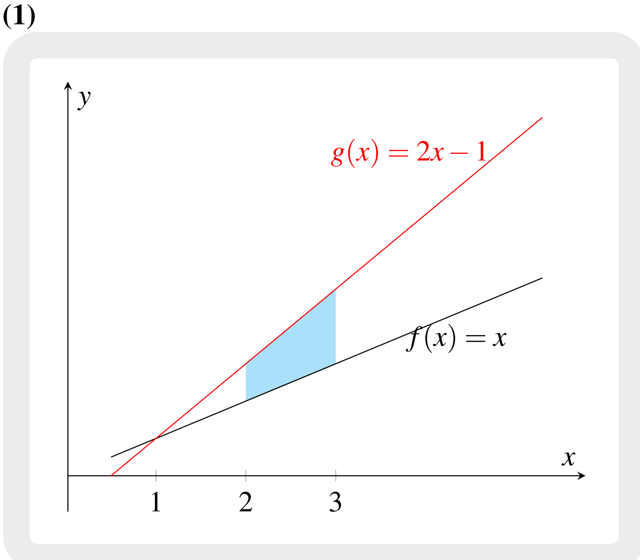


Figure 5.8: The area of the region bounded by the graphs of f and g.

■ **Example 5.5** Express the area of the shaded region as a definite integral then find the area.

(1)



(2)

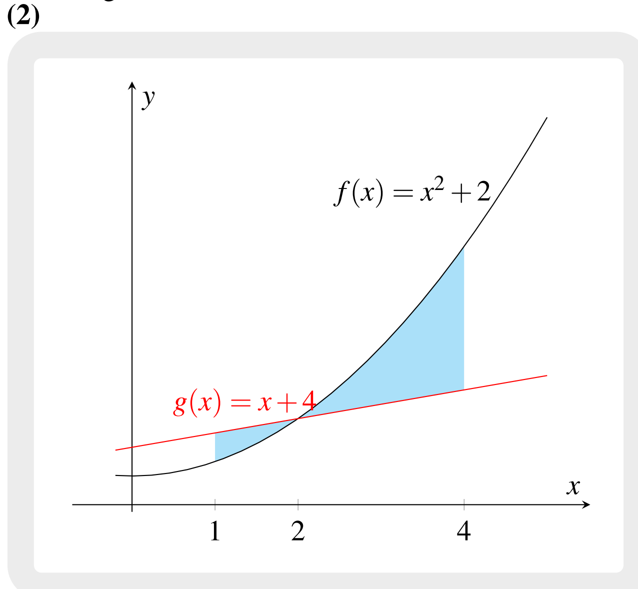


Figure 5.9

Solution:

(1) The area of the region bounded by the two curves $f(x)$ and $g(x)$ is

$$A = \int_2^3 (2x-1) - x\,dx = \int_2^3 (x-1)\,dx = \Big[\frac{x^2}{2} - x\Big]_2^3 = \Big[9\frac{9}{2} - 3) - (2-2)\Big] = \frac{3}{2}\,.$$

(2) We have two regions:

Region (1) is in the interval $[1,2]$.

Upper graph: $y = x+4$
Lower graph: $y = x^2+2$

$$\begin{aligned} A_1 &= \int_1^2 \left((x+4)-(x^2+2)\right)\, dx \\ &= \int_1^2 \left(2+x-x^2\right)\, dx \\ &= \left[2x+\frac{x^2}{2}-\frac{x^3}{3}\right]_1^2 = \frac{13}{6}\,. \end{aligned}$$

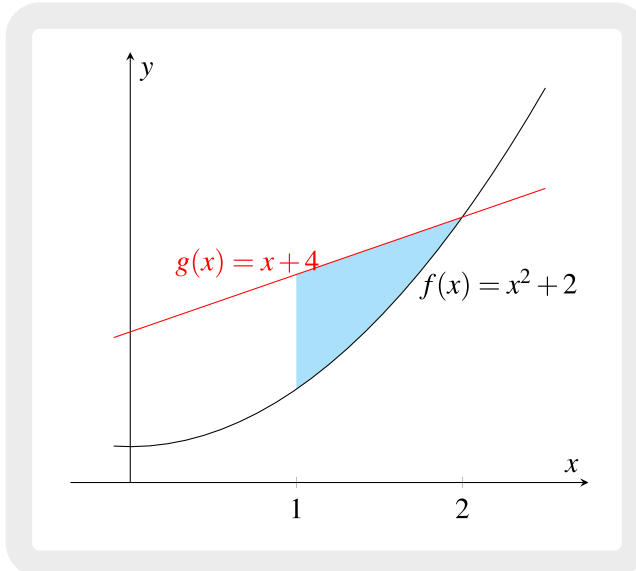


Figure 5.10

Region (2) is in the interval $[2,4]$.

Upper graph: $y = x^2+2$
Lower graph: $y = x+4$

$$\begin{aligned} A_2 &= \int_2^4 \left((x^2+2)-(x+4)\right)\, dx \\ &= \int_2^4 \left(x^2+x-2\right)\, dx \\ &= \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_2^4 = \frac{26}{3}\,. \end{aligned}$$

The total area is $A = A_1 + A_2 = \frac{13}{6} + \frac{26}{3} = \frac{65}{6}$.

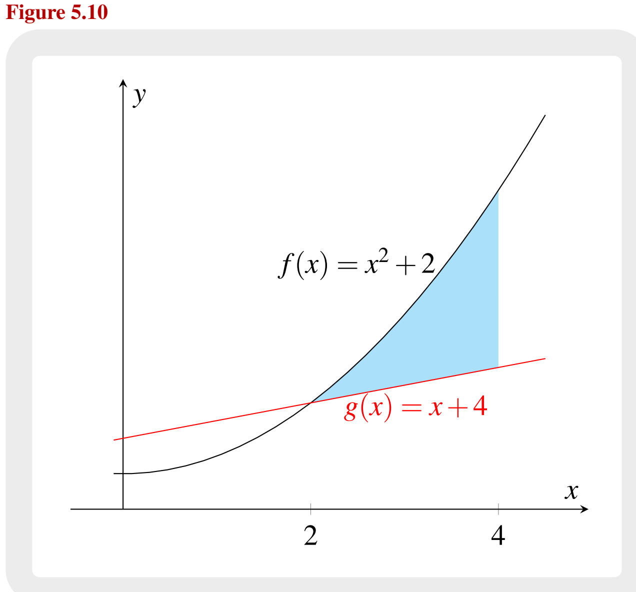


Figure 5.11

■ **Example 5.6** Sketch the region bounded by the graphs of $y = x^2$ and $y = x+6$ over $[-2,3]$, then find its area.

Solution: The region bounded by the given graphs is shown in the figure.

The area of the region is

$$\begin{aligned} A &= \int_{-2}^{3} (x+6-x^2)\, dx \\ &= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} \\ &= \left[\frac{27}{2} + \frac{22}{3}\right] \\ &= \frac{125}{6}\,. \end{aligned}$$

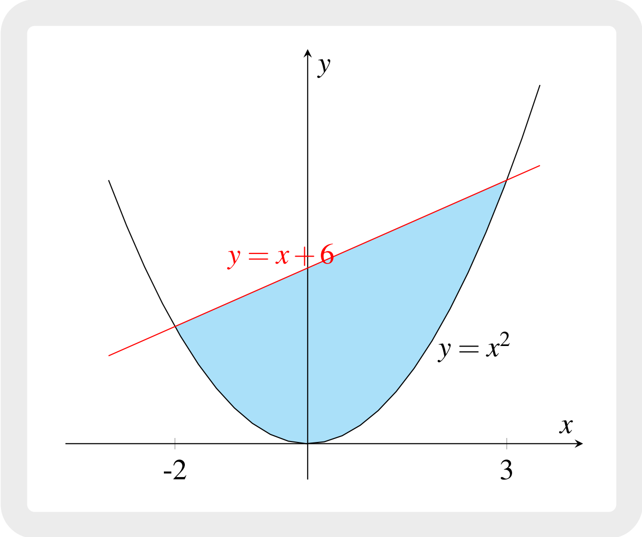


Figure 5.12

■ **Example 5.7** Sketch the region bounded by the graphs of $y = x^3$ and $y = x$, then find its area.

Solution:
The figure on the right shows the region bounded by the two functions. The region is divided into two regions as follows:
Region (1) is in the interval $[-1, 0]$
Upper graph: $y = x^3$
Lower graph: $y = x$

$$A_1 = \int_{-1}^{0} (x^3 - x)\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] = \frac{1}{4}\,.$$

Region (2) is in the interval $[0, 1]$
Upper graph: $y = x$
Lower graph: $y = x^3$

$$A_2 = \int_{0}^{1} (x - x^3)\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] = \frac{1}{4}\,.$$

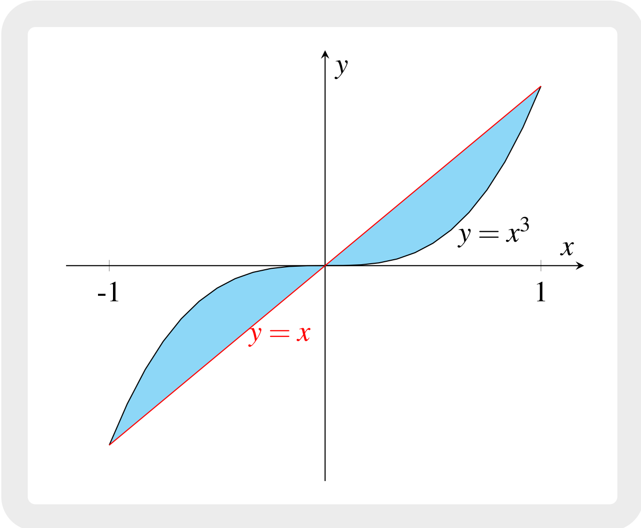


Figure 5.13

The total area is $A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

■ **Example 5.8** Sketch the region bounded by the graphs of $y = x^2$ and $x = y^2$ over $[0, 1]$, then find its area.

Solution:
The region bounded by the graphs of $y = x^2$ and $x = y^2$ over $[0, 1]$ is displayed in the figure.

The area of the region is

$$
\begin{aligned}
A &= \int_0^1 (\sqrt{x} - x^2)\, dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 \\
&= \left[\frac{2}{3} - \frac{1}{3} \right] \\
&= \frac{1}{3}\,.
\end{aligned}
$$

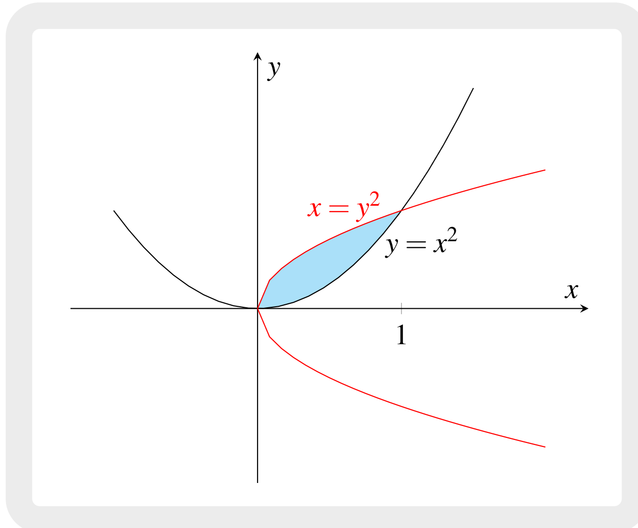


Figure 5.14

■ **Example 5.9** Sketch the region determined by the graphs of $y = \sin\ x$, $y = \cos\ x$, and y-axis, then find its area.

Solution:
The figure on the right shows the region bounded by the two functions. Over the interval $[0, \frac{\pi}{4}]$, the two curves intersect at $\frac{\pi}{4}$.

Hence,

$$
\begin{aligned}
\text{Area: } A &= \int_0^{\frac{\pi}{4}} \big(\cos\ x - \sin\ x\big)\, dx \\
&= \Big[\sin\ x + \cos\ x\Big]_0^{\frac{\pi}{4}} \\
&= \Big[\big(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\big) - (1)\Big] \\
&= \sqrt{2} - 1.
\end{aligned}
$$

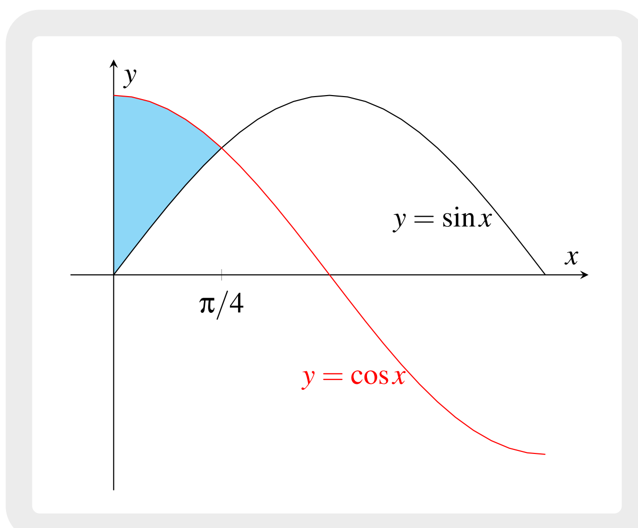


Figure 5.15

■ **Example 5.10** Sketch the region bounded by the graphs of $x = 2y$ and $x = \frac{y}{2} + 3$, then find its area.

Solution:

First, we find the intersection points:

$$
\begin{aligned}
2y &= \frac{y}{2} + 3 \\
\Rightarrow 4y &= y + 6 \\
\Rightarrow y &= 2.
\end{aligned}
$$

The two curves intersect at $(4,2)$.

The area of the region bounded by the curves $x = 2y$ and $x = \frac{y}{2} + 3$ over the interval $[0,2]$ is

$$
\begin{aligned}
A &= \int_0^2 (\frac{y}{2} + 3 - 2y)\ dy \\
&= \int_0^2 (-\frac{3}{2}y + 3)\ dy \\
&= \left[-\frac{3}{4}y^2 + 3y\right]_0^2 \\
&= -3 + 6 = 3.
\end{aligned}
$$

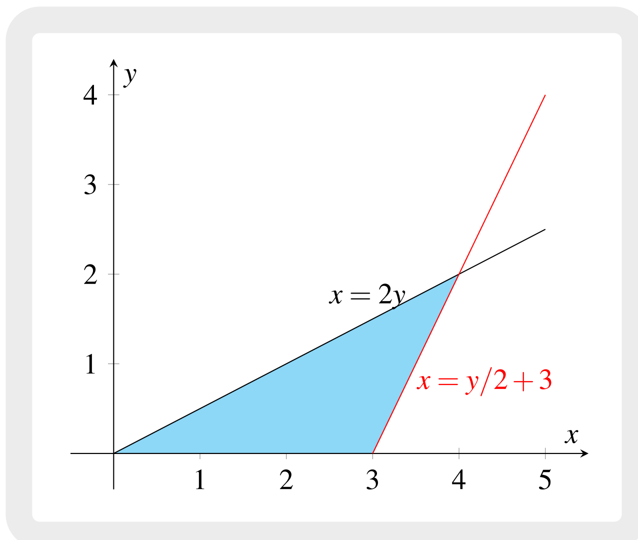


Figure 5.16

5.2 Solids of Revolution

Definition 5.1 If R is a plane region, the solid of revolution S is a solid generated from revolving R about a line in the same plane where the line is called the axis of revolution.

In the following examples, we show some simple solids of revolution.

■ **Example 5.11** Let $y = f(x) \geq 0$ be continuous for every $x \in [a,b]$. Let R be a region bounded by the graph of f and the x-axis from $x = a$ to $x = b$. The region revolution about x-axis generates a solid given in Figure 5.17 (right).

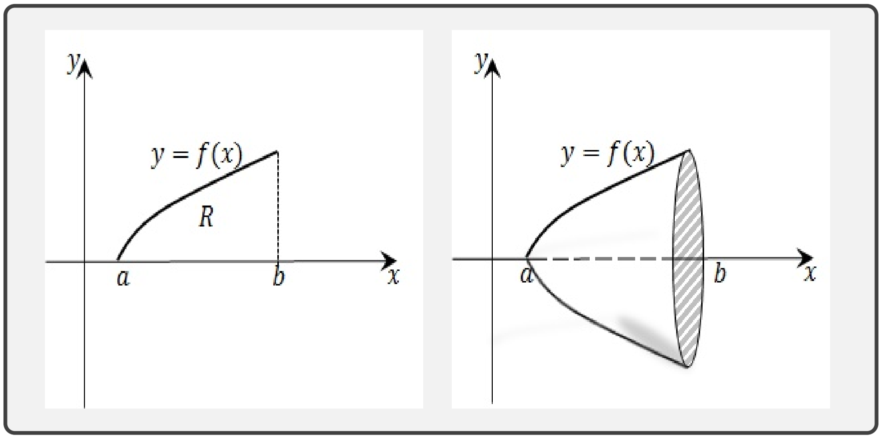


Figure 5.17: Revolution of a region about x-axis. The figure on the left shows the region under the continuous function $y = f(x)$ over the interval $[a,b]$. The figure on the right shows the solid S generated by revolving the region about x-axis.

■ **Example 5.12** Let $y = f(x)$ be a constant function from $x = a$ to $x = b$, as in Figure 5.18. The region R is a rectangle and by revolving it about x-axis, we obtain a circular cylinder.

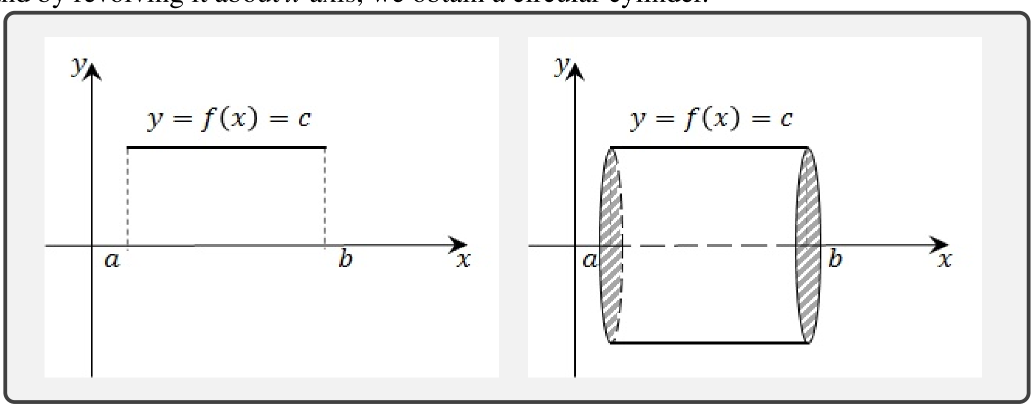


Figure 5.18: Revolution of a rectangular region about x-axis. The figure on the left shows the region under the constant function $f(x) = c$ over the interval $[a,b]$. The figure on the right shows the circular cylinder generated by revolving the region about x-axis.

■ **Example 5.13** Consider a region R bounded by the graph of $x = f(y)$ from $y = c$ to $y = d$. Revolution of R about y-axis generates a solid given in Figure 5.19.

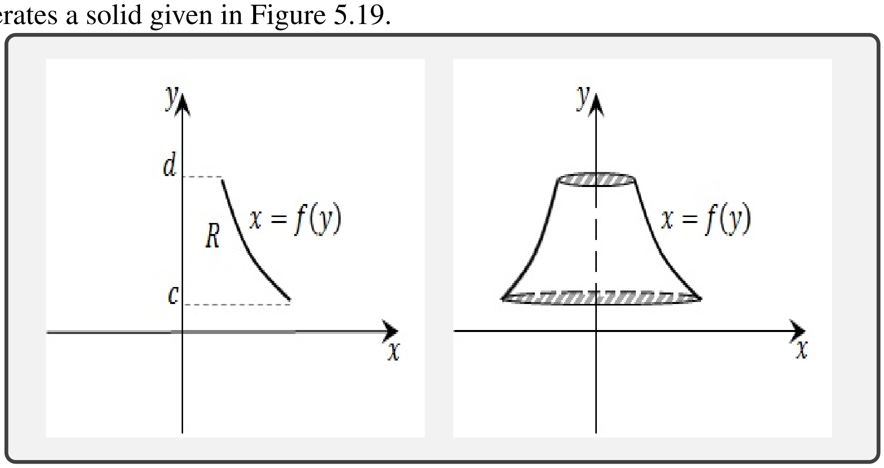


Figure 5.19: Revolution of a region about y-axis. The figure on the left displays the region under the function $x = f(y)$ over the interval $[c,d]$. The figure on the right displays the solid S generated by revolving the region about y-axis.

5.3 Volumes of Revolution Solids

One of the interesting applications of specific integrations is computing the solids revolution. In this section, we study three methods to compute the volumes of the revolution solids known as disk method, washer method and cylindrical shells method.

5.3.1 Disk Method

Let f be continuous on $[a,b]$ and let R be a region bounded by the graph of f and x-axis form $x = a$ to $x = b$. Let S be a solid generated by revolving R about x-axis. Assume that P is a partition of $[a,b]$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a mark where $\omega_k \in [x_{k-1}, x_k]$. From each subinterval $[x_{k-1}, x_k]$, we form a vertical rectangle, its high and width are $f(\omega_k)$ and Δx_k, respectively.

The revolution of the vertical rectangle about x-axis generates a circular disk as shown in Figure 5.21. Its radius and high are

$$r = f(\omega_k)\ , \quad h = \Delta x_k.$$

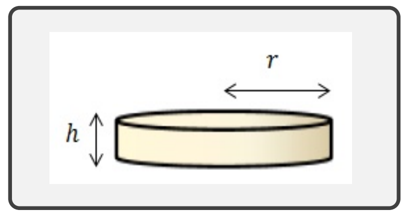


Figure 5.20

$$V = \pi r^2 h$$

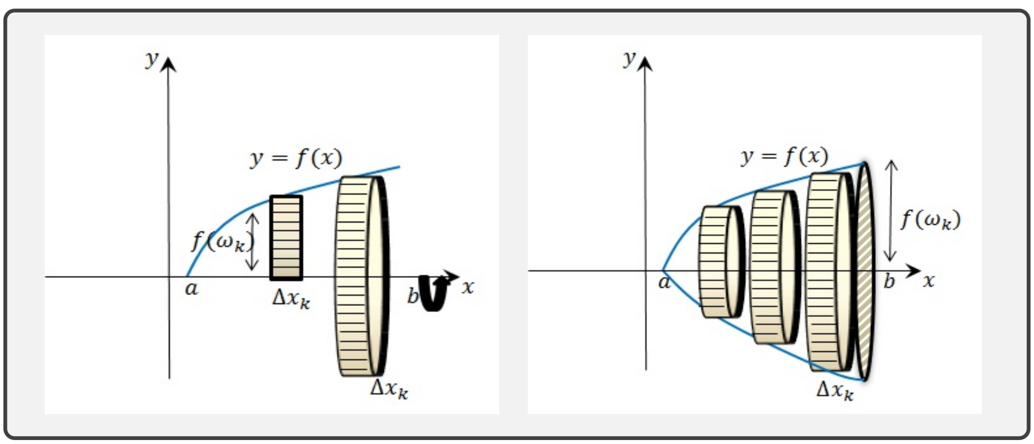


Figure 5.21: The volume of the revolution solid about x-axis by the disk method. The figure on the left shows the region R bounded by the function f on the interval $[a,b]$ and the figure on the right shows the solid S generated by revolving R about x-axis.

From Figure 5.21, the volume of each circular disk is

$$V_k = \pi(f(\omega_k))^2 \Delta x_k,\ \ k = 1, 2, \ldots, n\ .$$

The sum of the volumes of the circular disks approximates the volume of the revolution solid:

$$V = \sum_{k=1}^{n} V_k = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi\, \big(f(\omega_k)\big)^2\, \Delta x_k = \pi \int_a^b \Big[\, f(x)\, \Big]^2 dx.$$

Similarly, we can find the volume of the revolution generated by revolving a region about y-axis. Let f be continuous on $[c,d]$ and let R be a region bounded by the graph of f and y-axis from $y=c$ to $y=d$. Let S be a solid generated by revolving R about y-axis. Assume that P is a partition of $[c,d]$ and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a mark where $\omega_k \in [y_{k-1}, y_k]$. From each $[y_{k-1}, y_k]$, we form a horizontal rectangle, its high and width are $f(\omega_k)$ and Δy_k, respectively.
The revolution of each horizontal rectangle about y-axis generates a circular disk as shown in Figure 5.22. Its radius and high are

$$r = f(\omega_k)\ , \quad h = \Delta y_k\ .$$

Therefore, the volume of each circular disk is

$$V_k = \pi(f(\omega_k))^2 \Delta y_k, \ \ k = 1, 2, ..., n\ .$$

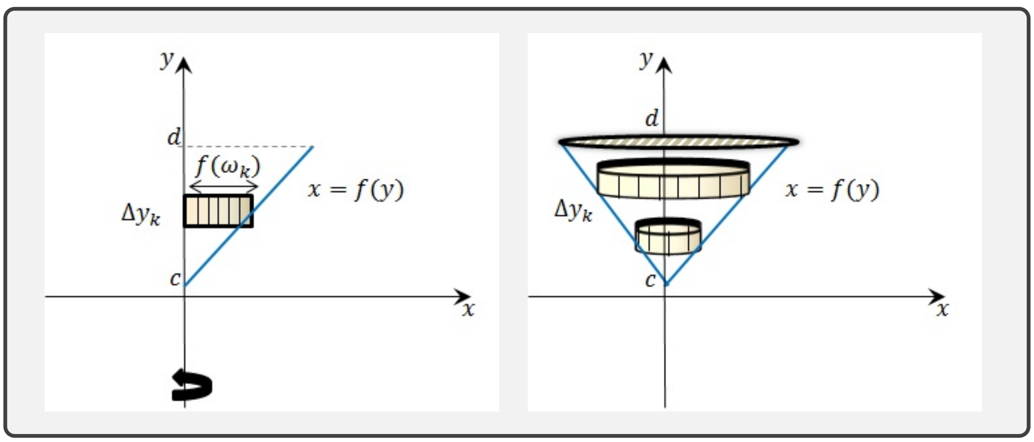


Figure 5.22: The volume of the revolution solid about y-axis by the disk method. The figure on the left shows the region R bounded by the function f on the interval $[c,d]$ and the figure on the right shows the solid S generated by revolving R about y-axis.

The volume of the revolution solid given in Figure 5.22 (right) is approximately the sum of the volumes of the circular disks:

$$\begin{aligned} V = \sum_{k=1}^{n} V_k &= \lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi(f(\omega_k))^2 \Delta y_k \\ &= \pi \int_c^d \Big[f(y)\Big]^2\, dy. \end{aligned}$$

These considerations are summarized in the following theorem:

Theorem 5.2

1. If R is a region bounded by the graph of f on the interval $[a,b]$, the volume of the revolution solid generated by revolving R about x-axis is

$$V = \pi \int_a^b \left[\, f(x)\, \right]^2 dx.$$

2. If R is a region bounded by the graph of f on the interval $[c,d]$, the volume of the revolution solid generated by revolving R about y-axis is

$$V = \pi \int_c^d \left[\, f(y)\, \right]^2 dy.$$

■ **Example 5.14** Sketch the region R bounded by the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 4$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution:
Figure 5.23 shows the region R and the solid S generated by revolving the region about x-axis.

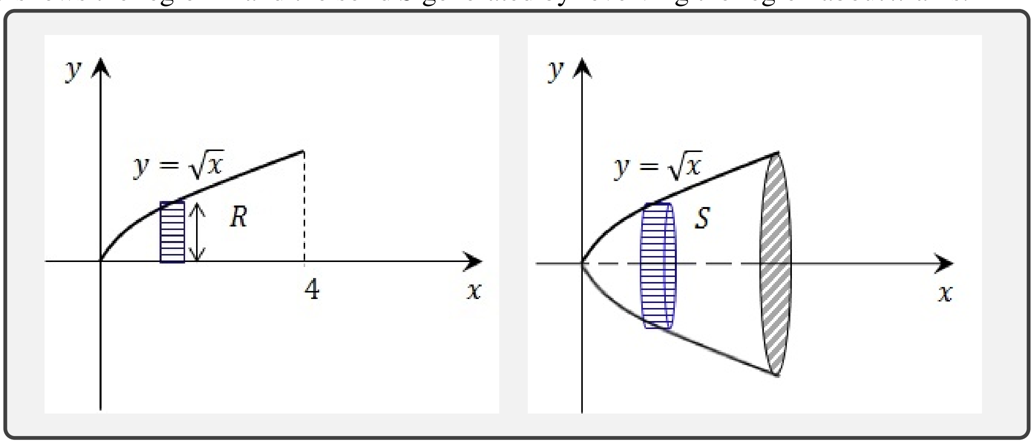


Figure 5.23

Since the revolution is about x-axis, we have a vertical disk with radius $y = \sqrt{x}$ and thickness dx. Thus, the volume of the solid S is

$$V = \pi \int_0^4 (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx = \frac{\pi}{2}\left[\, x^2\, \right]_0^4 = \frac{\pi}{2}\Big[16 - 0\Big] = 8\pi.$$

■ **Example 5.15** Sketch the region R bounded by the graph of $y = e^x$ on the interval $[0,1]$. Then, find the volume of the solid generated by revolving R about y-axis.

Solution:

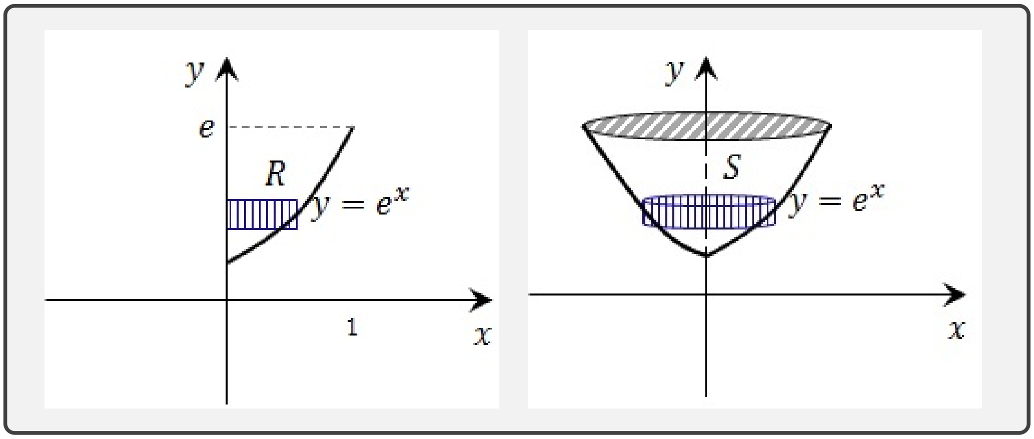


Figure 5.24

The figure shows the region R and the solid S generated by revolving the region about y-axis. Since the revolution is about y-axis, then we need to rewrite the function to become $x = f(y)$.

$$y = e^x \Rightarrow \ln y = \ln e^x \Rightarrow x = \ln y = f(y).$$

Also,

$$x = 0 \Rightarrow y = 1 \text{ and } x = 1 \Rightarrow y = e.$$

Now, we have a horizontal disk with radius $x = \ln y$ and thickness dy. Thus, the volume of the solid S is

$$V = \pi \int_1^e (\ln y)^2 \, dy = \pi \Big[2y + y\,(\ln y)^2 - 2y\,\ln y \Big]_1^e = \pi(e-2).$$

Use the integration by parts to evaluate the integral $\int (\ln y)^2 \, dy$

Example 5.16 Sketch the region R bounded by the graph of the equation $x = y^2$ on the interval $[0,1]$. Then, find the volume of the solid generated by revolving R about y-axis.

Solution:

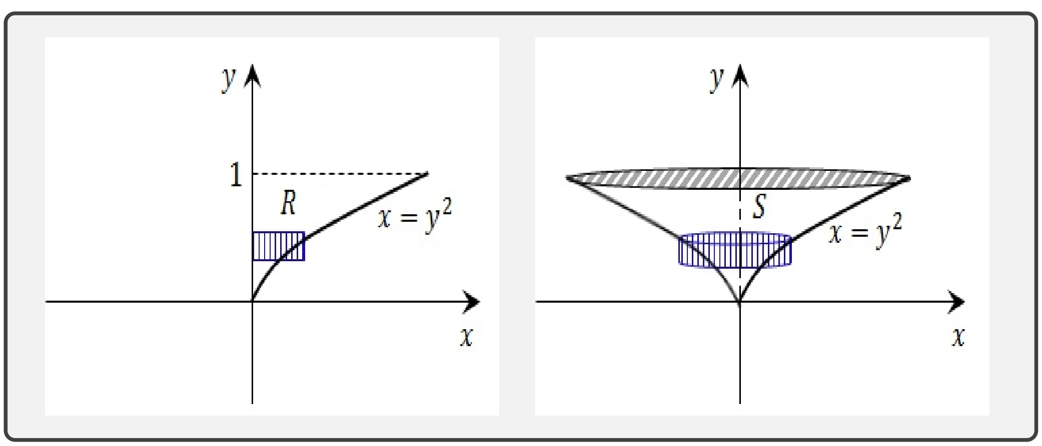


Figure 5.25

Since the revolution of R is about y-axis, we have a horizontal disk with radius $x = y^2$ and thickness dy. Thus, the volume of the solid S is

$$V = \pi \int_0^1 (y^2)^2 \, dy = \frac{\pi}{5}\Big[y^5 \Big]_0^1 = \frac{\pi}{5}\Big[1 - 0\Big] = \frac{\pi}{5}.$$

■ **Example 5.17** Sketch the region R bounded by the graph of the equation $y = \cos\ x$ from $x = 0$ to $x = \frac{\pi}{2}$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution:

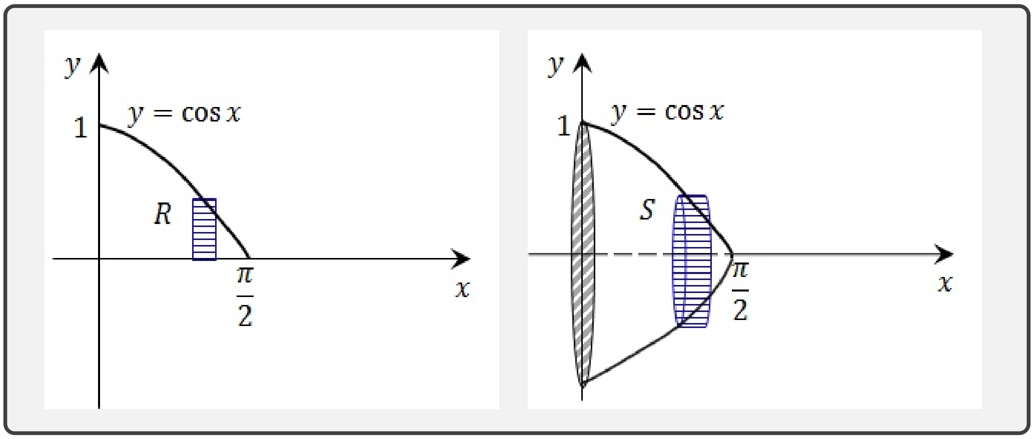


Figure 5.26

The figure shows the region R and the solid S generated by revolving the region about x-axis. Thus, the disk to evaluate the volume of the generated solid S is vertical with radius $y = \cos\ x$ and thickness dx. Hence,

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2\ x\, dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + \cos\ 2x)\ dx$$
$$= \frac{\pi}{2} \left[x + \frac{\sin\ 2x}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{4}\ .$$

Use the identity:

$$\cos^2\ x = \frac{1 + \cos\ x}{2}$$

5.3.2 Washer Method

The washer method is a generalization of the disk method for a region bounded by graphs of two functions f and g. Let R be a region bounded by the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$ as shown in Figure 5.27). The volume of the solid S generated by revolving the region R about x-axis can be found by calculating the difference between the volumes of the two solids generated by revolving the regions under f and g about x-axis as follows:
The outer radius: $y_1 = f(x)$
The inner radius: $y_2 = g(x)$
The thickness: dx
The volume of a washer is $dV = \pi \Big[(\text{the outer radius})^2 - (\text{the inner radius})^2 \Big].$ thickness.
This implies $dV = \pi \Big[(f(x))^2 - (g(x))^2 \Big] dx$.
Hence, the volume of the solid over the period $[a, b]$ is

$$V = \pi \int_a^b \left[\big(f(x)\big)^2 - \big(g(x)\big)^2 \right]\ dx.$$

Similarly, let R be a region bounded by the graphs of $f(y)$ and $g(y)$ such that $f(y) \geq g(y)$ for all $y \in [c, d]$ as shown in Figure 5.28. The volume of the solid S generated by revolving R about y-axis is

$$V = \pi \int_c^d \left[\big(f(y)\big)^2 - \big(g(y)\big)^2 \right]\ dy.$$

The following theorem summarizes the washer method.

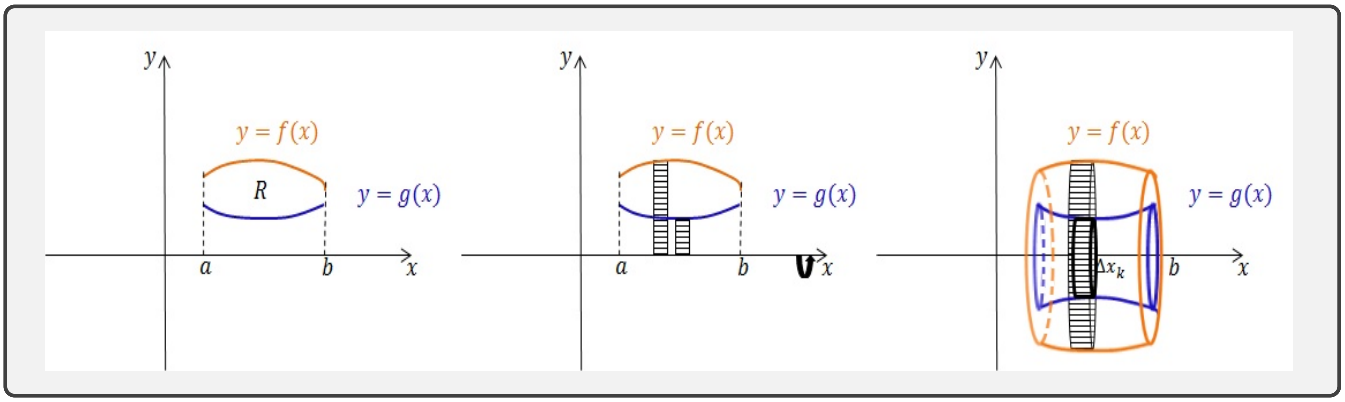


Figure 5.27: The volume of the revolution solid about x-axis by the washer method.

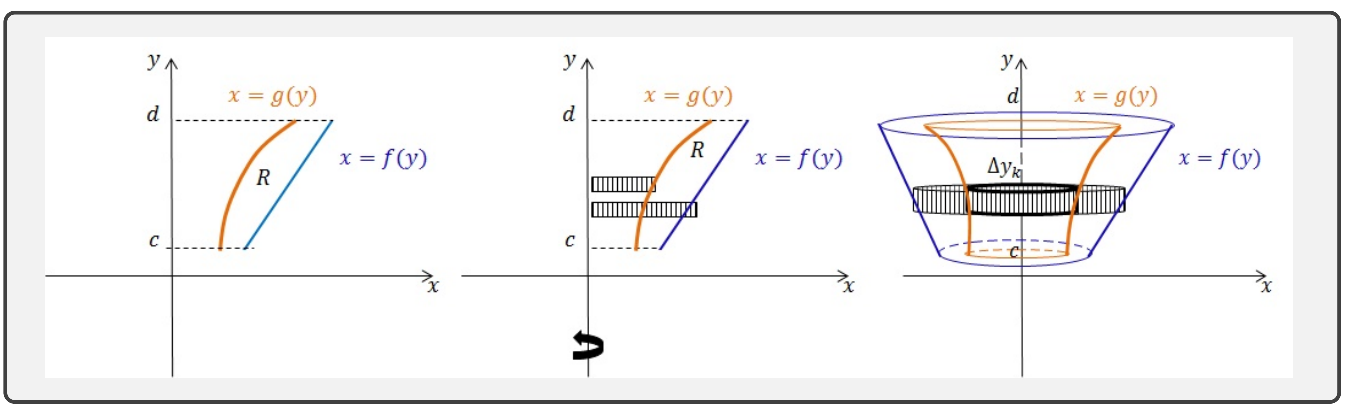


Figure 5.28: The volume of the revolution solid about y-axis by the washer method.

Theorem 5.3

1. If R is a region bounded by the graphs of f and g on the interval $[a,b]$ such that $f \geq g$, the volume of the revolution solid generated by revolving R about x-axis is
$$V = \pi \int_a^b \left[\left(f(x)\right)^2 - \left(g(x)\right)^2 \right] dx.$$
2. If R is a region bounded by the graphs of f and g on the interval $[c,d]$ such that $f \geq g$, the volume of the revolution solid generated by revolving R about y-axis is
$$V = \pi \int_c^d \left[\left(f(y)\right)^2 - \left(g(y)\right)^2 \right] dy.$$

Example 5.18 Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving R about x-axis.

Solution:
Let $f(x) = x^2$ and $g(x) = 2x$. First, we find the intersection points:

$$\begin{aligned} f(x) = g(x) \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x &= 0 \\ \Rightarrow x(x-2) &= 0 \\ \Rightarrow x = 0 \text{ or } x &= 2. \end{aligned}$$

Substituting $x = 0$ into $f(x)$ or $g(x)$ gives $y = 0$. Similarly, substitute $x = 2$ into any of the two functions gives $y = 2$. Thus, the two curves intersect in two points $(0,0)$ and $(2,4)$.

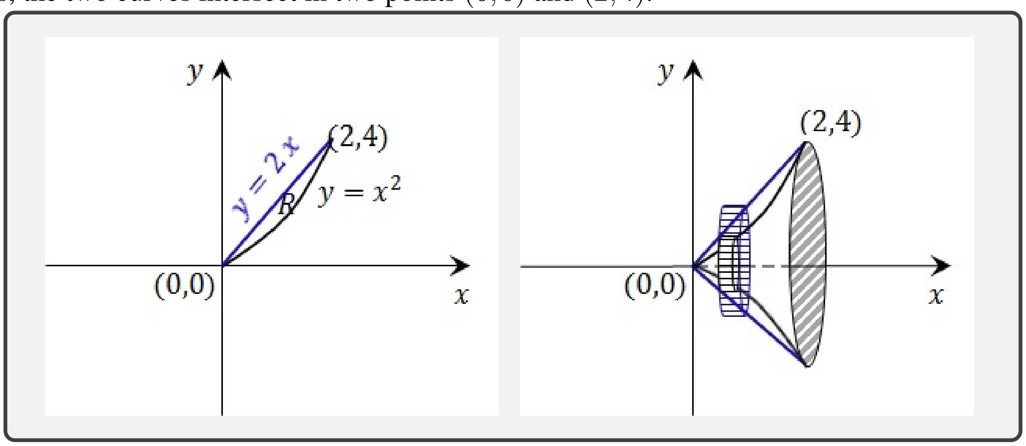


Figure 5.29

The figure shows the region R and the solid generated by revolving the region about x-axis. A vertical rectangle generates a washer where
the outer radius: $y_1 = 2x$,
the inner radius: $y_2 = x^2$ and
the thickness: dx.
The volume of the washer is $dV = \pi \left[(2x)^2 - (x^2)^2 \right] dx$.

Hence, the volume of the solid over the interval $[0,2]$ is

$$
\begin{aligned}
V = \pi\int_0^2 \left((2x)^2-(x^2)^2\right)\,dx = \pi\int_0^2 (4x^2-x^4)\,dx \\
&= \pi\left[\frac{4x^3}{3}-\frac{x^5}{5}\right]_0^2 \\
&= \pi\left[\frac{32}{3}-\frac{32}{5}\right] \\
&= \frac{64}{15}\pi.
\end{aligned}
$$

Example 5.19 Consider a region R bounded by the graphs of the functions $y=\sqrt{x}$, $y=6-x$ and x-axis. Revolve the region about y-axis and find the volume of the generated solid.

Solution:
Since the revolution is about y-axis, we need to rewrite the functions in terms of y i.e., $x=f(y)$ and $x=g(y)$.

$$y=\sqrt{x} \Rightarrow x=y^2=f(y)$$

$$y=6-x \Rightarrow x=6-y=g(y).$$

Now, we find the intersection points:

$$f(y)=g(y) \Rightarrow y^2=6-y \Rightarrow y^2+y-6=0 \Rightarrow y=-3 \text{ or } y=2.$$

Since $y=\sqrt{x}$, we ignore the value $y=-3$. By substituting $y=2$ into the two functions, we have $x=4$. Thus, the two curves intersect in one point $(4,2)$. The solid S generated by revolving the region R about y-axis is shown in Figure 5.30.
Since the revolution is about the y-axis, then we have a horizontal rectangle that generates a washer where
the outer radius: $x_1=6-y$,
the inner radius: $x_2=y^2$ and
the thickness: dy.
The volume of the washer is $dV=\pi\left[(6-y)^2-(y^2)^2\right]\,dy$.

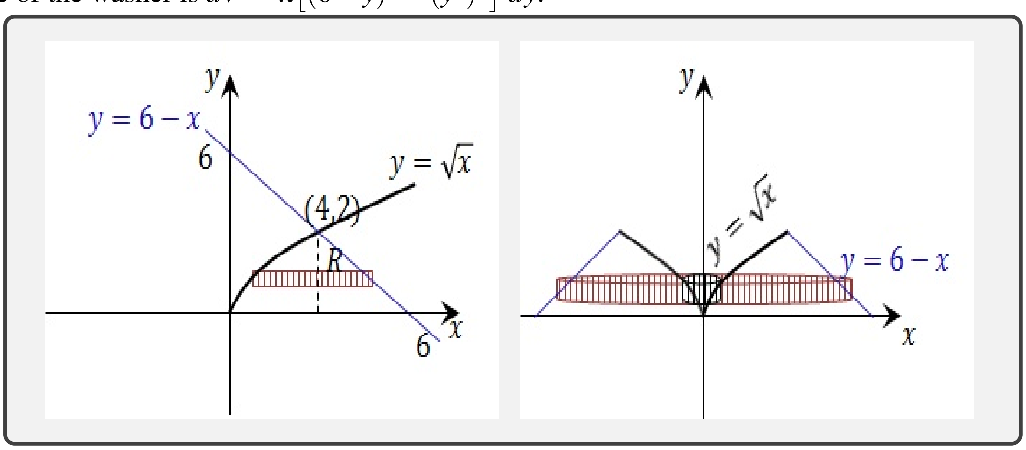


Figure 5.30

The volume of the solid over the interval $[0,2]$ is

$$V=\pi\int_0^2 \left[(6-y)^2-(y^2)^2\right]\,dy=\pi\Big[-\frac{(6-y)^3}{3}-\frac{y^5}{5}\Big]_0^2$$
$$=\pi\Big[(-\frac{64}{3}-\frac{32}{5})-(-\frac{216}{3}-0)\Big]$$
$$=\frac{664}{15}\pi.$$

■ **Example 5.20** Consider the same region as in Example 5.19 enclosed by the graphs of $y=\sqrt{x}$, $y=6-x$ and x-axis. Revolve the region about x-axis instead and find the volume of the generated solid.

Solution:
From the figure, we find that the solid is made up of two separate regions and each requires its own integral. Hence, we use the disk method to evaluate the volume of the solid generated by revolving each region.

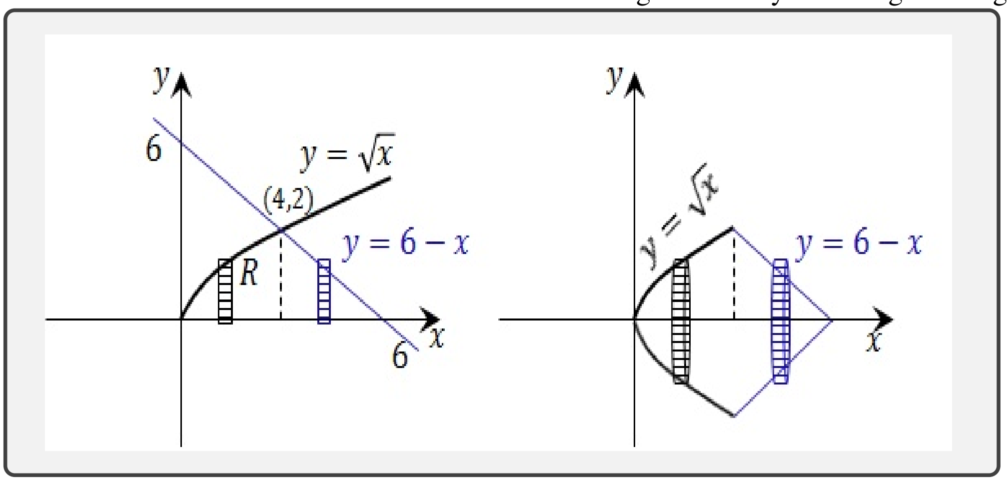


Figure 5.31

$$V=\pi\int_0^4(\sqrt{x})^2\,dx+\pi\int_4^6(6-x)^2\,dx$$
$$=\pi\int_0^4 x\,dx+\pi\int_4^6(6-x)^2\,dx$$
$$=\frac{\pi}{2}\Big[x^2\Big]_0^4-\frac{\pi}{3}\Big[(6-x)^3\Big]_4^6$$
$$=\frac{\pi}{2}(16-0)-\frac{\pi}{3}(0-8)$$
$$=\frac{32}{3}\pi.$$

Use the substitution method to do the second integral with

$u=6-x$ and $-du=dx$

The revolution of a region is not always about x-axis or y-axis, but it could be about a line paralleled to those axes. If the revolution axis is a line $y=y_0$, then evaluating the volume of the generated solid is similar to the case when the region revolves about x-axis. Whereas, if the revolution axis is a line $x=x_0$, then evaluating the volume of the generated solid is similar to the case when the region revolves about y-axis.

■ **Example 5.21** Let R be a region bounded by graphs of the functions $y=x^2$ and $y=4$. Evaluate the volume of the solid generated by revolving R about the given line.
(a) $y=4$ **(b)** $x=2$

Solution:

(a) We have a vertical circular disk with radius $4-y=4-x^2$ and thickness dx.

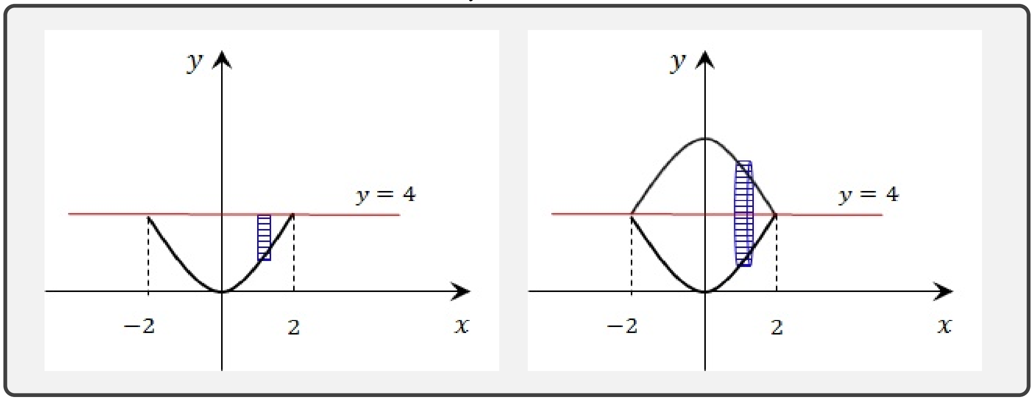


Figure 5.32

The volume of the disk is $dV=\pi(4-x^2)^2\,dx$.
Hence, the volume of the solid over the interval $[-2,2]$ is

$$\begin{aligned} V=\pi\int_{-2}^{2}(4-x^2)^2\,dx &= \pi\int_{-2}^{2}(16-8x^2+x^4)\,dx \\ &= \pi\left[16x-\frac{8x^3}{3}+\frac{x^5}{5}\right]_{-2}^{2} \\ &= \frac{512}{15}\pi. \end{aligned}$$

(b) A horizontal rectangle will generate a washer where
the outer radius: $2+\sqrt{y}$,
the inner radius: $2-\sqrt{y}$ and
the thickness: dy.

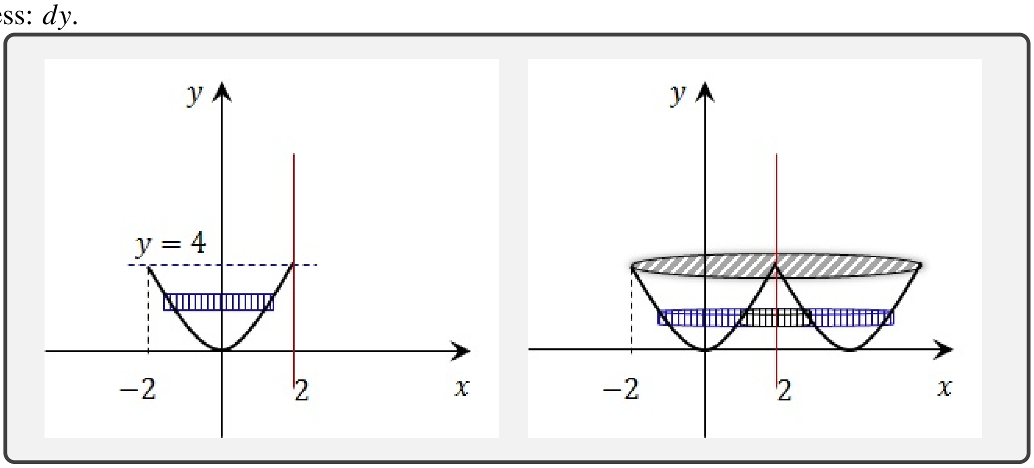


Figure 5.33

The volume of the washer is

$$dV=\pi\left[(2+\sqrt{y})^2-(2-\sqrt{y})^2\right]\,dy=8\pi\sqrt{y}\,dy.$$

Hence, the volume of the solid over the interval $[0,4]$ is

$$V=8\pi\int_0^4\sqrt{y}\,dx=\frac{16\pi}{3}\left[y^{\frac{3}{2}}\right]_0^4=\frac{128}{3}\pi.$$

■ **Example 5.22** Sketch the region R bounded by graphs of the equations $x=(y-1)^2$ and $x=y+1$. Then, find the volume of the solid generated by revolving R about $x=4$.

Solution:
First, we find the intersection points:

$$\begin{aligned}(y-1)^2=y+1 &\Rightarrow y^2-2y+1=y+1\\ &\Rightarrow y^2-3y=0\\ &\Rightarrow y=0 \text{ or } y=3.\end{aligned}$$

Substitute $y=0$ and $y=3$ into the two functions to have $x=1$ and $x=4$, respectively.

Thus, the two curves intersect in two points $(1,0)$ and $(4,3)$.

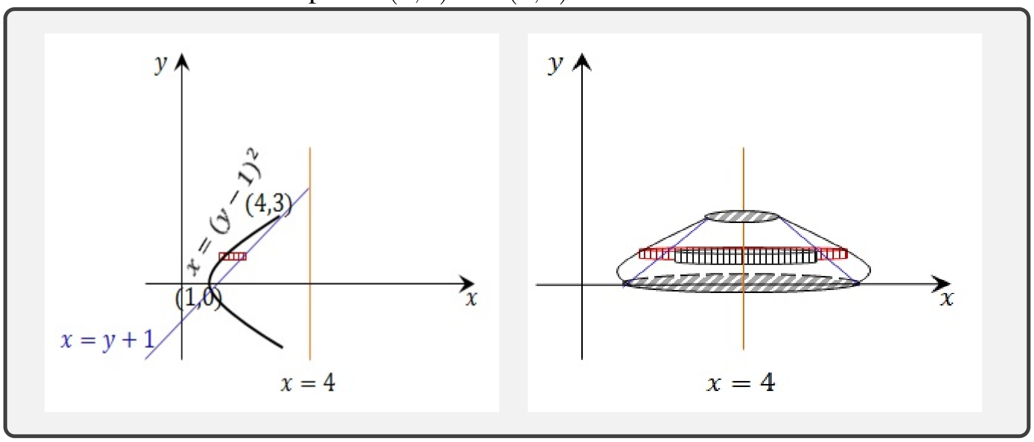


Figure 5.34

The figure shows the region R and the solid S. A horizontal rectangle generates a washer where
the outer radius: $4-(y-1)^2$,
the inner radius: $4-(y+1)=3-y$ and
the thickness: dy.
The volume of the washer is $dV=\pi\left[(4-(y-1)^2)^2-(3-y)^2\right]\,dy=\pi\left[16-8(y-1)^2+(y-1)^4-(3-y)^2\right]\,dy$.
Hence, the volume of the solid over the interval $[0,3]$ is

$$\begin{aligned}V&=\pi\Big(\int_0^3 16\,dy-8\int_0^3(y-1)^2\,dy+\int_0^3(y-1)^4\,dy-\int_0^3(3-y)^2\,dy\Big)\\ &=\pi\Big[16y-\frac{8(y-1)^3}{3}+\frac{(y-1)^5}{5}+\frac{(3-y)^3}{3}\Big]_0^3\\ &=\frac{108}{5}\pi.\end{aligned}$$

5.3.3 Method of Cylindrical Shells

In this section, we study a new method to evaluate the volume of revolution solid called cylindrical shells method. In the washer method, we assume that the rectangle from each subinterval is vertical to the revolution axis while in the cylindrical shells method, the rectangle will be parallel to the revolution axis.

Figure 5.35 shows a cylindrical shell. Let
r_1 be the inner radius of the shell,
r_2 be the outer radius of the shell,
h be high of the shell,
$\Delta r = r_2 - r_1$ be the thickness of the shell,
$r = \frac{r_1+r_2}{2}$ be the average radius of the shell.

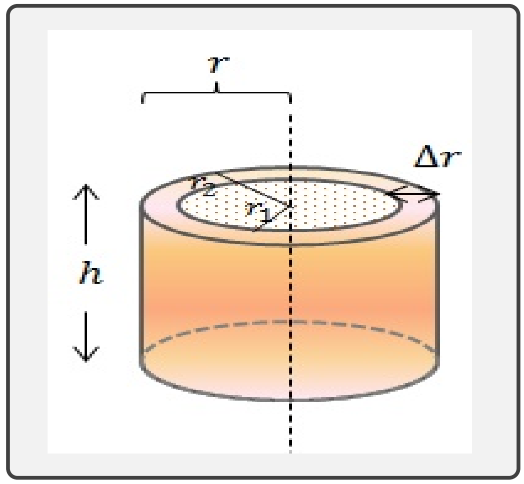


Figure 5.35

The volume of the cylindrical shell is

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi(\frac{r_2 + r_1}{2})h(r_2 - r_1) \\ &= 2\pi r h \Delta r. \end{aligned}$$

$$\underbrace{V}_{\text{volume of the shell}} = \underbrace{V_2}_{\text{volume of the outer cylinder}} - \underbrace{V_1}_{\text{volume of the inner cylinder}}$$

Now, let R be a region R on the interval $[a,b]$ and S be a solid generated by revolving the region about y-axis (Figure 5.36). Let P be a partition of the interval $[a,b]$ and let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be a mark on P where ω_k is the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about y-axis generates a cylindrical shell where
the high = $f(\omega_k)$,
the average radius = ω_k,
the thickness = Δx_k.

Hence, the volume of the cylindrical shell is $V_k = 2\pi\omega_k f(\omega_k)\Delta x_k$. To evaluate the volume of the whole solid, we sum the volumes of all cylindrical shells. This implies

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k.$$

From the Riemann sum

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k = \int_a^b x f(x)\, dx\,.$$

This implies

$$V = 2\pi \int_a^b x f(x)\, dx.$$

Similarly, we can find that if the revolution of the region is about x-axis, the volume of the revolution solid is

$$V = 2\pi \int_c^d y f(y)\, dy.$$

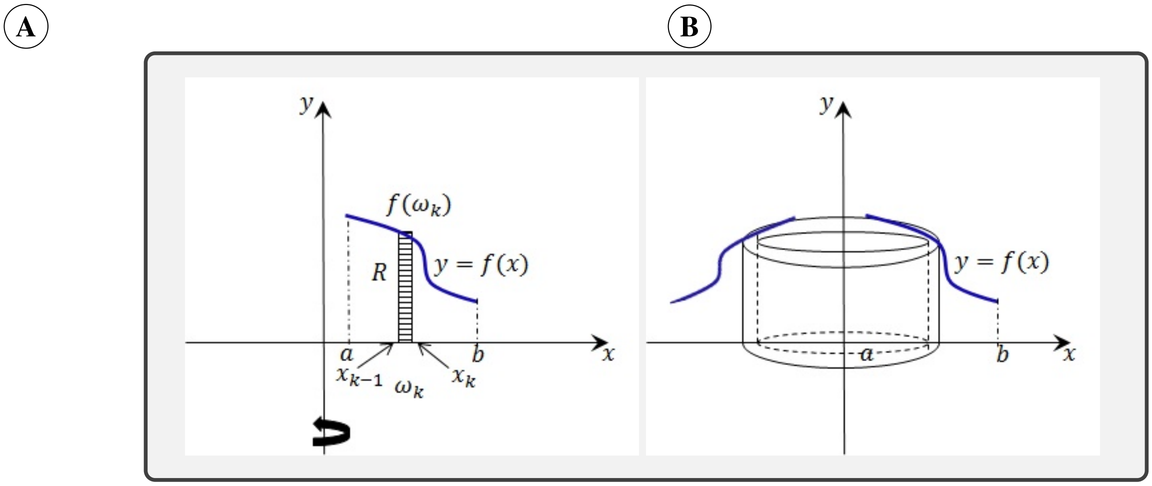


Figure 5.36: The volume of the revolution solid about y-axis by the cylindrical shells method.

Theorem 5.4

1. If R is a region bounded by the graph of f on the interval $[a,b]$, the volume of the revolution solid generated by revolving R about y-axis is
$$V = 2\pi \int_a^b x f(x)\, dx.$$
2. If R is a region bounded by the graph of f on the interval $[a,b]$, the volume of the revolution solid generated by revolving R about x-axis is
$$V = 2\pi \int_c^d y f(y)\, dy.$$

The importance of the cylindrical shells method appears when solving equations for one variable in terms of another (i.e., solving x in terms of y). For example, let S be a solid generated by revolving the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about y-axis. By the washer method, we have to solve the cubic equation for x in terms of y, but this is not simple.

■ **Example 5.23** Sketch the region R bounded by the graph of $y = 2x - x^2$ and x-axis. Then, by the cylindrical shells method, find the volume of the solid generated by revolving R about y-axis.

Solution:
The figure shows the region R and the solid S generated by revolving the region about y-axis.

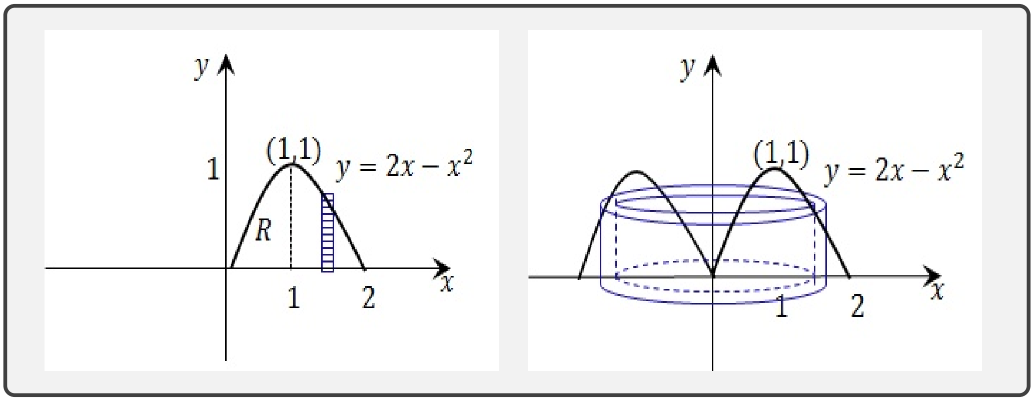


Figure 5.37

Since the revolution is about y-axis, the rectangle is vertical and by revolving it, we obtain a cylindrical shell where
the high: $y = 2x - x^2$,
the average radius: x,
the thickness: dx.

The volume of the cylindrical shell is $dV = 2\pi x(2x - x^2)\ dx = 2\pi(2x^2 - x^3)\ dx$.
Thus, the volume of the solid over the interval $[0,2]$ is

$$
\begin{aligned}
V &= 2\pi \int_0^2 (2x^2 - x^3)\ dx \\
&= 2\pi \Big[\frac{2x^3}{3} - \frac{x^4}{4}\Big]_0^2 \\
&= 2\pi\big(\frac{16}{3} - \frac{16}{4}\big) = \frac{8\pi}{3}.
\end{aligned}
$$

■ **Example 5.24** Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $x = 2$, and y-axis. Then, find the volume of the solid generated by revolving R about x-axis.

Solution:

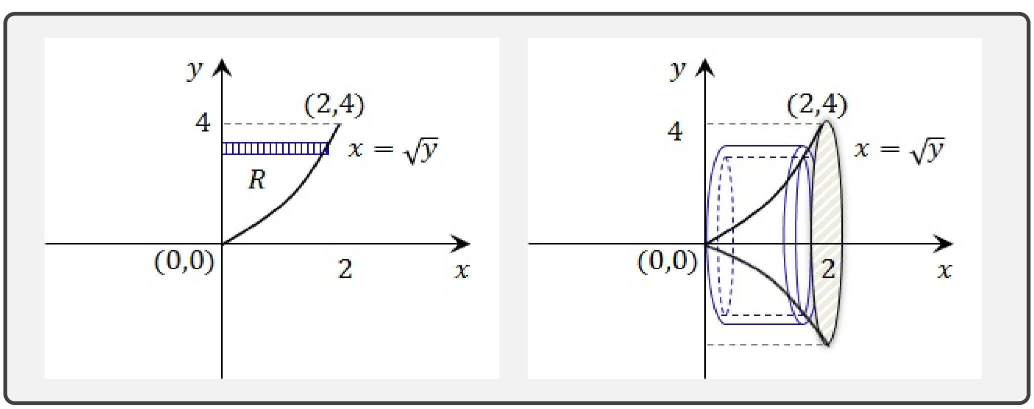


Figure 5.38

Since the revolution is about x-axis, the rectangle is horizontal and by revolving it, we have a cylindrical shell

where
the high: $x = \sqrt{y}$,
the average radius: y,
the thickness: dy.
The volume of the cylindrical shell is $dV = 2\pi\, y\, \sqrt{y}\, dy$.
Thus, the volume of the solid over the interval $[0,4]$ is

$$\begin{aligned} V = 2\pi \int_0^4 y\sqrt{y}\, dy &= 2\pi \int_0^4 y^{\frac{3}{2}}\, dy \\ &= \frac{4\pi}{5}\left[\, y^{\frac{5}{2}}\, \right]_0^4 \\ &= \frac{4\pi}{5}\left[32 - 0\right] = \frac{128\pi}{5}. \end{aligned}$$

Exercises

1 - 36 ■ Sketch the region bounded by the graphs of the given equations, then find its area.

1 $y = x+2$ and x-axis over $[-2,1]$

2 $y = x^3$ and x-axis over $[0,2]$

3 $y = x^2,\ y = 4$

4 $x = y^3$ and y-axis from $y = 0$ to $y = 2$

5 $x = y^2$ and y-axis from $y = -1$ to $y = 1$

6 $y = x^3,\ y = -x,\ y = 8$

7 $y = x+1,\ y = 2x$ and x-axis

8 $x = y^3 - 1$ and y-axis from $y = 1$ to $y = 2$

9 $y = x,\ x = 2-y,\ y = 0$

10 $y = x,\ y = x-1$ over $[0,2]$

11 $x^2 + y = 4,\ y = 0$

12 $x = 2y,\ y+6 = 2x,\ x = 0$

13 $y = x^2,\ y = \sqrt{x}$

14 $x = y^2,\ y = x+1,\ y = 1,\ y = 2$

15 $y - 1 = 3x$ and $y - 2 = x$ from $x = 0$ to $x = 1$

16 $y = \sqrt{x+1}$ and $y = x-1$ over $[1,3]$

17 $y = x^3,\ y = x^2$

18 $y = (x+1)^2$ and x-axis over $[-2,0]$

19 $y = x^2+1$ and $y = x+1$ from $x = 0$ to $x = 1$

20 $x = \sqrt{y}$ and $2x = y$ from $y = 0$ to $y = 4$

21 $y = \sqrt{x}+1,\ y = x+1$

22 $y = x^2,\ y = \sqrt{x}$

23 $y = x^2+1,\ y = 2x$ and $x = 0$

24 $y = \sin\ x$ and $y = \cos\ x$ from $x = 0$ to $x = \frac{\pi}{2}$

25 $y = e^x$ and x-axis over $[0, \ln 4]$

26 $y = 3x,\ y = -x+2$ and x-axis

27 $y = e^{-x}$ from $x = -1$ to $x = 2$

28 $y = \sin\ x$ and $y = \cos\ x$ over $[0, \frac{\pi}{6}]$

29 $y = e^x$ and y-axis from $x = 0$ to $x = \ln 2$

30 $y = x,\ y = -x+2$ and x-axis

31 $y = \cos\ 2x$ and x-axis over $[0, \frac{\pi}{4}]$

32 $y = \sin\ x,\ x = \frac{\pi}{4},\ x = \frac{\pi}{2}$

33 $y = \sec^2\ x,\ y = 0,\ x = \frac{-\pi}{4},\ x = \frac{\pi}{4}$

34 $y = \tan\ x$ and x-axis from $x = 0$ to $x = \frac{\pi}{4}$

35 $y = x^2 - 1,\ x = 1,\ x = 2$

36 $y = \ln x,\ y = 0,\ x = e^2$

37 - 54 ■ Sketch the region R bounded by the graphs of the given equations and find the volume of the solid generated by revolving R about x-axis.

37 $y = 2x$ and x-axis over $[0, 1]$

38 $y = x,\ x + y = 4$ and y-axis

39 $y = x^2,\ y = 4 - x^2$

40 $y = \frac{1}{x}$ and x-axis over $[1, 3]$

41 $y = x^2,\ y = \sqrt{x}$

42 $y = x^2,\ y = 1 - x^2$

43 $y = x^2,\ y = x^3$

44 $y = 1 + x^3,\ x = 1,\ x = 2,\ y = 0$

45 $x = y + 1,\ x = 2y - 3,\ x = 1,\ x = 3$

46 $x = y,\ x = \sqrt{y}$

47 $y = x^3,\ y = x^2,\ x = 1,\ x = 3/2$

48 $y = 4x - x^2$ and x-axis

49 $y = e^x$ over $[0, 2]$

50 $y = x^2,\ y = 9$

51 $y = x^2,\ y = x$

52 $y = \ln x$ over $[1, 4]$

53 $y = \sin\ x$ and $y = \cos\ x$ from $x = 0$ to $x = \frac{\pi}{4}$ (use $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$)

54 $y = \sin\ x$ from $x = 0$ to $x = \frac{\pi}{2}$

55 - 72 ■ Sketch the region R bounded by the graphs of the given equations and find the volume of the solid generated by revolving R about y-axis.

55 $x = 3y$ and x-axis over $[0, 1]$

56 $x = y^2,\ x = 2y$

57 $y = x^3$ and y-axis over $[0, 1]$

58 $y = x^2$, $y = 0$ and $x = 2$

59 $x = y^2,\ y = x - 2$

60 $y = \cos\ x,\ x = 0,\ x = \frac{\pi}{2}$

61 $y = \cos\ x,\ y = \sin\ x,\ x = 0,\ x = \frac{\pi}{4}$

62 $y^2 = 1 - x,\ x = 0$

63 $x = 3y$ and $x = y + 2$ from $y = 0$ to $y = 1$

64 $x = y,\ x = y + 1,\ y = 0,\ y = 2$

65 $y = x^2 - 1,\ y = 0,\ x = 1,\ x = 2$

66 $x = y^3,\ x = y,\ y = 0,\ y = 1$

67 $(x - 2)^2 + y = 1,\ y = 0$

68 $y = 1 - x^2,\ y = 1 - x$

69 $y = x^2 + 1$ and x-axis over $[0, 1]$

70 $y = 6 - 3x$ and y-axis

71 $y = 1 - x^2,\ x = 0,\ x = 1$

72 $y = (x - 1)^2$ and x-axis over $[0, 2]$

73 - 78 ■ Choose the correct answer.

73 The area of the region bounded by the graphs of the functions $y = x^2$ and $y = 2 - x^2$ is equal to
(a) 2 (b) 4 (c) $\frac{3}{8}$ (d) $\frac{8}{3}$

74 The area of the region bounded by the graphs of the functions $x = -y^2$ and $x = -1$ is equal to
(a) $\frac{4}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{6}$ (d) $\frac{8}{3}$

75 The area of the region bounded by the graphs of the functions $y = x$ and $y = -x$ and $y = 1$ is equal to
(a) 1 (b) 0 (c) 2 (d) $\frac{1}{2}$

76 The area of the region bounded by the graphs of the functions $y = 2x$ and $y = x$ and $0 \leq x \leq 1$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) 2 (d) $\frac{1}{3}$

77 The area of the region bounded by the graphs of the functions $y = \cos\ x,\ y = \sin\ x,\ x = 0$ and $x = \frac{\pi}{4}$ is equal to
(a) $\sqrt{2} - 1$ (b) 0 (c) $\sqrt{2} + 1$ (d) $1 - \sqrt{2}$

78 The area of the region bounded by the graphs of the functions $x = y^2$ and $x = 2 - y^2$ is equal to
(a) $\frac{1}{3}$ (b) 8 (c) 1 (d) $\frac{8}{3}$

Chapter 6

PARTIAL DERIVATIVES

6.1 Functions of Several Variables

Definition 6.1

1. A function of two variables is a rule that assigns an ordered pair (x_1, x_2) to a real number w:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x_1, x_2) \longrightarrow w\ .$$

2. A function of three variables is a rule that assigns an ordered triple (x_1, x_2, x_3) to a real number w:

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(x_1, x_2, x_3) \longrightarrow w\ .$$

■ **Example 6.1**

(1) $f(x,y) = x^2 + y^2$ is a function of two variables x and y. The function $f(x,y)$ takes $(x,y) \in \mathbb{R}^2$ to $\omega \in \mathbb{R}$. For example, $f(1,2) = 1^2 + 2^2 = 5$ i.e., the function f takes $(1,2) \in \mathbb{R}^2$ to $5 \in \mathbb{R}$.

(2) $f(x,y,z) = x^2 + y^2 + z$ is a function of three variables x, y and z. The function $f(x,y,z)$ takes $(x,y,z) \in \mathbb{R}^3$ to $\omega \in \mathbb{R}$. For example, the function f takes $(1,2,-1) \in \mathbb{R}^3$ to $4 \in \mathbb{R}$.

Definition 6.2 A function of n variables is a rule that assigns an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ to a real number w:

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x_1, x_2, \ldots, x_n) \longrightarrow w\ .$$

■ **Example 6.2** $f(x,y,z,u,v) = x^2 + y^2 - 7zu + v^2$ is a function of five variables. The function $f(x,y,z,u,v)$ takes $(x,y,z,u,v) \in \mathbb{R}^5$ to $\omega \in \mathbb{R}$. For example, the function f takes $(1,0,1,1,2) \in \mathbb{R}^5$ to $-2 \in \mathbb{R}$.

6.2 Partial Derivatives

For one variable $y = f(x)$, the derivative dy/dx gives the change rate of y with respect to x. A similar thing occurs with functions of more than one variable. For example, for a function of two variables $\omega = f(x,y)$, the independent variables are x and y while ω is the dependent variable i.e. as x and y vary the value of ω traces out a surface.

6.2.1 Partial Derivatives of Functions of Several Variables

Definition 6.3 Let $w = f(x,y)$ be a function of two variables.

1. The partial derivative of $\omega = f(x,y)$ with respect to x denoted $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, f_x or w_x is calculated by applying the rules of differentiation to x holding y constant.
2. The partial derivative of $\omega = f(x,y)$ with respect to y denoted $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, f_y or w_y is calculated by applying the rules of differentiation to y holding x constant.

■ **Example 6.3** If $f(x,y) = x^2y^3 + xy\ln(x+y)$, calculate **(1)** f_x and **(2)** f_y.

Solution:

(1) $f_x = 2xy^3 + y\ln(x+y) + xy\left(\frac{1}{x+y}\right) = 2xy^3 + y\ln(x+y) + \frac{xy}{x+y}$.

(2) $f_y = 3x^2y^2 + x\ln(x+y) + xy\left(\frac{1}{x+y}\right) = 3x^2y^2 + x\ln(x+y) + \frac{xy}{x+y}$.

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}$$

■ **Example 6.4** If $f(x,y) = \frac{2x}{y} + \sin(xy)$, calculate **(1)** f_x and **(2)** f_y.

Solution:

(1) $f_x = \frac{2}{y} + y\cos(xy)$.

(2) $f_y = -\frac{2x}{y^2} + x\cos(xy)$.

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\sin(u)) = \cos(u)\, u'$$

Definition 6.4 Let $w = f(x,y,z)$ be a function of three variables.

1. The partial derivative of $\omega = f(x,y,z)$ with respect to x denoted $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, f_x or w_x is calculated by applying the rules of differentiation to x holding y and z constants.
2. The partial derivative of $\omega = f(x,y,z)$ with respect to y denoted $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, f_y or w_y is calculated by applying the rules of differentiation to y holding x and z constants.
3. The partial derivative of $\omega = f(x,y,z)$ with respect to z denoted $\frac{\partial f}{\partial z}$, $\frac{\partial w}{\partial z}$, f_z or w_z is calculated by applying the rules of differentiation to z holding x and y constants.

■ **Example 6.5** If $f(x,y) = z^2y^3 - y^2(x^3+z)$, calculate **(1)** f_x **(2)** f_y **(3)** f_z.

Solution:

(1) $f_x = 0 - y^2(3x^2) = -3y^2x^2$.

(2) $f_y = 3z^2y^2 - 2y(x^3 + z)$.

(3) $f_z = 2zy^3 - y^2(1) = 2zy^3 - y^2$.

6.2.2 Second Partial Derivatives

In derivative calculus with one variable, we saw that the second derivative is often useful. It tells how the curve is sharp and determines the maximum and minimum points. In a more complicated case, the second derivative will be used for multi-variable functions. With two variables $f(x,y)$, there are four possible second derivatives:

$$\frac{\partial^2 f}{\partial x^2}, \ \frac{\partial^2 f}{\partial x \partial y}, \ \frac{\partial^2 f}{\partial y \partial x}, \ \frac{\partial^2 f}{\partial y^2} \ .$$

Therefore, as number of variables increases, the number of second derivatives increases. Now, let $\omega = f(x,y)$ be a function of x and y, then

- $\frac{\partial^2 f}{\partial x^2}$ means the second derivative with respect to x holding y constant.
- $\frac{\partial^2 f}{\partial y^2}$ means the second derivative with respect to y holding x constant.
- $\frac{\partial^2 f}{\partial x \partial y}$ means differentiate first with respect to y and then with respect to x.

Definition 6.5 Let $w = f(x,y)$ be a function of two variables, then

1. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x} f_x = f_{xx}$.
2. $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y} f_y = f_{yy}$.
3. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x} f_y = f_{yx}$.
4. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y} f_x = f_{xy}$.

The second partial derivatives of functions of three variables are defined in the same manner given in the previous definition.

Theorem 6.6

(1) Let $f(x,y)$ be a function of two variables. If the second partial derivatives f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$.

(2) Let $f(x,y,z)$ be a function of three variables. If the partial derivatives f_{xy}, f_{yx}, f_{xz}, and f_{zx} exist and are continuous, then $f_{xy} = f_{yx}, f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

■ **Example 6.6** If $f(x,y) = x^3 + 2x^2y^2 + y^3$, calculate **(1)** f_{xy} **(2)** f_{yx}.

Solution:

(1) $f_x = 3x^2 + 4xy^2$, then $f_{xy} = 8xy$.

(2) $f_y = 4x^2y + 3y^2$, then $f_{yx} = 8xy$.

From this example, we have $f_{xy} = f_{yx}$.

■ **Example 6.7** If $f(x,y,z) = z^3x + y^2(x+yz)$, calculate

(1) f_x, f_y and f_z at $(1,1,1)$.

(2) f_{xx}, f_{yy} and f_{zz}.

(3) f_{xy}, f_{yz} and f_{zx} at $(0,-1,1)$.

Solution:

(1) $f_x = z^3 + y^2$, $f_y = 2y(x+yz) + y^2z = 2xy + 3y^2z$ and $f_z = 3xz^2 + y^3$. At $(1,1,1)$, we have $f_x = 2$, $f_y = 5$ and $f_z = 4$.

(2) $f_{xx} = 0$, $f_{yy} = 2x + 6yz$ and $f_{zz} = 6xz$.

(3) $f_{xy} = 2y$, $f_{yz} = 3y^2$ and $f_{zx} = 3z^2$. At $(0,-1,1)$, we have $f_{xy} = -2$, $f_{yz} = 3$ and $f_{zx} = 3$.

6.3 Chain Rule for Partial Derivatives

A chain rule for ordinary derivatives is to differentiate a function of a function (composite functions). If $f(x)$ and $g(x)$ are two functions, then the composite function of the two functions is $(f \circ g)(x) = f\big(g(x)\big)$. For example, if $f(x) = \cos\ x$ and $g(x) = x^2$, then $(f \circ g) = f\big(g(x)\big) = \cos\ x^2$. To differentiate such function, we apply the chain rule given in the following definition.

Definition 6.7 If g is a differentiable function at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f\big(g(x)\big)$ is differentiable at x as follows:

$$\frac{dF}{dx} = \frac{df}{dg(x)}\ \frac{dg(x)}{dx}\ .$$

■ **Example 6.8** If $y = \cos\ x^2$, calculate $\frac{dy}{dx}$.

Solution:

Let $f(x) = \cos\ x$ and $g(x) = x^2$, then $(f \circ g)(x) = f\big(g(x)\big) = \cos\ x^2$.
It follows that

$$\frac{df}{dg(x)} = -\sin\ \big(g(x)\big) \quad \text{and} \quad \frac{dg(x)}{dx} = 2x\ .$$

By applying the chain rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{df}{dg(x)}\ \frac{dg(x)}{dx}\\ &= -\sin\ \big(g(x)\big)\ \ (2x)\ \ = -2x\ \ \sin\ x^2\ .\end{aligned}$$

In the following, we expanded the chain rule for composite functions of two or three functions. Thus, we need to use the chain rule more than once.

1. If $w = f(x,y)$, $x = g(t)$, and $y = h(t)$ such that f, g and h are differentiable, then

$$\frac{df}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial x}\ \frac{dx}{dt} + \frac{\partial w}{\partial y}\ \frac{dy}{dt}\ .$$

2. If $w = f(x,y)$, $x = g(t,s)$, and $y = h(t,s)$ such that f, g and h are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t}\,.$$

$$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s}\,.$$

3. If $w = f(x,y,z)$, $x = g(t,s)$, $y = h(t,s)$, and $z = k(t,s)$ such that f, g, h and k are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial t}\,.$$

$$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial s}\,.$$

Note that the previous result can be proven by repeating the chain rule.

■ **Example 6.9** If $f(x,y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1) $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}$
$= y\, s^2 + (x + 2y)(1)$
$= (s+t)s^2 + s^2 t + 2s + 2t$
$= s^3 + 2s^2 t + 2s + 2t$.

(2) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s}$
$= y\,(2st) + (x + 2y)(1)$
$= (s+t)(2st) + s^2 t + 2s + 2t$
$= 3s^2 t + 2st^2 + 2s + 2t$.

■ **Example 6.10** If $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts, y = s + t$, and $z = \frac{s}{t}$, calculate **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1) $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\,\frac{\partial z}{\partial t}$
$= (1 + y\cos(xy) - z\sin(xz))s + x\cos(xy)(1) - x\sin(xz)(\frac{-s}{t^2})$
$= s + \big((s+t)s + ts\big)\cos(ts(s+t)) + \big((\frac{s}{t^2})ts - (\frac{s}{t})s\big)\sin(s^2)$
$= s + (s^2 + 2ts)\cos(ts(s+t))$.

(2) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\,\frac{\partial z}{\partial s}$
$= (1 + y\cos(xy) - z\sin(xz))t + x\cos(xy)(1) - x\sin(xz)(\frac{1}{t})$
$= t + \big((s+t)s + ts\big)\cos(ts(s+t)) - \big((\frac{s}{t})t + (\frac{s}{t})t\big)\sin(s^2)$
$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)$.

6.4 Implicit Differentiation

Sometimes a function can be defined implicitly by an equation of the form $f(x,y) = 0$. We can solve y in terms of x to have a function $y = y(x)$ such that $f(x,y(x)) = 0$ for all x. For example, consider the following equation

$2y+8x=6$. We can rewrite the equation as $y=3-4x$ which is in the form $y=f(x)$. By taking the derivative, we have $\frac{dy}{dx}=-4$.
Alternatively, we know that y is a function of x i.e. $y=y(x)$. By differentiating the equation $2y+8x=6$ implicitly, we have

$$\begin{aligned} 2\frac{dy}{dx}+8\frac{dx}{dx} &= \frac{d6}{dx} \\ 2\frac{dy}{dx}+8 &= 0\,. \end{aligned}$$

Now, rearrange to have $\frac{dy}{dx}$,

$$2\frac{dy}{dx}+8=0 \Rightarrow \frac{dy}{dx}=-\frac{8}{2}=-4$$

and this what we obtained before.
Suppose we cannot find y explicitly as a function of x, only implicitly through the equation $F(x,y)=0$. For example, consider a circle of radius r centered at the origin and represented by the formula $x^2+y^2=r^2$. The graph of the circle is not the graph of a function because it fails the vertical line test. By solving y in terms of x, we have $y=\pm\sqrt{r^2-x^2}$. This formula of the circle cannot be expressed as one function, so how we can find $\frac{dy}{dx}$. The answer is by implicit differentiation.

We know that $F(x,y)=0$ defines y as a function of x, $y=y(x)$. Now, differentiate both sides of $F(x,y(x))=0$ by using the chain rule. This implies

$$\frac{\partial F}{\partial x}(1)+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y}\,.$$

The following definition summarizes the implicit differentiation.

Definition 6.8

1. Suppose that the equation $F(x,y)=0$ defines y implicitly as a function of x, $y=f(x)$ such that f is differentiable. Then,
$$\frac{dy}{dx}=-\frac{F_x}{F_y}\,.$$
2. Suppose that the equation $F(x,y,z)=0$ defines z implicitly as a function of x and y, $z=f(x,y)$ such that f is differentiable. Then,
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial x}=-\frac{F_y}{F_z}\,.$$

■ **Example 6.11** Let $y^2-xy+3x^2=0$, find $\frac{dy}{dx}$.

Solution:
Let $F(x,y)=y^2-xy+3x^2=0$, then $F_x=-y+6x$ and $F_y=2y-x$. Hence,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{-y+6x}{2y-x}=\frac{y-6x}{2y-x}\,.$$

■ **Example 6.12** Let $F(x,y,z) = x^2y + z^2 + \sin(xyz) = 0$, calculate **(1)** $\frac{\partial z}{\partial x}$ **(2)** $\frac{\partial z}{\partial y}$.

Solution:

$F_x = 2xy + yz\cos(xyz)$, $F_y = x^2 + xz\cos(xyz)$ and $F_z = 2z + xy\cos(xyz)$. Hence,

(1) $\dfrac{\partial z}{dx} = -\dfrac{F_x}{F_z} = -\dfrac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}$.

(2) $\dfrac{\partial z}{dy} = -\dfrac{F_y}{F_z} = -\dfrac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}$.

Exercises

1 - 26 ■ Find f_x, f_y, f_{xx} and f_{yy}.

1 $f(x,y) = 2x^4y^3 - xy^2 + 3y + 1$

2 $f(x,y) = 4e^{x^2y^3}$

3 $f(x,y) = 3x + 4y$

4 $f(x,y) = xy^3 + x^2y^2$

5 $f(x,y) = x^3y + e^x$

6 $f(x,y) = xe^{2x+3y}$

7 $f(x,y) = \frac{x-y}{x+y}$

8 $f(x,y) = 2x\sin(x^2y)$

9 $f(x,y) = x^2\sin y + y^2\cos x$

10 $f(x,y) = x^3 + xy^2 + y$

11 $f(x,y) = x^2y^2 + xy^2$

12 $f(x,y) = x^3 + x + 2y^2 + y$

13 $f(x,y) = yx^3 + xy^4 - 3x - 3y$

14 $f(x,y) = \frac{y}{x}\ln x$

15 $f(x,y) = \frac{1}{x^2+y^2}$

16 $f(x,y) = x^2 + xy - y^2$

17 $f(x,y) = \ln(x^2 - y)$

18 $f(x,y) = x\cos y + ye^x$

19 $f(x,y) = y\sin xy$

20 $f(x,y) = 4x^2 - 8xy^4 + 7y^3 - 3$

21 $f(x,y) = \sin xy$

22 $f(x,y) = x^3 + 3x^2y + y^2 + 4x + 2$

23 $f(x,y) = x^2y + 4xy^3$

24 $f(x,y) = x^2\tan y + y^2$

25 $f(x,y) = x^3\ln y + xy^4$

26 $f(x,y) = x^3y - y^3x$

27 - 41 ■ Find f_{xy}, f_{xz}, f_{yz}, and f_{zz} at the given point.

27 $f(x,y,z) = x\cos z + x^2y^3e^z$, $(1,1,0)$

28 $f(x,y,z) = 2y - \sin(xz)$, $(0,1,0)$

29 $f(x,y,z) = \ln(z + xy^2)$, $(1,1,1)$

30 $f(x,y,z) = x^2 + xy + y^2z^3$, $(1,-1,1)$

31 $f(x,y,z) = xy + yz$, $(2,2,1)$

32 $f(x,y,z) = x^3z + x + y^2z$, $(1,-2,1)$

33 $f(x,y,z) = x^2y + xz^3$, $(-3,2,1)$

34 $f(x,y,z) = xyz - e^{xz}$, $(0,1,0)$

35 $f(x,y,z) = x^2 + yz + 2z^3$, $(1,0,0)$

36 $f(x,y,z) = \cos xy + 2z^2 + xy^2z^3$, $(0,0,-1)$

37 $f(x,y,z) = 4x^3y + zx + y$, $(1,1,1)$

38 $f(x,y,z) = 3x^2 + 2y^2 + xy^3 + z^2$, $(1,-1,1)$

39 $f(x,y,z) = x^2 + xy^2 + y^2z^3$, $(1,1,1)$

40 $f(x,y,z) = x^3 + x^2y^2 + 2y^3 + 2x + z^3$, $(2,2,1)$

41 $f(x,y,z) = xyz + y^2 + x^3 + z$, $(1,-1,2)$

42 - 57 ■ Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at the given point.

42 $f(x,y) = x^2 + 2xy$, $x = 2s + t$, $y = s\ln t$, $(1,1)$

43 $f(x,y) = x^2y$, $x = st^2$, $y = st^3$, $(-1,1)$

44 $f(x,y) = xy$, $x = st$, $y = t^2 - s^2$, $(1,1)$

45 $f(x,y) = x^2y$, $x = \sin(st)$, $y = t^2 - s^2$, $(0,1)$

46 $f(x,y) = xy^2$, $x = \sin t$, $y = s(t^2+1)$, $(1,1)$

47 $f(x,y) = \cos(x^2y) + y^3$, $x = s + t^2$, $y = st$, $(1,-1)$

48 $f(x,y) = e^{x^2+y^2}$, $x = 3t$, $y = s + t$, $(-1,1)$

49 $f(x,y) = xy\ln(xy)$, $x = s + t$, $y = 2st$, $(1,1)$

50 $f(x,y) = \frac{1}{xy}$, $x = st$, $y = s^2t$, $(1,1)$

51 $f(x,y,z) = y + \cos(xy) + \sin(xz)$, $x = st$, $y = s + t$, $z = t(s+1)$ $(1,1)$

52 $f(x,y,z) = y + \tan(xz) + \cos y$, $x = s^2t$, $y = \frac{s}{t}$, $z = st$ $(0,1)$

53 $f(x,y,z) = xyz$, $x = t^2 + s$, $y = st$, $z = t^3$ $(1,1)$

54 $f(x,y,z) = (x+y)z$, $x = t + s$, $y = st$, $z = t^2$ $(1,-1)$

55 $f(x,y,z) = (\cos x + y)z$, $x = 2t + s$, $y = st$, $z = 3t$ $(1,1)$

56 $f(x,y,z) = e^{x+y+z}$, $x = 3t + s$, $y = s^2$, $z = t^2$ $(1,1)$

57 $f(x,y,z) = x^2yz$, $x = s\sin t$, $y = t^2 + 1$, $z = s + 1$ $(1,1)$

58 - 75 ■ By using the implicit function differentiation, find $\frac{dy}{dx}$.

58 $x^3 - 3xy^2 + y^3 = 5$

59 $x - \sqrt{xy} + 3y = 4$

60 $4x + 6y = 5$

61 $x^2 + y^2 = 1$

62 $4y + 2x = 8$

63 $y^2 + x^2 = 16$

64 $5y^3 + 4x^5 = 20$

65 $x^2 + y^3 = 2$

66 $y^2 - x^3(2-x) = 0$

67 $y\sin y + x = 1$

68 $x^2 + y^2 - 4 = 0$

69 $\sqrt{xy} - y^2 + 2x = 2$

70 $y^{\frac{1}{2}} - 2x^2 + 5y = 1$

71 $x^2y^3 + x = 2$

72 $\ln x + \ln y = 4$

73 $\sin^{-1} x - y = 0$

74 $\sqrt{1 + x^2y^2} = 2xy$

75 $2x^3 + x^2y + y^3 = 1$

Chapter 7

DIFFERENTIAL EQUATIONS

7.1 Definition of Differential Equations

A differential equation is an equation which contains derivatives of the unknown. There are two classes of the differential equations: ordinary differential equations (O.D.E.) and partial differential equations (P.D.E.). In this book, we only consider the ordinary differential equations.

Definition 7.1 An equation that involves $x, y, y', y'', y''', y^{(4)}, \ldots, y^{(n)}$ for a function $y(x)$ with n^{th} derivative of y with respect to x is an ordinary differential equation of order n.

Example 7.1

(1) $y' = x^2 + 5$ is a differential equation of order 1.

(2) $y'' + x(y')^3 - y = x$ is a differential equation of order 2.

(3) $(y^{(4)})^3 + x^2 y'' = 2x$ is a differential equation of order 4.

Remark 7.2 $y = y(x)$ is called a solution of a differential equation if it satisfies that differential equation.

Example 7.2 Verify that $y = 3x^2 + 4x$ is a solution of the differential equation $y' = 6x + 4$.

Solution:
We want to see whether the function y satisfies the equation. By taking the derivative, we obtain $y' = 6x + 4$ and this is the given differential equation.

Note:

1. $y = y(x) + c$ is the general solution of the differential equation.

2. If an initial condition was added to the differential equation to assign a certain value for c, then $y = y(x)$ is called the particular solution of the differential equation.

Example 7.3 Verify that $y = 4x^3 + 2x^2 + x$ is a solution of the differential equation $y' = 12x^2 + 4x + 1$. Then, with the initial condition $y(0) = 2$, find the particular solution of the equation.

Solution:
First, we want to check whether the function y satisfies the differential equation. By taking the derivative, we have $y' = 12x^2 + 4x + 1$ and this is the given differential equation. Hence, $y = 4x^3 + 2x^2 + x + c$ is the general solution of the given differential equation. Now, since $y(0) = 2$, then

$$y(0) = 40^3 + 20^2 + 0 + c = 2 \Rightarrow c = 2 \ .$$

Therefore, $y = 4x^3 + 2x^2 + x + 2$ is the particular solution of the differential equation $y' = 12x^2 + 4x + 1$.

7.2 Separable Differential Equations

A differential equation is separable if the equation can be written in the form

$$M(x) + N(y)y' = 0$$

where $M(x)$ and $N(y)$ are continuous functions and $y' = \frac{dy}{dx}$.
To solve the separable differential equation, we have the following steps:

1. Write the equation as $M(x)dx + N(y)dy = 0$. This implies $N(y)dy = -M(x)dx$.
2. Integrate the left-hand side with respect to y and the right-hand side with respect to x: $\int N(y)dy = \int -M(x)dx$.
3. Solve for y to write the solution in the form $y = y(x)$.

Example 7.4 Solve the differential equation $y' - y^2e^x = 0$.

Solution: Manipulate the differential equation to become $N(y)dy = -M(x)dx$.

$$\begin{aligned}
y' - y^2e^x = 0 \Rightarrow \frac{dy}{dx} = y^2e^x \Rightarrow & \frac{dy}{y^2} = e^x\, dx \\
\Rightarrow & \int y^{-2}\, dy = \int e^x\, dx \qquad \text{integrate both sides} \\
\Rightarrow & \frac{y^{-1}}{-1} = e^x + c \\
\Rightarrow & y = -\frac{1}{e^x + c} \ . \qquad \text{solve for } y
\end{aligned}$$

Example 7.5 Solve the differential equation $\frac{dy}{dx} = y^2e^x$, with $y(0) = 1$.

Solution: Write the differential equation in the form $N(y)dy = -M(x)dx$.

$$\begin{aligned}
\frac{dy}{dx} = yx \Rightarrow \frac{dy}{y} = x\, dx & \\
& \Rightarrow \int \frac{1}{y}\, dy = \int x\, dx \qquad \text{integrate both sides} \\
& \Rightarrow \ln|y| = x^2 + c \\
& \Rightarrow y = e^{x^2+c} \ . \qquad \text{solve for } y
\end{aligned}$$

With $y(0)=1$, we have $1=e^c$. This implies $c=\ln(1)=0$. Hence, the particular solution is $y=e^{x^2}$.

■ **Example 7.6** Solve the differential equation $dy-(1+y^2)\sin x\,dx=0$.

Solution: Write the differential equation in the form $N(y)dy=-M(x)dx$.

$$\begin{aligned} dy-(1+y^2)\sin x\,dx=0 \Rightarrow & \frac{dy}{1+y^2}=\sin x\,dx \\ & \Rightarrow \int \frac{1}{1+y^2}\,dy=\int \sin x\,dx && \text{integrate both sides} \\ & \Rightarrow \tan^{-1} y=-\cos x+c \\ & \Rightarrow y=\tan(-\cos x+c)\,. && \text{solve for } y \text{ by taking tan function for both sides} \end{aligned}$$

■ **Example 7.7** Solve the differential equation $\frac{dy}{dx}-\frac{1}{2}y=\frac{3}{2}$, with $y(0)=4$.

Solution: Manipulate the differential equation to become $N(y)dy=-M(x)dx$.

$$\begin{aligned} \frac{dy}{dx}-\frac{1}{2}y=\frac{3}{2}\Rightarrow 2\frac{dy}{dx}-y=3\Rightarrow & \frac{dy}{3+y}=2\,dx \\ & \Rightarrow \int \frac{1}{3+y}\,dy=\int 2\,dx && \text{integrate both sides} \\ & \Rightarrow \ln|3+y|=2x+c \\ & \Rightarrow y=e^{2x+c}-3\,. && \text{solve for } y \end{aligned}$$

With $y(0)=4$, we have $4=e^c-3$. Hence, $e^c=7$ and this implies $c=\ln(7)$. Therefore, the particular solution is $y=7e^{2x}-3$.

■ **Example 7.8** Solve the differential equation $e^{-y}\sin x-y'\cos^2 x=0$

Solution: Write the differential equation in the form $N(y)dy=-M(x)dx$.

$$\begin{aligned} e^{-y}\sin x-y'\cos^2 x=0\Rightarrow e^{-y}-\frac{\cos^2 x}{\sin x}\frac{dy}{dx}=0\Rightarrow & e^{y}=\frac{\sin x}{\cos^2 x}\,dx \\ & \Rightarrow \int e^y\,dy=\int \tan x\,\sec x\,dx && \text{integrate both sides} \\ & \Rightarrow e^y=\sec x+c \\ & \Rightarrow y=\ln|\sec x+c|\,. && \text{solve for } y \text{ by taking ln for both sides} \end{aligned}$$

■ **Example 7.9** Solve the differential equation $y'=1-y+x^2-yx^2$.

Solution: Write the differential equation in the form $N(y)dy = -M(x)dx$.

$$
\begin{aligned}
y' = 1 - y + x^2 - yx^2 \Rightarrow y' = (1-y) + x^2(1-y) &\Rightarrow dy = (1-y)(1+x^2)dx \\
&\Rightarrow \int \frac{1}{1-y}\, dy = \int (1+x^2)\, dx \qquad \text{integrate both sides} \\
&\Rightarrow -\ln|1-y| = x + \frac{x^3}{3} + c \\
&\Rightarrow 1 - y = e^{-(x+\frac{x^3}{3}+c)} \qquad \text{solve for } y \\
&\Rightarrow y = 1 - e^{-(x+\frac{x^3}{3}+c)}\ .
\end{aligned}
$$

7.3 First-Order Linear Differential Equations

The first-order linear differential equation has the form

$$y' + P(x)y = Q(x)\ ,$$

where $P(x)$ and $Q(x)$ are continuous functions of x.

To solve the first-order linear equation, first rewrite the equation (if necessary) in the standard form above, then multiply both sides by the integrating factor $\mu(x) = e^{\int P(x)\, dx}$. This implies

$$
\begin{aligned}
y' + P(x)y = Q(x) &\Rightarrow \mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x) \\
&\Rightarrow \mu(x)\frac{dy}{dx} + e^{\int P(x)\, dx}P(x)y = \mu(x)Q(x) \\
&\Rightarrow \mu(x)\frac{dy}{dx} + y\frac{d}{dx}\left(e^{\int P(x)\, dx}\right) = \mu(x)Q(x) \\
&\Rightarrow \mu(x)\frac{dy}{dx} + y\frac{d\mu(x)}{dx} = \mu(x)Q(x) \\
&\Rightarrow \frac{d}{dx}\big(\mu(x)y\big) = \mu(x)Q(x) \\
&\Rightarrow \mu(x)y = \int \mu(x)Q(x)\, dx \\
&\Rightarrow y = \frac{1}{\mu(x)}\int \mu(x)Q(x)\, dx\, .
\end{aligned}
$$

From this, to solve the first-order linear differential equation, we do the following steps:

1. Compute the integrating factor $\mu(x) = e^{\int P(x)\, dx}$.

2. Find the general solution by using the formula:

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)Q(x)\, dx\, .$$

■ **Example 7.10** Solve the differential equation $x\frac{dy}{dx} + y = x^2 + 1$.

Solution: Write the differential equation in the form $y' + P(x)y = Q(x)$.

$$x\frac{dy}{dx} + y = x^2 + 1 \Rightarrow y' + \frac{1}{x}y = \frac{x^2+1}{x}\, .$$

From this, we have $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2+1}{x}$. Hence, the integrating factor is $\mu(x) = e^{\int \frac{1}{x}\, dx} = e^{\ln|x|} = x$. The general solution of the first-order linear differential equation is

$$
\begin{aligned}
y(x) &= \frac{1}{x}\int x\left(\frac{x^2+1}{x}\right)\, dx \\
&= \frac{1}{x}\int (x^2+1)\, dx \\
&= \frac{1}{x}\left(\frac{x^3}{3} + x\right) + c \\
&= \frac{x^2}{3} + 1 + \frac{c}{x}\, .
\end{aligned}
$$

■ **Example 7.11** Solve the differential equation $y' - \frac{2}{x}y = x^2e^x$, with $y(1) = e$.

Solution: The differential equation is in the form $y' + P(x)y = Q(x)$ where $P(x) = -\frac{2}{x}$ and $Q(x) = x^2e^x$. Hence, the integrating factor is $\mu(x) = e^{-2\int \frac{1}{x}\, dx} = e^{-2\ln|x|} = x^{-2}$.

The general solution of the first-order linear differential equation is

$$\begin{aligned} y(x) &= x^2 \int \frac{1}{x^2}(x^2 e^x)\, dx \\ &= x^2 \int e^x\, dx \\ &= x^2 \big(e^x + c\big)\ . \end{aligned}$$

With $y(1) = e$, we have $e = 1 + c$ and this implies $c = e - 1$. The particular solution is $y = x^2(e^x + e - 1)$.

■ **Example 7.12** Solve the differential equation $y' + y = \cos(e^x)$.

Solution: The differential equation takes the form $y' + P(x)y = Q(x)$ where $P(x) = 1$ and $Q(x) = \cos(e^x)$. Hence, the integrating factor is $\mu(x) = e^{\int 1\, dx} = e^x$.
The general solution of the first-order linear differential equation is

$$\begin{aligned} y(x) &= e^{-x} \int e^x \cos(e^x)\, dx \\ &= e^{-x} \big(\sin(e^x) + c\big)\ . \end{aligned}$$

Use integration by substitution with $u = e^x$ and $du = e^x\, dx$

■ **Example 7.13** Solve the differential equation $xy' - 3y = x$.

Solution: The differential equation is in the form $y' + P(x)y = Q(x)$ where $P(x) = -3$ and $Q(x) = x$. Hence, the integrating factor is $\mu(x) = e^{\int -3\, dx} = e^{-3x}$.
The general solution of the first-order linear differential equation is

$$\begin{aligned} y(x) &= e^{3x} \int x\, e^{-3x}\, dx \\ &= e^{3x} \big(-\frac{x^2}{3} e^{-3x} - \frac{1}{9} e^{-3x} + c\big)\ . \end{aligned}$$

Use integration by parts with $u = x$ and $dv = e^{-3x}\, dx$

Exercises

1 - 16 ■ Solve the differential equation.

1 $x^2dy + y^2dx = 0$

2 $\cos^2 x dy - y^2 dx = 0$

3 $x\frac{dy}{dx} - 2y = x^3 \sec x \tan x$

4 $y' = 1 + y$

5 $y' + 3y = e^{-2x}$

6 $\frac{dy}{dx} = (1 + y^2)\sin x$

7 $y' + y = e^{2x}$

8 $xy' - y = x^3 e^x$

9 $xy' - y = x^2 e^{-x}, x > 0$

10 $2y' - y = 4$

11 $y' = \frac{3x^2+2x-1}{2y}$

12 $y' = y\cos x$

13 $xy' + 2y = 4x^3$

14 $y' - y = e^{-x}$

15 $y' + \frac{2x}{1+x^2}y = 1$

16 $\frac{dy}{dx} + y - \frac{1}{e^x+1} = 0$

17 - 40 ■ Solve the differential equation with the given initial condition.

17 $y' + 2y = x,\ y(0) = 1$

18 $\frac{dy}{dx} + 2y = e^{-x},\ y(0) = \frac{3}{4}$

19 $\frac{dy}{dx} - 2xy = x,\ y(0) = 0$

20 $(1 + x^2)y' + 4xy = \frac{x}{(1+x^2)^2},\ y(0) = 1$

21 $xy' + y = \sin x,\ y(\frac{\pi}{3}) = 2$

22 $y' - \frac{1}{3}y = e^{-x},\ y(0) = a$

23 $xy' + 2y = 4x^2,\ y(1) = 2$

24 $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y},\ y(0) = -1$

25 $y' = 3x^2 + 3x^2 y,\ y(0) = 0$

26 $xy' + y = x^3,\ y(-1) = 3$

27 $y' - \frac{1}{x}y = x,\ y(1) = 2$

28 $y' - y^2 = 0,\ y(0) = 1$

29 $y' = \frac{1+3x^2}{3y^2},\ y(0) = 1$

30 $\cos x y' + \sin x y = 2\cos^3 x \sin x,\ y(\frac{\pi}{4}) = 3\sqrt{2}, 0 \le x < \frac{\pi}{2}$

31 $\frac{dy}{dx} = 5y^2 x,\ y(1) = \frac{1}{25}$

32 $y' = \frac{3x^2+4x-4}{2y},\ y(1) = 3$

33 $y' = \frac{xy^2}{\sqrt{1+x^2}},\ y(0) = -1$

34 $y' = e^{-y}(2x - 4),\ y(5) = 0$

35 $\frac{dy}{dx} = \frac{y^2}{x},\ y(1) = 2$

36 $y' = e^{x - \sin y}\sec(y),\ y(0) = 0$

37 $xy' + y = x^3,\ y(1) = -3$

38 $y' = \frac{1+2x}{\tan y},\ y(0) = 0$

39 $y' = 2x^2 + 2x^2 y^2,\ y(0) = 0$

40 $xy' + 2y = x^2 - x + 1,\ y(1) = \frac{1}{2}$

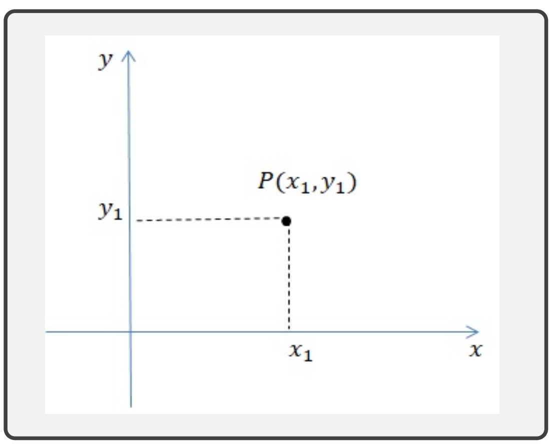
Chapter 8

Polar Coordinates System

8.1 Polar Coordinates

So far we have used functions of the form $y = f(x)$ or $x = f(y)$ to describe curves by determining points (x, y) in a Cartesian (rectangular) coordinate system. In this chapter, we are going to study a new coordinate system called a polar coordinate system. Figure 8.1 shows the Cartesian and polar coordinates system.

Definition 8.1 The polar coordinate system is a two-dimensional system consisted of a pole and polar axis (half line). Each point P in a polar plane is determined by a distance r from a fixed point O called the pole (or origin) and an angle θ from a fixed direction.



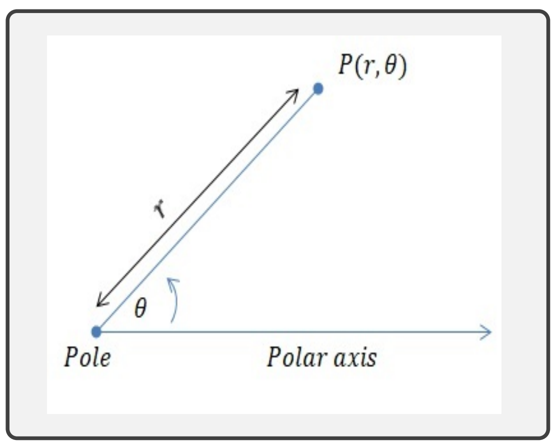


Figure 8.1: The Cartesian coordinate system (on the left) and the polar coordinate system (on the right).

Note:

1. The point P in the polar coordinate system is represented by the ordered pair (r, θ) where r and θ are called polar coordinates.

2. The angle θ takes positive numbers if it is measured counterclockwise from the polar axis, but if the angle is measured clockwise, it takes negative numbers.

3. In the polar coordinate system, if $r > 0$, the point $P(r,\theta)$ will be in the same quadrant as θ. However, if $r < 0$, the point will be in the quadrant on the opposite side of the pole. That is, the points $P(r,\theta)$ and $P(-r,\theta)$ lie in the same line through the pole O, but on opposite sides.

4. In the Cartesian coordinate system, every point has only one representation while in the polar coordinate system, each point has many representations. The following formula gives all representations of the point $P(r,\theta)$ in the polar coordinate system

$$\boxed{P(r,\theta+2n\pi) = P(r,\theta) = P(-r,\theta+(2n+1)\pi), \quad n \in \mathbb{Z}\,.}$$

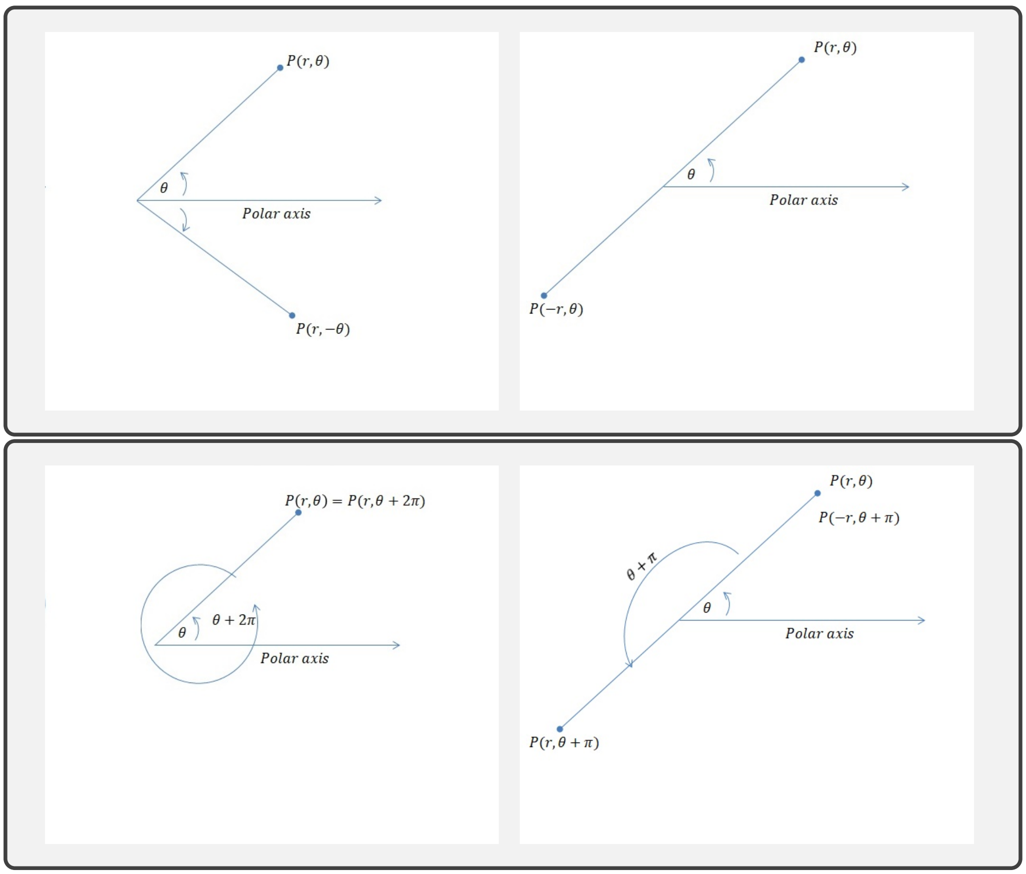


Figure 8.2

■ **Example 8.1** Plot the points whose polar coordinates are given.

(1) $(2, 5\pi/4)$

(2) $(2, -3\pi/4)$

(3) $(2, 13\pi/4)$

(4) $(-2, \pi/4)$

Solution:

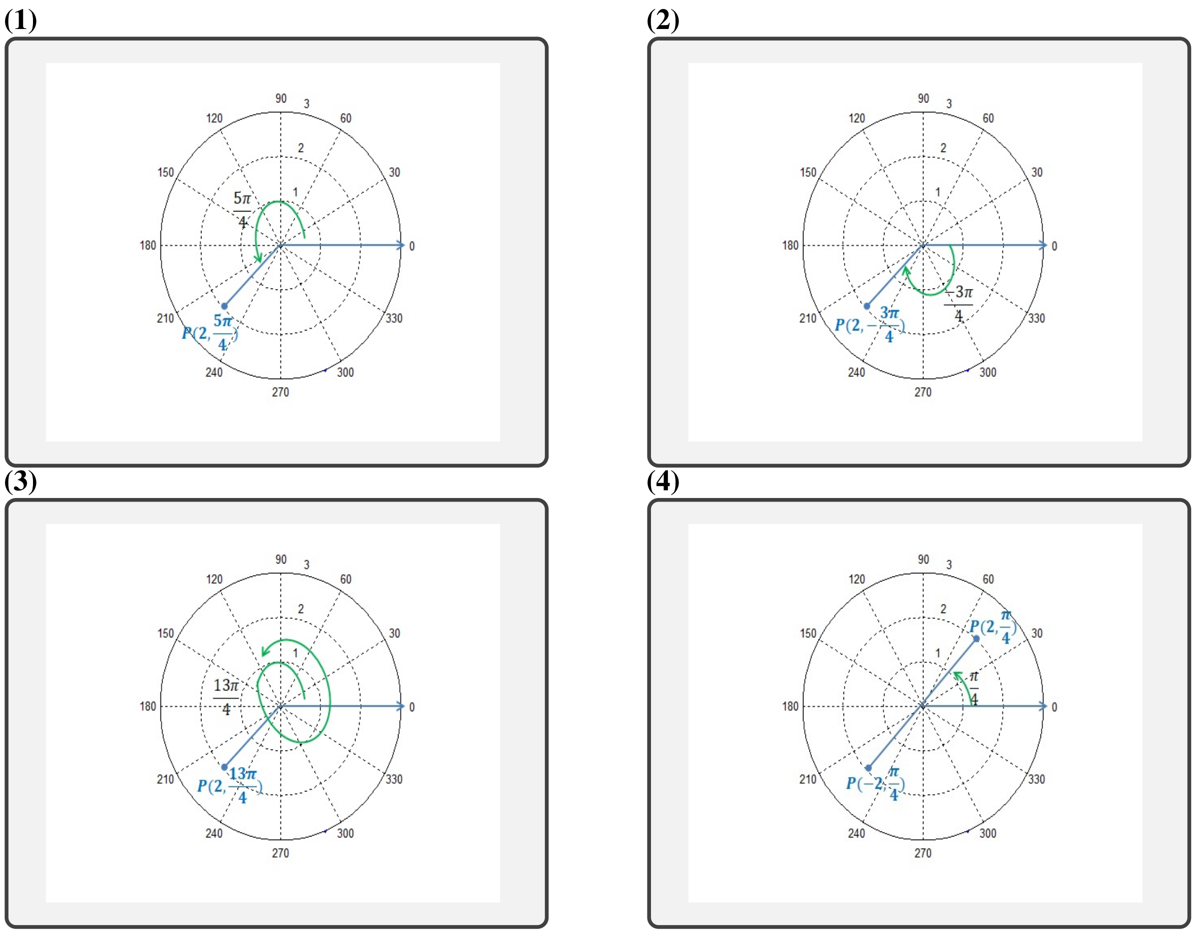


Figure 8.3

8.2 The Relationship between Cartesian and Polar Coordinates

Let (x,y) be a Cartesian coordinate and (r,θ) be a polar coordinate of the same point P. Let the pole be at the origin of the Cartesian coordinates system, and let the polar axis lies on the positive x-axis and the line $\theta = \frac{\pi}{2}$ lies on the positive y-axis as shown in Figure 8.4.

From the right triangle, we have

$$\cos\ \theta = \frac{x}{r} \Rightarrow x = r\cos\ \theta \ \text{ and}$$

$$\sin\ \theta = \frac{y}{r} \Rightarrow y = r\sin\ \theta.$$

Hence,

$$\begin{aligned} x^2+y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2 \qquad \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

This implies, $x^2+y^2 = r^2$ and $\tan\ \theta = \frac{y}{x}$ for $x \neq 0$.

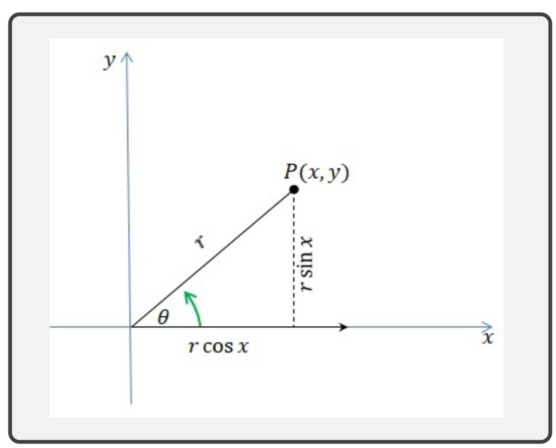


Figure 8.4: The relationship between the Cartesian and polar coordinates.

The previous relationships can be summarized as follows:

$$x = r\cos\theta,\ \ y = r\sin\theta$$
$$\tan\theta = \frac{y}{x} \text{ for } x \neq 0$$
$$x^2 + y^2 = r^2$$

Example 8.2 Convert from polar coordinates to Cartesian coordinates.

(1) $(1, \pi/4)$

(2) $(2, \pi)$

(3) $(2, -2\pi/3)$

(4) $(4, 3\pi/4)$

Solution:

(1) From the polar point $(1, \pi/4)$, we have $r = 1$ and $\theta = \frac{\pi}{4}$. Hence,

$$x = r\cos\theta = (1)\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$y = r\sin\theta = (1)\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Therefore, in the Cartesian coordinates, the point $(1, \pi/4)$ is represented by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(2) From the polar point $(2, \pi)$, we have $r = 2$ and $\theta = \pi$. Hence,

$$x = r\cos\theta = 2\cos\pi = -2,$$

$$y = r\sin\theta = 2\sin\pi = 0.$$

Hence, the polar point $(2, \pi)$ is $(-2, 0)$ in the Cartesian coordinates.

(3) From the polar point $(2, -2\pi/3)$, we have $r = 2$ and $\theta = \frac{-2\pi}{3}$. Hence,

$$x = r\cos\theta = 2\cos\frac{-2\pi}{3} = -1,$$

$$y = r\sin\theta = 2\sin\frac{-2\pi}{3} = -\sqrt{3}.$$

Therefore, the Cartesian coordinate $(-1, -\sqrt{3})$ is the point corresponding to the polar point $(2, -2\pi/3)$.

(4) From the polar point $(4, 3\pi/4)$, we have $r = 4$ and $\theta = \frac{3\pi}{4}$. Hence,

$$x = r\cos\theta = 4\cos\frac{3\pi}{4} = -2\sqrt{2},$$

$$y = r\sin\theta = 4\sin\frac{3\pi}{4} = 2\sqrt{2}.$$

In the Cartesian coordinates, the point $(4, 3\pi/4)$ is represented by $(-2\sqrt{2}, 2\sqrt{2})$.

■ **Example 8.3** For the given Cartesian point, find one representation in the polar coordinates.

(1) $(1,-1)$

(2) $(2\sqrt{3},-2)$

(3) $(-2,2)$

(4) $(1,1)$

Solution:

(1) From the given Cartesian point, we have $x=1$ and $y=-1$. Hence,

$$x^2+y^2=r^2 \Rightarrow r=\sqrt{2},$$

$$\tan\theta=\frac{y}{x}=-1 \Rightarrow \theta=-\frac{\pi}{4}\,.$$

In the polar coordinates, the Cartesian point $(1,-1)$ can be represented by $(\sqrt{2},-\frac{\pi}{4})$.
Remember, there are infinitely polar representations of the point (x,y) (see Note 4 on page 125).

(2) From the Cartesian point, we have $x=2\sqrt{3}$ and $y=-2$. Hence,

$$x^2+y^2=r^2 \Rightarrow r=4,$$

$$\tan\theta=\frac{y}{x}=\frac{-1}{\sqrt{3}} \Rightarrow \theta=\frac{5\pi}{6}\,.$$

Therefore, the polar point $(4,\frac{5\pi}{6})$ is one representation of the Cartesian point $(2\sqrt{3},-2)$.

(3) From the Cartesian point, we have $x=-2$ and $y=2$. Hence,

$$x^2+y^2=r^2 \Rightarrow r=2\sqrt{2},$$

$$\tan\theta=\frac{y}{x}=-1 \Rightarrow \theta=\frac{3\pi}{4}\,.$$

The polar point $(2\sqrt{2},\frac{3\pi}{4})$ is one representation of the Cartesian point $(-2,2)$.

(4) From the Cartesian point, we have $x=1$ and $y=1$. Hence,

$$x^2+y^2=r^2 \Rightarrow r=\sqrt{2},$$

$$\tan\theta=\frac{y}{x}=1 \Rightarrow \theta=\frac{\pi}{4}\,.$$

The Cartesian point $(1,1)$ can be represented by $(\sqrt{2},\frac{\pi}{4})$ in the polar coordinates.

In the Cartesian coordinates, the function $y=f(x)$ is a dependent relation which can be represented by a curve in the Cartesian plane. In polar coordinates, the function $r=f(\theta)$ is a dependent relation between coordinates r and θ which also can be represented by a curve called a polar curve. For example, $r=2\cos\,\theta$ is a polar equation represents the dependent relation between coordinates r and θ:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{3}$	$2/\sqrt{2}$	1	0

Table 8.1

■ **Example 8.4** Find a polar equation that has the same graph as the equation in x and y.

(1) $x = -5$

(2) $y = 3$

(3) $x^2 + y^2 = 2$

(4) $y^2 = 9x$

Solution:

(1) $x = 7 \Rightarrow r\cos\ \theta = -5 \Rightarrow r = -5\sec\theta.$

(2) $y = -3 \Rightarrow r\sin\ \theta = 3 \Rightarrow r = 3\csc\theta.$

(3)
$$\begin{aligned} x^2 + y^2 = 2 &\Rightarrow r^2\cos^2\ \theta + r^2\sin^2\ \theta = 2 \\ &\Rightarrow r^2(\cos^2\ \theta + \sin^2\ \theta) = 2 \\ &\Rightarrow r^2 = 2\ . \end{aligned}$$

(4)
$$\begin{aligned} y^2 = 9x &\Rightarrow r^2\sin^2\ \theta = 9r\cos\ \theta \\ &\Rightarrow r\sin^2\ \theta = 9\cos\ \theta \\ &\Rightarrow r = 9\cot\ \theta\csc\ \theta. \end{aligned}$$

■ **Example 8.5** Find an equation in x and y that has the same graph as the polar equation.

(1) $r = 4$

(2) $r = 3\sin\ \theta$

(3) $r = 6\cos\ \theta$

(4) $r = \sec\ \theta$

Solution:

(1) $r = 4 \Rightarrow \sqrt{x^2 + y^2} = 4 \Rightarrow x^2 + y^2 = 4.$

(2) $r = 3\sin\theta \Rightarrow r = 3\frac{y}{r} \Rightarrow r^2 = 3y \Rightarrow x^2 + y^2 = 3y \Rightarrow x^2 + y^2 - 3y = 0.$

(3) $r = 6\cos\ \theta \Rightarrow r = 6\ \frac{x}{r} \Rightarrow r^2 = 6x \Rightarrow x^2 + y^2 - 6x = 0.$

(4) $r = \sec\ \theta \Rightarrow r = \frac{1}{\cos\ \theta} \Rightarrow r\cos\ \theta = 1 \Rightarrow x = 1.$

8.3 Polar Curves

Before starting sketching the polar curves, we study symmetry in the polar coordinates system.

■ Symmetry in Polar Coordinates

Theorem 8.2

1. **Symmetry about the polar axis.**
   The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if replacing (r,θ) with $(r,-\theta)$ or with $(-r,\pi-\theta)$ does not change the polar equation.

2. **Symmetry about the vertical line $\theta = \frac{\pi}{2}$.**
   The graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \frac{\pi}{2}$ if replacing (r,θ) with $(r,\pi-\theta)$ or with $(-r,-\theta)$ does not change the polar equation.

3. **Symmetry about the pole $\theta = 0$.**
   The graph of $r = f(\theta)$ is symmetric with respect to the pole if replacing (r,θ) with $(-r,\theta)$ or with $(r,\theta+\pi)$ does not change the polar equation.

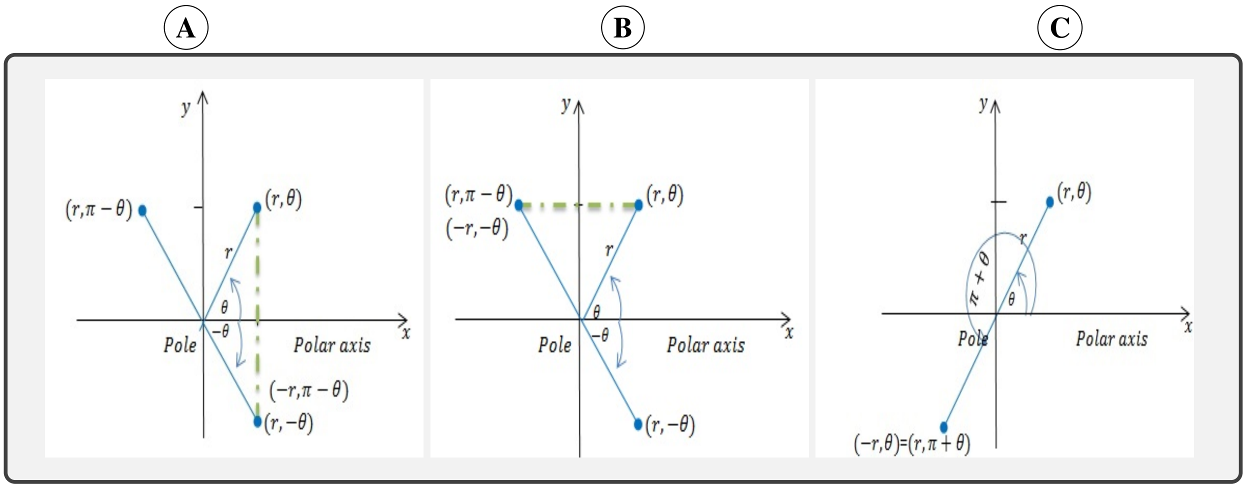


Figure 8.5: Symmetry of the curves in the polar coordinates system. (A) symmetry about the polar axis, (B) symmetry about the vertical line $\theta = \frac{\pi}{2}$, and (C) symmetry about the pole $\theta = 0$.

Example 8.6

(1) The graph of $r = 4\cos\,\theta$ is symmetric about the polar axis. By replacing (r,θ) with $(r,-\theta)$, we have

$$4\cos\,(-\theta) = 4\cos\,\theta = r, \;\text{ thus }\; (r,\theta) = (r,-\theta)\;.$$

Also, by replacing (r,θ) with $(-r,\pi-\theta)$, we have

$$-4\cos(\pi-\theta) = 4\cos\,\theta = r, \;\text{ thus }\; (r,\theta) = (-r,\pi-\theta)\;.$$

(2) The graph of $r = 2\sin\,\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$. By replacing (r,θ) with $(r,\pi-\theta)$, we obtain

$$2\sin\,(\pi-\theta) = 2\sin\,\theta = r, \;\text{ so }\; (r,\theta) = (r,\pi-\theta)\;.$$

Also, by replacing (r,θ) with $(-r,-\theta)$, we have

$$-2\sin\,(-\theta) = 2\sin\,\theta = r, \;\text{ so }\; (r,\theta) = (-r,-\theta)\;.$$

(3) The graph of $r^2 = a^2\sin\,2\theta$ is symmetric about the pole. If we replace (r,θ) with $(-r,\theta)$, we have

$$(-r)^2 = a^2\sin\,2\theta \;\text{ and this implies }\; r^2 = a^2\sin\,2\theta, \;\text{ thus }\; (r,\theta) = (-r,\theta)\;.$$

Also, if we replace (r,θ) with $(r,\theta+\pi)$, we have

$$r^2 = a^2\sin\,\big(2(\pi+\theta)\big) = a^2\sin\,(2\pi+2\theta) = a^2\sin\,2\theta, \;\text{ thus }\; (r,\theta) = (r,\theta+\pi)\;.$$

.

Some Special Polar Curves

Lines in polar coordinates system

1. The polar equation of a straight line $ax+by=c$ is $r = \frac{c}{a\cos\,\theta+b\sin\,\theta}$.
Since $x = r\cos\,\theta$ and $y = r\sin\,\theta$, then

$$ax+by=c \Rightarrow r(a\cos\,\theta+b\sin\,\theta) = c \Rightarrow r = \frac{c}{(a\cos\,\theta+b\sin\,\theta)}$$

2. The polar equation of a vertical line $x = k$ is $r = k\sec\,\theta$.
Let $x = k$, then $r\cos\,\theta = k$. This implies $r = \frac{k}{\cos\,\theta} = k\sec\,\theta$.

3. The polar equation of a horizontal line $y = k$ is $r = k\csc\,\theta$.
Let $y = k$, then $r\sin\,\theta = k$. This implies $r = \frac{k}{\sin\,\theta} = k\csc\,\theta$.

4. The polar equation of a line that passes the origin point and makes an angle θ_0 with the positive x-axis is $\theta = \theta_0$.

■ **Example 8.7** Sketch the graph of $\theta = \frac{\pi}{4}$.

Solution:

We are looking for a graph of the set of polar points:

$$\{(r,\theta)\,|, r \in \mathbb{R}\}$$

θ	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
r	-6	-3	-1	1	3	6

Table 8.2

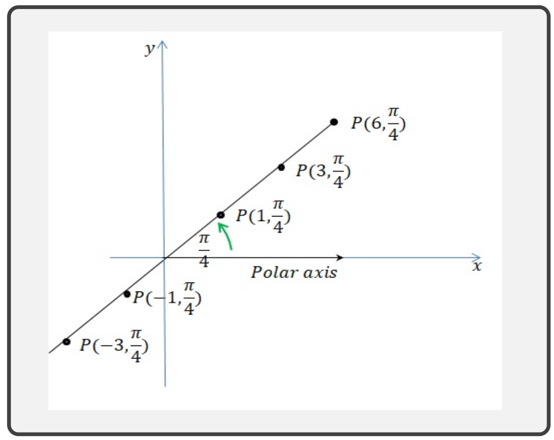


Figure 8.6

■ Circles in polar coordinates system

1. The circle equation with center at the pole O and radius $|a|$ is $r = a$.
2. The circle equation with center at $(a,0)$ and radius $|a|$ is $r = 2a\cos\,\theta$.
3. The circle equation with center at $(0,a)$ and radius $|a|$ is $r = 2a\sin\,\theta$.

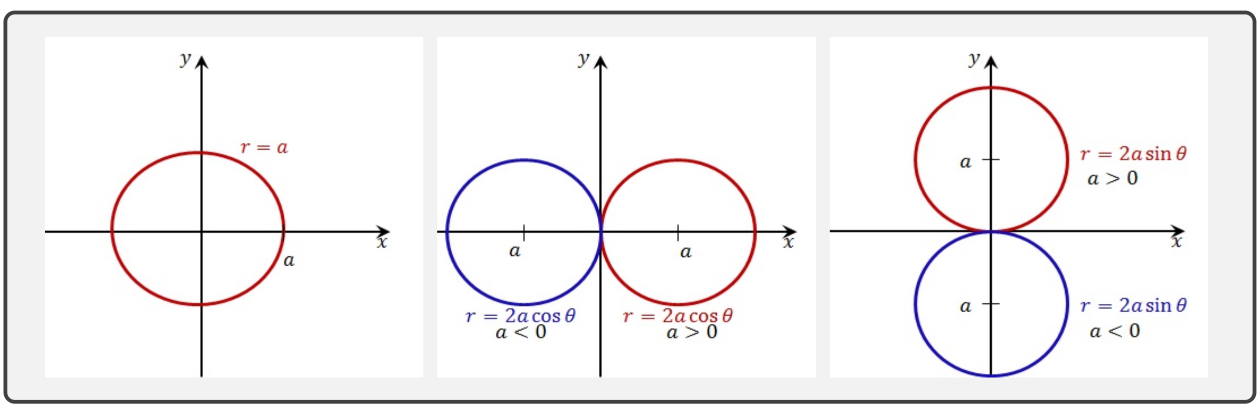


Figure 8.7: Circles in polar coordinates.

■ **Example 8.8** Sketch the graph of $r = 4\sin\,\theta$.

Solution:
Note that the graph of $r = 4\sin\,\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since $4\sin\,(\pi - \theta) = 4\sin\,\theta$.

Therefore, we restrict our attention to the interval $[0, \pi/2]$ and by the symmetry, we complete the graph. The following table displays the polar coordinates of some points on the curve:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	2	$4/\sqrt{2}$	$2\sqrt{3}$	4

Table 8.3

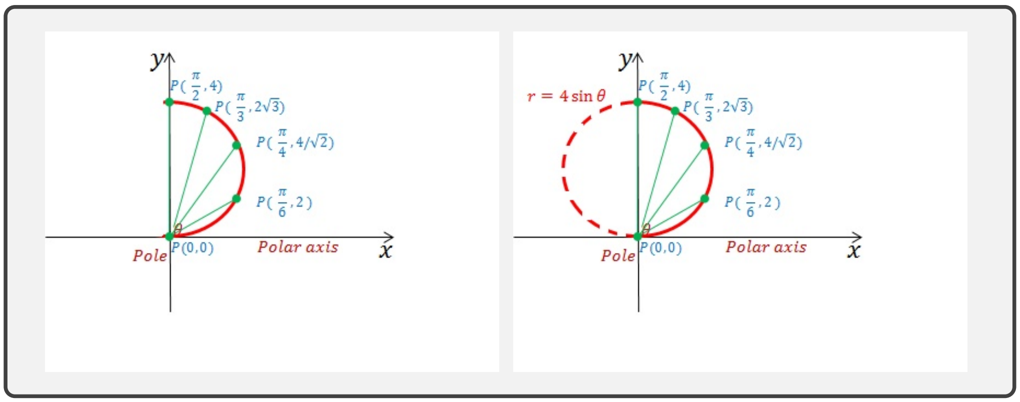
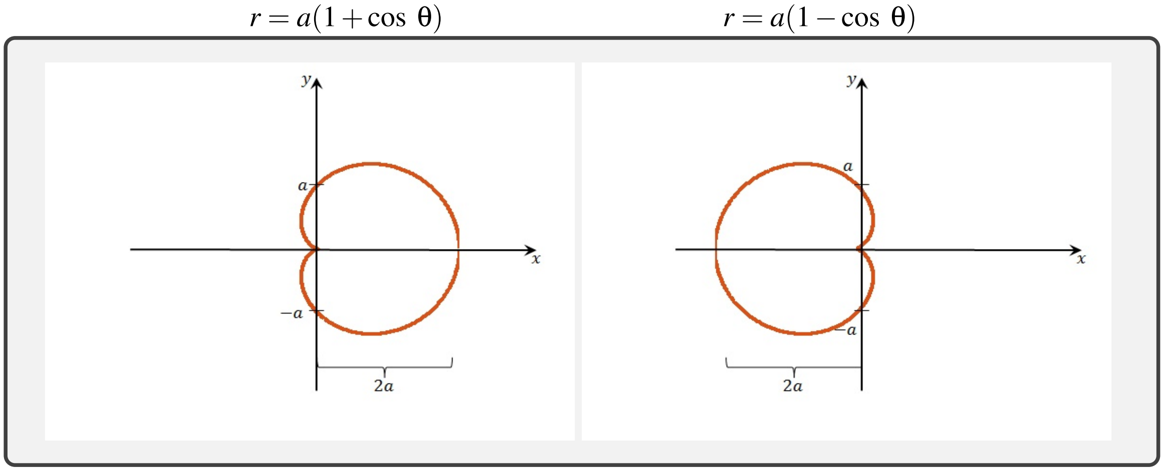


Figure 8.8: The graph of the polar curve $r = 4\sin\ \theta$.

■ Cardioid curves

1. $r = a(1 \pm \cos\ \theta)$ **2.** $r = a(1 \pm \sin\ \theta)$

$r = a(1 + \cos\ \theta)$ $r = a(1 - \cos\ \theta)$



$r = a(1 + \sin\ \theta)$ $r = a(1 - \sin\ \theta)$

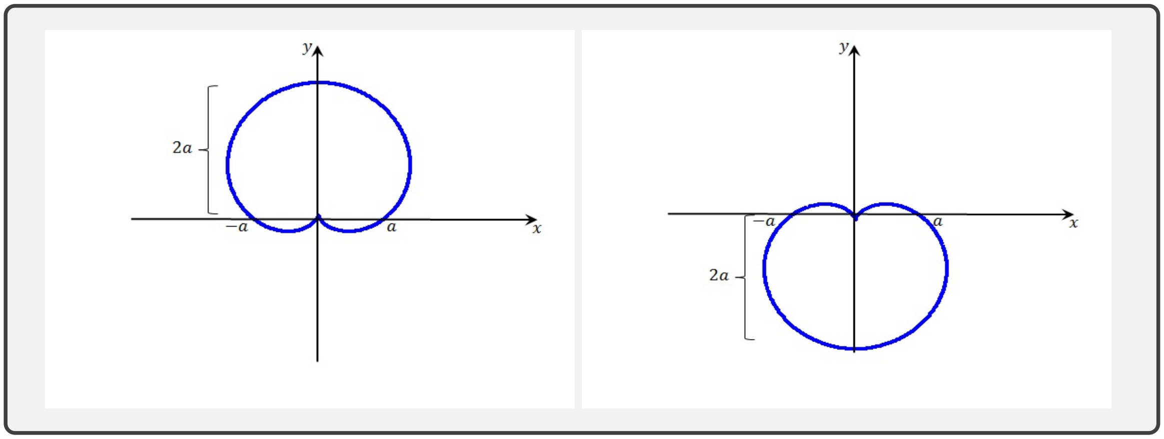


Figure 8.9: Cardioid curves.

■ **Example 8.9** Sketch the graph of $r = a(1-\cos\,\theta)$ where $a > 0$.

Solution:
The curve is symmetric about the polar axis since $\cos\,(-\theta) = \cos\,\theta$. Therefore, we restrict our attention to the interval $[0, \pi]$ and by the symmetry, we complete the graph. The following table displays some solutions of the equation $r = a(1-\cos\,\theta)$:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a/2$	a	$3a/2$	$2a$

Table 8.4

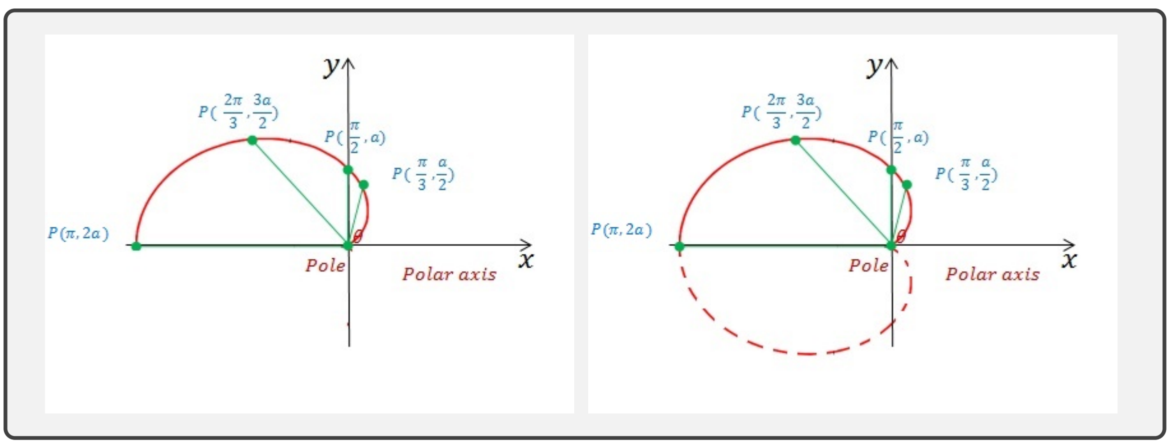
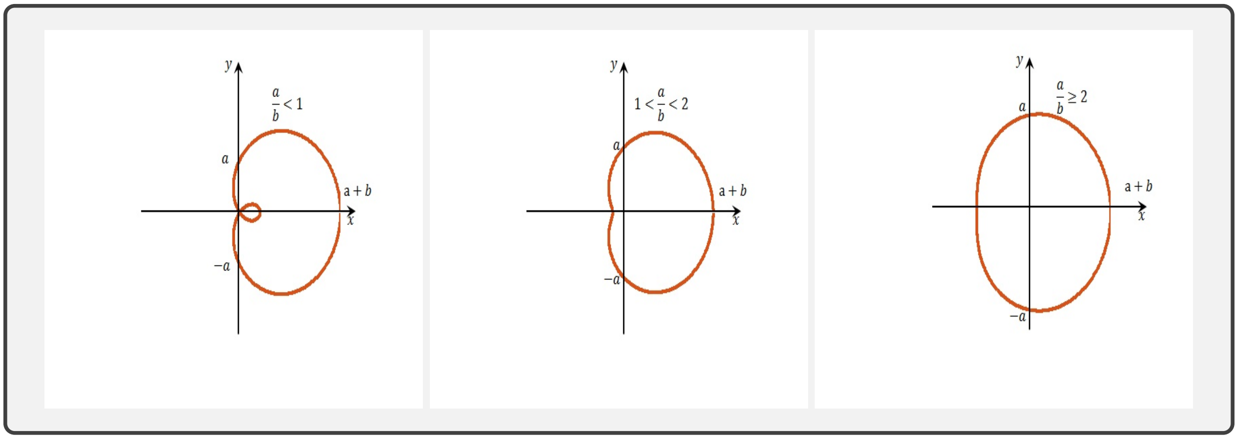


Figure 8.10: The graph of $r = a(1-\cos\,\theta)$ where $a > 0$.

■ Limaçons curves

1. $r = a \pm b\cos\,\theta$ **2.** $r = a \pm b\sin\,\theta$

$$r = a + b\cos\,\theta$$



$$r = a - b\cos\ \theta$$

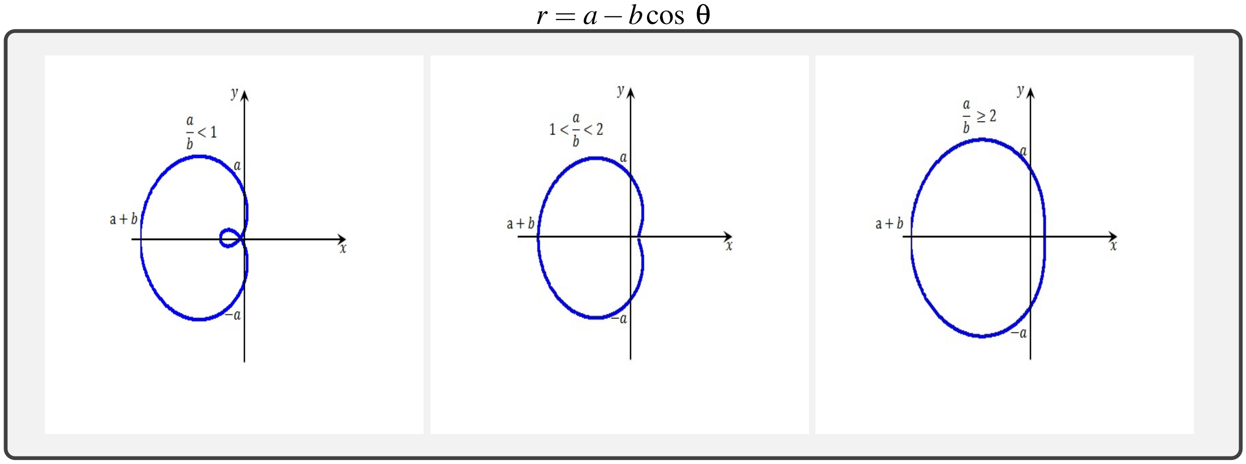
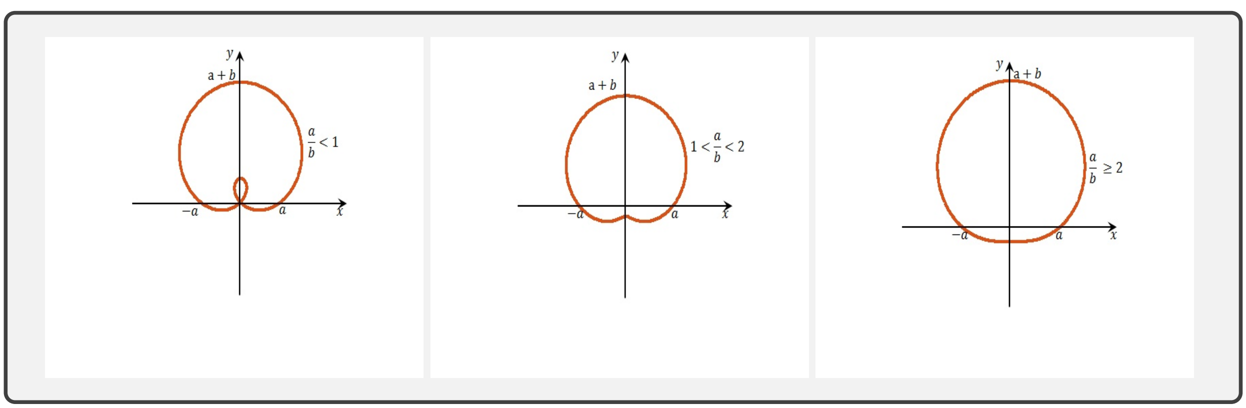


Figure 8.11: Limaçons curves $r = a \pm b\cos\ \theta$.

$$r = a + b\sin\ \theta$$



$$r = a - b\sin\ \theta$$

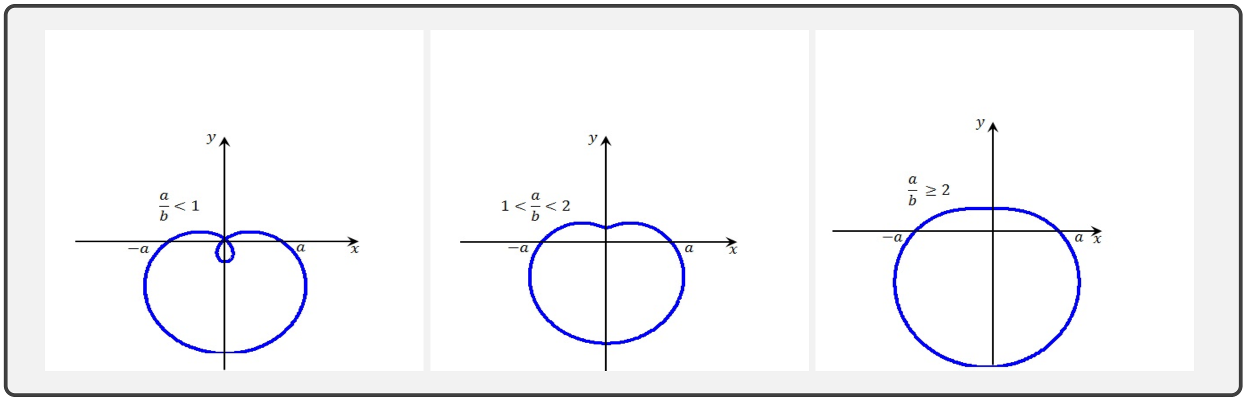
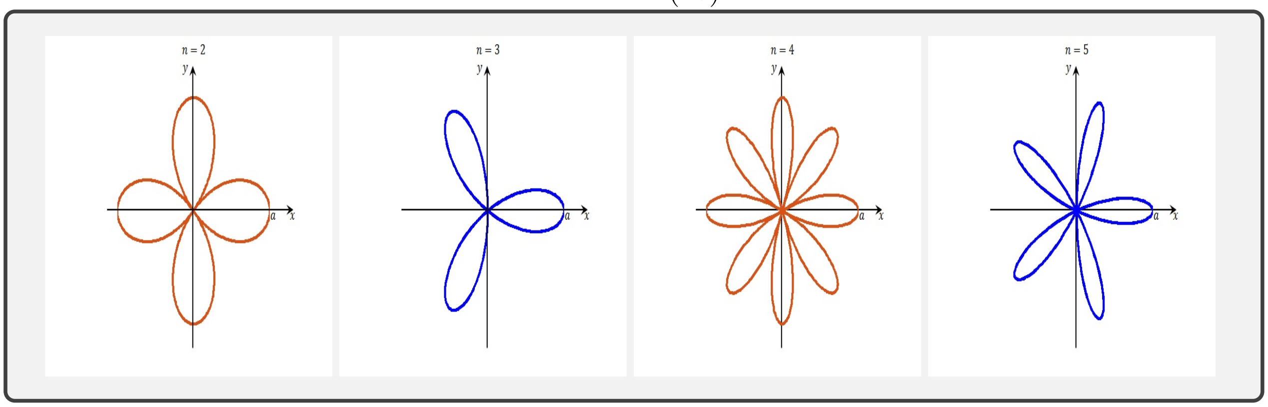


Figure 8.12: Limaçons curves $r = a \pm b \sin\ \theta$.

■ Roses

1. $r = a\ \cos\ (n\theta)$ **2.** $r = a\ \sin\ (n\theta)$ where $n \in \mathbb{N}$.

$r = a\ \cos\ (n\theta)$



$r = a\ \sin\ (n\theta)$

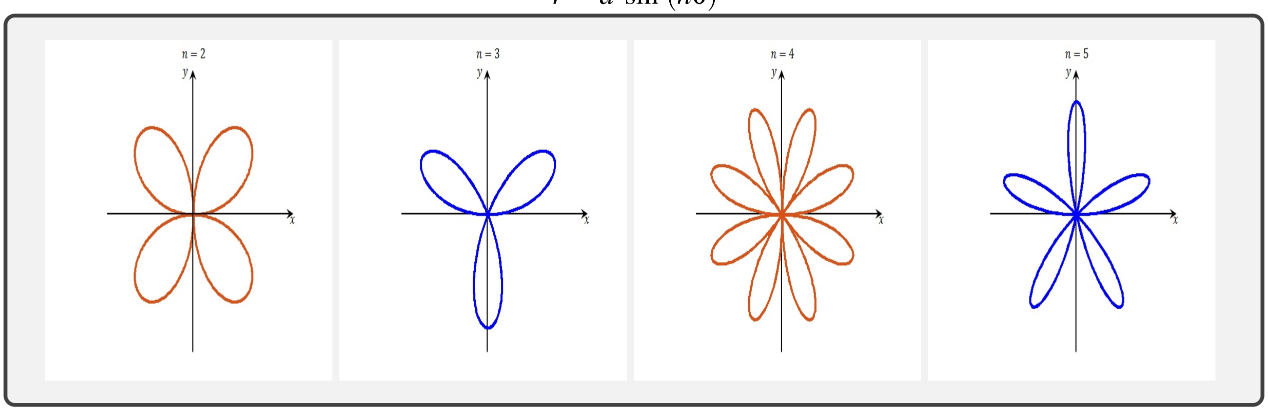


Figure 8.13: Roses in polar coordinates.

Note that if n is odd, there are n petals; however, if n is even, there are $2n$ petals.

■ Spiral of Archimedes

$r = a\,\theta$

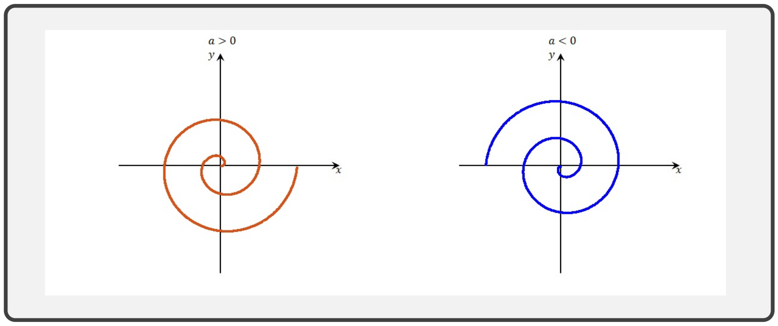


Figure 8.14: Spiral of Archimedes.

8.4 Area in Polar Coordinates

In chapter 5, we have seen how to compute area of the region under a function $f(x)$ over the interval $[a,b]$. Now, consider what happens if we use a polar function $r = f(\theta)$ for θ in the interval $[\alpha, \beta]$.

Let $r = f(\theta)$ be a continuous function on the interval $[\alpha, \beta]$ such that $0 \leq \alpha \leq \beta \leq 2\pi$. Let $f(\theta) \geq 0$ over that interval and R be a polar region bounded by the polar equations $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ as shown in Figure 8.15.

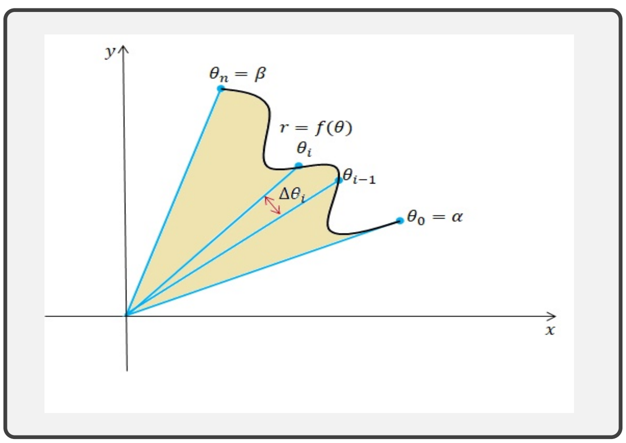


Figure 8.15: Areas in polar coordinates.

To find the area of R, let $P = \{\theta_1, \theta_2, ..., \theta_n\}$ be a regular partition of the interval $[\alpha, \beta]$. Consider the interval $[\theta_{i-1}, \theta_i]$ where $\Delta\theta_i = \theta_i - \theta_{i-1}$. By choosing $\omega_i \in [\theta_{i-1}, \theta_i]$, we have a circular sector where its angle and radius are $\Delta\theta_i$ and $f(\omega_i)$, respectively. The area between θ_{i-1} and θ_i can be approximated by the area of a circular sector.

Let $f(u_i)$ and $f(v_i)$ be the maximum and minimum values of f on $[\theta_{i-1}, \theta_i]$. From Figure 8.16, we have

$$\underbrace{\frac{1}{2}\left[f(u_i)\right]^2 \Delta\theta_i}_{\text{Area of the sector of radius} f(u_i)} \leq \Delta A_i \leq \underbrace{\frac{1}{2}\left[f(v_i)\right]^2 \Delta\theta_i}_{\text{Area of the sector of radius} f(v_i)}$$

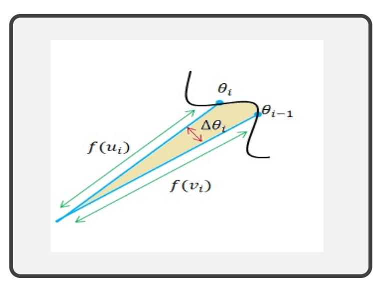


Figure 8.16

By summing from $i = 1$ to $i = n$, we obtain

$$\sum_{i=1}^{n} \frac{1}{2}\left[f(u_i)\right]^2 \Delta\theta_i f(u_i) \leq \underbrace{\sum_{i=1}^{n} \Delta A_i}_{=A} \leq \sum_{i=1}^{n} \frac{1}{2}\left[f(v_i)\right]^2 \Delta\theta_i f(v_i)$$

The limit of the sums as the norm $||P||$ approaches zero,

$$\lim_{||P||\to 0} \sum_{i=1}^{n} \frac{1}{2}\left[f(u_i)\right]^2 \Delta\theta_i f(u_i) = \lim_{||P||\to 0} \sum_{i=1}^{n} \frac{1}{2}\left[f(u_i)\right]^2 \Delta\theta_i f(v_i) = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta \, .$$

Therefore,

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta}$$

Similarly, assume f and g are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) \geq g(\theta)$. The area of the region bounded by the polar graphs of f and g on the interval $[\alpha, \beta]$ is

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] \, d\theta}$$

■ **Example 8.10** Find the area of the region bounded by the graph of the polar equation.

(1) $r = 3$ **(2)** $r = 2\cos\,\theta$ **(3)** $r = 4\sin\,\theta$ **(4)** $r = 6 - 6\sin\,\theta$

Solution:

(1) From Figure 8.17, the area is

$$A = \frac{1}{2}\int_0^{2\pi} 3^2 \, d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}\Big[\,\theta\,\Big]_0^{2\pi} = 9\pi.$$

Note that we can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.

$$A = 4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} 3^2 \, d\theta\Big) = 2\int_0^{\frac{\pi}{2}} 9 \, d\theta = 18\Big[\,\theta\,\Big]_0^{\frac{\pi}{2}} = 9\pi.$$

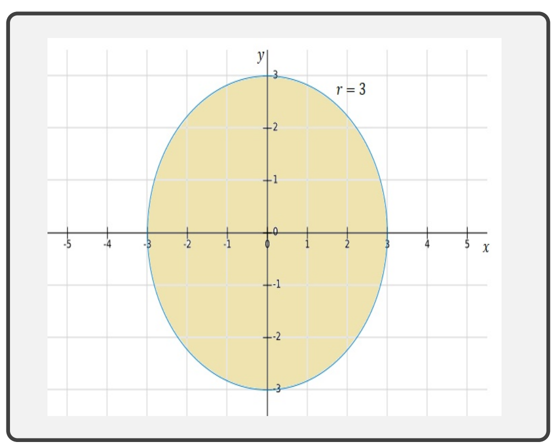


Figure 8.17

(2) We find the area of the upper half circle and multiply the result by 2 as follows:

$$
\begin{aligned}
A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos\,\theta)^2\,d\theta\right) &= \int_0^{\frac{\pi}{2}} 4\cos^2\,\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (1+\cos\,2\theta)\,d\theta \\
&= 2\left[\theta + \frac{\sin\,2\theta}{2}\right]_0^{\frac{\pi}{2}} \\
&= 2\left[\frac{\pi}{2} - 0\right] \\
&= \pi.
\end{aligned}
$$

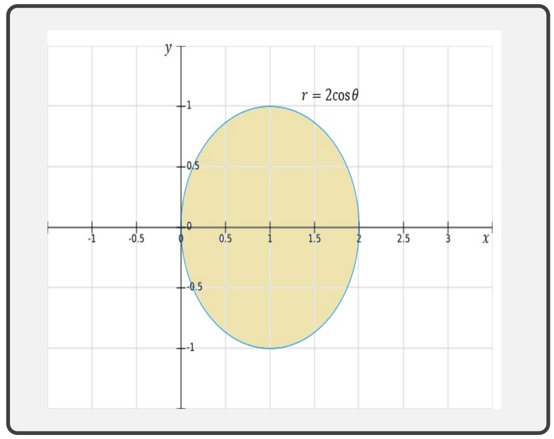


Figure 8.18

(3) From Figure 8.19, the area of the region is

$$
\begin{aligned}
A = \frac{1}{2}\int_0^{\pi} (4\sin\,\theta)^2\,d\theta = \frac{16}{4}\int_0^{\pi} (1-\cos\,2\theta)\,d\theta &\\
= 4\left[\theta - \frac{\sin\,2\theta}{2}\right]_0^{\pi} &\\
= 4\left[\pi - 0\right] &\\
= 4\pi. &
\end{aligned}
$$

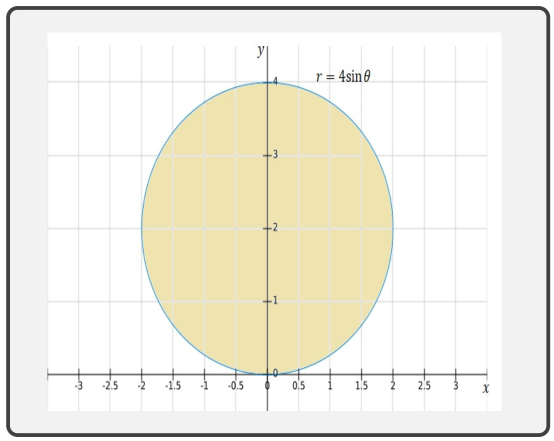


Figure 8.19

(4) From Figure 8.20, the area of the region is

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} 36(1-\sin\,\theta)^2\,d\theta \\
&= 18\int_0^{2\pi} (1 - 2\sin\,\theta + \sin^2\,\theta)\,d\theta \\
&= 18\left[\theta + 2\cos\,\theta + \frac{\theta}{2} - \frac{\sin\,2\theta}{4}\right]_0^{2\pi} \\
&= 18\left[(2\pi + 2 + \pi) - 2\right] \\
&= 54\pi.
\end{aligned}
$$

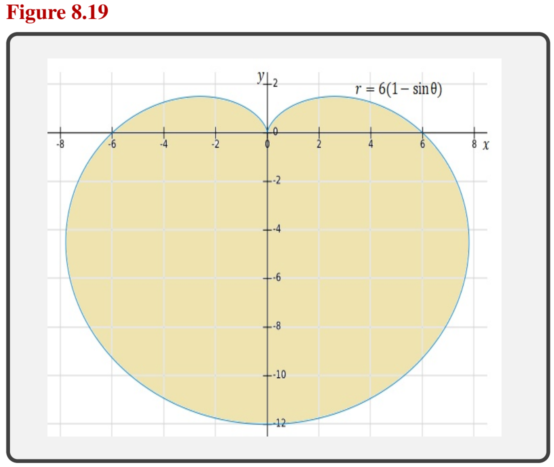


Figure 8.20

■ **Example 8.11** Find the area of the region that is between the curves $r = 2$ and $r = 3$ in the first quadrant.

Solution: The region bounded by the two curves $r_1 = 2$ and $r_2 = 3$ is displayed in the figure.

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (r_2^2 - r_1^2)\, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 5\, d\theta \\
&= \frac{5}{2}\Big[\theta\Big]_0^{\frac{\pi}{2}} \\
&= \frac{5}{2}\left[\frac{\pi}{2} - 0\right] \\
&= \frac{5\pi}{4}\, .
\end{aligned}
$$

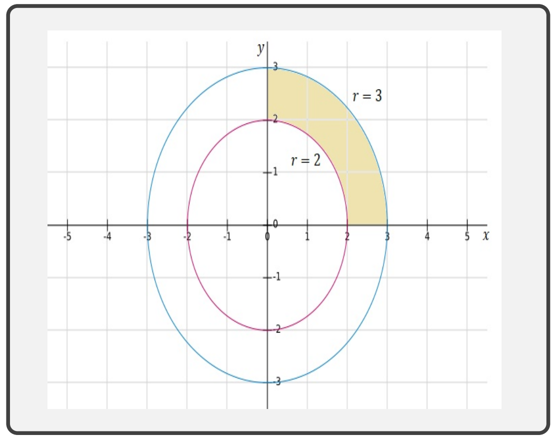


Figure 8.21

■ **Example 8.12** Find the area of the region that is inside the graphs of the equations $r = \sin\theta$ and $r = \sqrt{3}\cos\theta$.

Solution:
First, we find the intersection points of the two curves

$$\sin\ \theta = \sqrt{3}\cos\ \theta \Rightarrow \tan\ \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}\, .$$

The origin O is in each circle, but it cannot be found by solving the equations. Therefore, when looking for the intersection points of the polar graphs, we sometimes take under consideration the graphs. The region is divided into two small regions: below and above the line $\frac{\pi}{3}$.

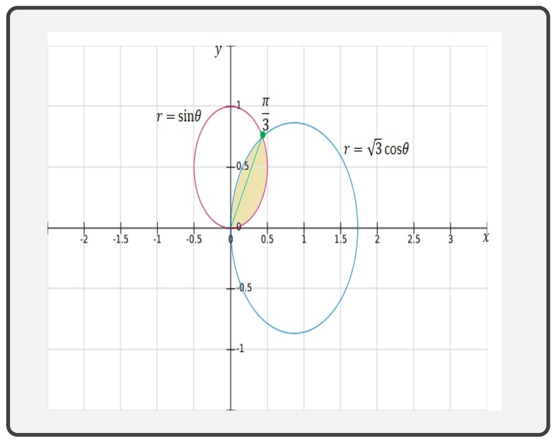


Figure 8.22

Region(1) is in the interval $[0, \frac{\pi}{3}]$.

$$
\begin{aligned}
A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\ \theta\, d\theta &= \frac{1}{4}\int_0^{\frac{\pi}{3}} (1 - \cos\ 2\theta)\, d\theta \\
&= \frac{1}{4}\left[\theta - \frac{\sin\ 2\theta}{2}\right]_0^{\frac{\pi}{3}} \\
&= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sin\ \frac{2\pi}{3}}{2}\right] \\
&= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right].
\end{aligned}
$$

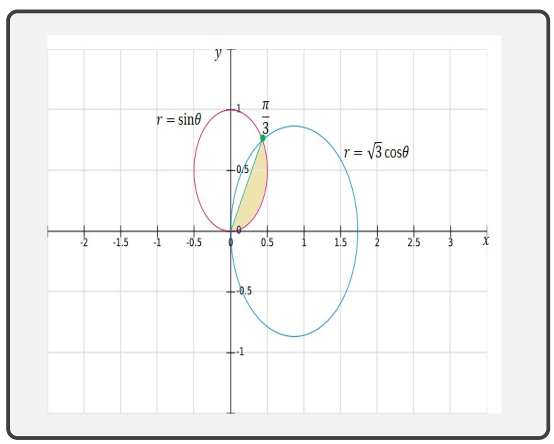


Figure 8.23

Region(2) is in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$.

$$\begin{aligned}
A_2 = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\,\theta)^2\, d\theta &= \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos\,2\theta)\, d\theta \\
&= \frac{3}{4}\Big[\theta + \frac{\sin\,2\theta}{2}\Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \frac{3}{4}\Big[\big(\frac{\pi}{2} - 0\big) - \big(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\big)\Big] \\
&= \frac{3}{4}\Big[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\Big].
\end{aligned}$$

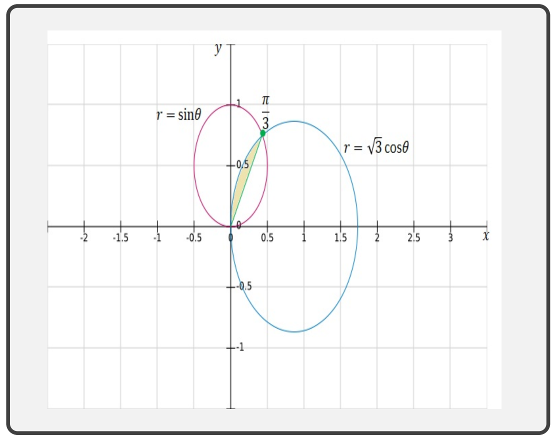


Figure 8.24

Total area $A = A_1 + A_2 = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$.

■ **Example 8.13** Find the area of the region that is outside the graph of $r = 3$ and inside the graph of $r = 2 + 2\cos\theta$.

Solution: The intersection point of the two curves in the first quadrant is

$$2 + 2\cos\,\theta = 3 \Rightarrow \cos\,\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

As shown in the figure, we find the area in the first quadrant, then we double the result to find the area of the whole region.

$$\begin{aligned}
A &= 2\Big(\frac{1}{2}\int_0^{\frac{\pi}{3}} \big(4(1+\cos\,\theta)^2 - 9\big)\, d\theta\Big) \\
&= \int_0^{\frac{\pi}{3}} \big(4(1+2\cos\,\theta+\cos^2\,\theta) - 9\big)\, d\theta \\
&= \int_0^{\frac{\pi}{3}} (8\cos\,\theta + 4\cos^2\,\theta - 5)\, d\theta \\
&= \Big[8\sin\,\theta + \sin\,2\theta - 3\theta\Big]_0^{\frac{\pi}{3}} \\
&= \frac{9}{2}\sqrt{3} - \pi.
\end{aligned}$$

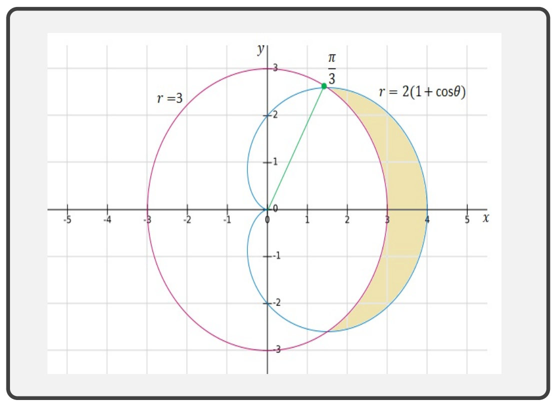


Figure 8.25

Exercises

1 - 8 ■ Find the corresponding Cartesian coordinates for the given polar coordinates.

1 $(2, \frac{\pi}{3})$

2 $(1, \frac{\pi}{2})$

3 $(-2, \frac{\pi}{6})$

4 $(3, \pi)$

5 $(\frac{1}{2}, \frac{3\pi}{2})$

6 $(3, 2\pi)$

7 $(-7, \frac{3\pi}{4})$

8 $(3, \frac{\pi}{3})$

9 - 16 ■ Find the corresponding polar coordinates for the given Cartesian coordinates for $r \geq 0$ and $0 \leq \theta \leq \pi$.

9 $(1, 1)$

10 $(1, \sqrt{3})$

11 $(-1, 1)$

12 $(\sqrt{3}, 3)$

13 $(\sqrt{3}, 1)$

14 $(\frac{1}{2}, \frac{1}{2})$

15 $(-1, \sqrt{3})$

16 $(3, 0)$

17 - 24 ■ Find a polar equation that has the same graph as the equation in x and y and vice versa.

17 $x = 4$

18 $x^2 + y^2 = 5$

19 $r = \csc\ \theta$

20 $r = 6\cos\ \theta$

21 $x^2 = 2y$

22 $x^2 - y^2 = 9x$

23 $r = \frac{3}{1-\sin\ \theta}$

24 $r = 2 - 3\sin\ \theta$

25 - 28 ■ Sketch the curve of the polar equations.

25 $r = 3\sec\ \theta$

26 $r = 4\cos\ \theta$

27 $r = 2 + 2\sin\ \theta$

28 $r = 3 + 2\cos\ \theta$

29 - 34 ■ Find the area of the region bounded by the graph of the polar equation.

29 $r = 3\sin\ \theta$

30 $r = 1 + \sin\ \theta$

31 $r = 2$

32 $r = 4\cos\ \theta$

33 $r = 6(1 + \cos\ \theta)$

34 $r = 2(1 - \sin\ \theta)$

35 - 41 ■ Find the area of the region bounded by the graph of the polar equations.

35 inside $r = 2$

36 between $r = 2$ and $r = 3$

37 inside $r = 1 + \cos\ \theta$ and outside $r = 3\cos\ \theta$

38 inside $r = 2 + 2\cos\ \theta$ and outside $r = 3$

39 outside $r = 2 - 2\cos\theta$ and inside $r = 4$

40 inside both graphs $r = 1 + \cos\theta$ and $r = 1$

41 inside both graphs $r = 2\cos\theta$ and $r = 2\sin\theta$

Appendix

Appendix (1): Integration Rules and Integrals Table

Integration Rules:

$$\int \big(f(x) \pm g(x)\big)\; dx = \int f(x)\; dx \pm \int g(x)\; dx$$

$$\int kf(x)\; dx = k\int f(x)\; dx$$

$$\int f'\big(g(x)\big)g'(x)\; dx = f\big(g(x)\big) + c$$

$$\int_a^b f'(x)\; dx = f(b) - f(a)$$

Elementary Integrals:

$$\int x^r\; dx = \frac{x^{r+1}}{r+1} \text{ if } r \neq -1$$

$$\int \sin x\; dx = \cos x$$

$$\int \cos x\; dx = -\sin x$$

$$\int \sec^2 x\; dx = \tan x$$

$$\int \csc^2 x\; dx = -\cot x$$

$$\int \sec x\; \tan x\; dx = \sec x$$

$$\int \csc x\; \cot x\; dx = -\csc x$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\; dx = \sin^{-1} \frac{x}{a}$$

$$\int \frac{1}{a^2 + x^2}\; dx = \frac{1}{a}\; \tan^{-1} \frac{x}{a}$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\; dx = \frac{1}{a}\; \sec^{-1} \mid \frac{x}{a} \mid$$

Inverse Trigonometric Integrals:

$$\int \sin^{-1} x\; dx = x\sin^{-1} x + \sqrt{1 - x^2} + c$$

$$\int \tan^{-1} x\; dx = x\tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + c$$

$$\int \sec^{-1} x\; dx = x\sec^{-1} x - \ln \mid x + \sqrt{x^2 - 1} \mid + c$$

$$\int x^n\; \sin^{-1} x\; dx = \frac{x^{n+1}}{n+1} \sin^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1 - x^2}}\; dx + c \text{ if } n \neq -1$$

$$\int x^n\; \tan^{-1} x\; dx = \frac{x^{n+1}}{n+1} \tan^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1 + x^2}\; dx + c \text{ if } n \neq -1$$

$$\int x^n\; \sec^{-1} x\; dx = \frac{x^{n+1}}{n+1} \sec^{-1} x - \frac{1}{n+1} \int \frac{x^n}{\sqrt{x^2 - 1}}\; dx + c \text{ if } n \neq -1$$

Trigonometric Integrals:

$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$

$\int \tan^2 x\,dx = \tan x - x + c$

$\int \cot^2 x\,dx = -\cot x - x + c$

$\int \sec^3 x\,dx = \frac{1}{2}\ \sec x\ \tan x + \frac{1}{2}\ln|\sec x\ +\ \tan x| + c$

$\int \sec^3 x\,dx = \frac{1}{2}\ \csc x\ \cot x + \frac{1}{2}\ln|\csc x\ -\ \cot x| + c$

$\int \sin^n x\,dx = -\frac{1}{n}\ \sin^{n-1} x\ \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx + c$

$\int \cos^n x\,dx = \frac{1}{n}\ \cos^{n-1} x\ \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx + c$

$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx + c$ if $n \neq 1$

$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx + c$ if $n \neq 1$

$\int \sec^n x\,dx = \frac{1}{n-1}\ \sec^{n-2} x\ \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx + c$ if $n \neq 1$

$\int \csc^n x\,dx = -\frac{1}{n-1}\ \csc^{n-2} x\ \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx + c$ if $n \neq 1$

$\int \sin^n x\ \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x\ \cos^m x\,dx + c$ if $n \neq m$

$\int \sin^n x\ \cos^m x\,dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x\ \cos^{m-2} x\,dx + c$ if $m \neq n$

$\int x^n\ \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx + c$

$\int x^n\ \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx + c$

Miscellaneous Integrals:

$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + c$

$\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left(\ln|ax+b| + \frac{b}{ax+b}\right) + c$

$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left(\frac{ax+b}{n+2} - \frac{b}{n-1}\right) + c$

$\int \frac{a}{(a^2 \pm x^2)^n}\,dx = \frac{1}{2a^2(n-1)}\left(\frac{x}{(a^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{1}{(a^2 \pm x^2)^{n-1}}\,dx\right)$ if $n \neq -1$

$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(3ax-2b)(ax+b)^{3/2} + c$

$\int x^n\sqrt{ax+b}\,dx = \frac{2}{a(2n+3)}\left(x^n(ax+b)^{3/2} - nb\int x^{n-1}\sqrt{ax+b}\,dx\right)$

$\int \frac{x}{\sqrt{ax+b}}\,dx = \frac{2}{3a^2}(ax-2b)\sqrt{ax+b} + c$

$\int \frac{x^n}{\sqrt{ax+b}}\,dx = \frac{2}{a(2n+1)}\left(x^n\sqrt{ax+b} - nb\int \frac{x^{n-1}}{\sqrt{ax+b}}\,dx\right)$

$\int \frac{1}{x\sqrt{ax+b}}\,dx = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c$ if $b > 0$

$$\int \frac{1}{x\sqrt{ax+b}}\,dx = \frac{1}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} + c \text{ if } b<0$$

$$\int \frac{1}{x^n\sqrt{ax+b}}\,dx = -\frac{\sqrt{ax+b}}{b(n-1)x^{n-1}} - \frac{(2n-3)a}{2(n-1)b}\int \frac{1}{x^{n-1}\sqrt{ax+b}}\,dx \text{ if } n\neq 1$$

$$\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int x\sqrt{2ax-x^2}\,dx = \frac{2x^2-ax-3a^3}{6}\sqrt{2ax-x^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -\frac{2\sqrt{2ax-x^2}}{x} - \cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int \frac{x}{\sqrt{2ax-x^2}}\,dx = -\sqrt{2ax-x^2} + a\cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int \frac{x^2}{\sqrt{2ax-x^2}}\,dx = -\frac{(x+3a)}{2}\sqrt{2ax-x^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right) + c$$

$$\int \frac{1}{x\sqrt{2ax-x^2}}\,dx = -\frac{\sqrt{2ax-x^2}}{ax} + c$$

Appendix (2): Answers to Exercises

Chapter 1:

1 $(y-2)^2 = 2(x+4)$

2 $(y-6)^2 = -\frac{4}{3}(x-2)$

3 $(x+1)^2 = (y-1)$

4 $(x-2)^2 = 12(y-1)$

5 $(y+2)^2 = -2x$

6 $(x-5)^2 = -\frac{1}{2}(y-3)$

7 $(x-1)^2 = 4(y-2)$

8 $(x-4)^2 = -12(y+3)$

9 $(x+3)^2 = -4(y+5)$

10 $(y-1)^2 = 8(x-2)$

11 $(x-4)^2 = \frac{2}{3}(y+5)$

12 $(x+8)^2 = -20(y-2)$

13 $(x-3)^2 = 8(y-4)$

14 $\frac{x^2}{16} + \frac{y^2}{9} = 1$

15 $\frac{x^2}{9} + \frac{y^2}{5} = 1$

16 $\frac{(x-2)^2}{4} + (y-2)^2 = 1$

17 $\frac{(x-1)^2}{9} + \frac{(y+1)^2}{4} = 1$

18 $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{16} = 1$

19 $\frac{(x-2)^2}{4} + 4\frac{(y-\frac{1}{2})^2}{25} = 1$

20 $\frac{(x+1)^2}{16} + \frac{(y-4)^2}{25} = 1$

21 $\frac{(x-7)^2}{25} + \frac{(y+2)^2}{16} = 1$

22 $\frac{y^2}{4} - \frac{x^2}{9} = 1$

23 $\frac{y^2}{36} - \frac{x^2}{28} = 1$

24 $\frac{(x-6)^2}{25} - \frac{(y-1)^2}{9} = 1$

25 $\frac{x^2}{16} - \frac{y^2}{16} = 1$

26 $x^2 - \frac{y^2}{4} = 1$

27 $\frac{y^2}{25} - \frac{x^2}{9} = 1$

28 $4\frac{y^2}{49} - \frac{x^2}{49} = 1$

29 $\frac{(y-5)^2}{36} - \frac{(x-3)^2}{4} = 1$

30 $\frac{(x-4)^2}{25} - (y-2)^2 = 1$

31 $(x-7)^2 - \frac{(y+2)^2}{8} = 1$

32 $V(1,-1), F(1,1), D: y=-3$

33 $V(2,-2), F(2,-\frac{9}{4}), D: y=-\frac{7}{4}$

34 $V(-2,3), F(-2,\frac{8}{3}), D: y=\frac{10}{3}$

35 $V(-2,-5), F(-2,-\frac{9}{2}), D: y=-\frac{11}{2}$

36 $V(1,1), F(1,\frac{15}{16}), D: y=\frac{17}{16}$

37 $V(-3,-11), F(-3,-10\frac{15}{16}), D: y=-11\frac{1}{16}$

38 $V(5,-7), F(5,-6\frac{15}{16}), D: y=-7\frac{1}{16}$

39 $V(-4,9), F(-4,8\frac{19}{20}), D: y=9\frac{1}{20}$

40 $V(4,-9), F(4,-8\frac{3}{4}), D: y=-9\frac{1}{4}$

41 $V_1(5,0), V_2(-5,0), F_1(\sqrt{15},0), F_2(-\sqrt{15},0)$

42 $V_1(6,0), V_2(-6,0), F_1(4\sqrt{2},0), F_2(-4\sqrt{2},0)$

43 $V_1(10,0), V_2(-10,0), F_1(\sqrt{51},0), F_2(-\sqrt{51},0)$

44 $V_1(0,\sqrt{7}), V_2(0,-\sqrt{7}), F_1(0,\sqrt{2}), F_2(0,-\sqrt{2})$

45 $V_1(7,0), V_2(-7,0), F_1(\sqrt{13},0), F_2(-\sqrt{13},0)$

46 $V_1(1,2), V_2(-7,2), F_1(-3+\sqrt{7},2), F_2(-3-\sqrt{7},2)$

47 $V_1(0,9), V_2(0,-9), F_1(0,3\sqrt{5}), F_2(0,-3\sqrt{5})$

48 $V_1(0,\sqrt{30}), V_2(0,-\sqrt{30}), F_1(0,\sqrt{15}), F_2(0,-\sqrt{15})$

49 $V_1(\sqrt{55},0), V_2(-\sqrt{55},0), F_1(2\sqrt{7},0), F_2(-2\sqrt{7},0)$

50 $V_1(8,0), V_2(-8,0), F_1(3\sqrt{6},0), F_2(-3\sqrt{6},0)$

51 $V_1(3,-2+\sqrt{10}), V_2(3,-2-\sqrt{10}), F_1(3,-2+\sqrt{6}), F_2(3,-2-\sqrt{6})$

52 $V_1(0,7), V_2(0,-3), F_1(0,6), F_2(0,-2)$

53 $V_1(0,7), V_2(0,-7), F_1(0,3\sqrt{5}), F_2(0,-3\sqrt{5})$

54 $V(-7,9), F(-7,8\frac{7}{8}), D: y=9\frac{1}{8}$

55 $V(-3,-4), F(-3,-3\frac{3}{4}), D: y=-4\frac{1}{4}$

56 $V(4,-3), F(4,-2\frac{7}{8}), D: y=-3\frac{1}{8}$

57 $V(-4,-3), F(-4,-3\frac{1}{8}), D: y=-2\frac{6}{8}$

58 $V(5,3), F(5\frac{1}{4},3), D: x=4\frac{3}{4}$

59 $V(-3,-1), F(-3\frac{1}{8},-1), D: x=-2\frac{7}{8}$

60 $V_1(-3+\sqrt{5},4), V_2(-3-\sqrt{5},4), F_1(-1,4), F_2(-5,4)$

61 $V(-1,\frac{3}{2}), F(-1,1), D: y=2$

62 $V_1(5,0), V_2(-5,0), F_1(\sqrt{34},0), F_2(-\sqrt{34},0)$

63 $V_1(4,0), V_2(-4,0), F_1(5,0), F_1(-5,0)$

64 $V_1(0,7), V_2(0,-7), F_1(0,\sqrt{74}), F_2(0,-\sqrt{74})$

65 $V_1(2,0), V_2(-2,0), F_1(\sqrt{53},0), F_2(-\sqrt{53},0)$

66 $V_1(5,0), V_2(-5,0), F_1(\sqrt{106},0), F_2(-\sqrt{106},0)$

67 $V_1(0,8), V_2(0,-8), F_1(0,\sqrt{89}), F_2(0,-\sqrt{89})$

68 $V_1(6,0), V_2(-6,0), F_1(2\sqrt{14},0), F_2(-2\sqrt{14},0)$

69 $V_1(1,2), V_2(-7,2), F_1(2,2), F_2(-8,2)$

70 $V_1(7,0), V_2(-3,0), F_1(2+\sqrt{41},0), F_2(2-\sqrt{41},0)$

71 $V_1(6,13), V_2(6,-3), F_1(6,5+\sqrt{89}), F_2(6,5-\sqrt{89})$

72 $V_1(5,5), V_2(-13,5), F_1(-4+2\sqrt{34},5), F_2(-4-2\sqrt{34},5)$

73 $V_1(0,\sqrt{10}), V_2(0,-\sqrt{10}), F_1(0,\sqrt{35}), F_2(0,-\sqrt{35})$

74 $V_1(0,7), V_2(0,-7), F_1(0,\sqrt{39}), F_2(0,-\sqrt{39})$

75 $V_1(-4,-3+\sqrt{5}), V_2(-4,-3-\sqrt{5}), F_1(-4,-3+\sqrt{6}), F_2(-4,-3-\sqrt{6})$

Chapter 2:

1 $\begin{bmatrix} 3 & 0 \\ 1 & 11 \\ 15 & 14 \end{bmatrix}$

2 $\begin{bmatrix} 4 & 0 \\ 2 & 29 \\ 35 & 31 \end{bmatrix}$

3 $\begin{bmatrix} -7 & 0 \\ -1 & 3 \\ -5 & -8 \end{bmatrix}$

4 Not possible

5 $\begin{bmatrix} 28 & 34 \\ 81 & 50 \\ 29 & 58 \end{bmatrix}$

6 Not possible

7 $\begin{bmatrix} 1 & 5 & 0 \\ 3 & -4 & 9 \\ 2 & 6 & 2 \end{bmatrix}$

8 $\begin{bmatrix} 3 & 15 & 0 \\ 9 & -12 & 27 \\ 6 & 18 & 6 \end{bmatrix}$

9 -2

10 -16

11 $\begin{bmatrix} 2 & 0 \\ 4 & 11 \\ 3 & 11 \end{bmatrix}$

12 $\begin{bmatrix} 20 & -5 \\ 5 & 25 \\ 10 & 35 \end{bmatrix}$

13 $\begin{bmatrix} -16 & 5 \\ 3 & -3 \\ -4 & -13 \end{bmatrix}$

14 Not possible

15 $\begin{bmatrix} -5 & 3 & 1 \\ 46 & 27 & 16 \\ 65 & 39 & 23 \end{bmatrix}$

16 $\begin{bmatrix} 7 & 1 & 1 \\ 57 & 36 & 21 \\ 37 & 22 & 13 \end{bmatrix}$

17 Not possible

18 $\begin{bmatrix} -2 & 3 & 1 \\ 1 & 6 & 4 \end{bmatrix}$

19 $\begin{bmatrix} 8 & 2 & 4 \\ -2 & 10 & 14 \end{bmatrix}$

20 $\begin{bmatrix} 4 & -1 \\ 1 & 5 \\ 2 & 7 \end{bmatrix}$

21 -21

22 -6

23 -3

24 -6

25 11

26 -43

27 29

28 638

29 5

30 -104

31 73

32 12

Chapter 3:

1 $X = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

2 $X = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

3 $X = \begin{bmatrix} \frac{4}{3} \\ -3 \\ -\frac{11}{3} \end{bmatrix}$

4 $X = \begin{bmatrix} \frac{8}{7} \\ -\frac{9}{7} \\ -\frac{9}{7} \end{bmatrix}$

5 $X = \begin{bmatrix} \frac{1}{3} \\ 3 \\ \frac{7}{3} \end{bmatrix}$

6 $X = \begin{bmatrix} \frac{3}{10} \\ \frac{4}{5} \\ \frac{7}{10} \end{bmatrix}$

7 $X = \begin{bmatrix} -\frac{9}{4} \\ \frac{1}{2} \\ \frac{5}{4} \end{bmatrix}$

8 $X = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$

9 $X = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

10 $X = \begin{bmatrix} \frac{3}{10} \\ \frac{4}{5} \\ \frac{7}{10} \end{bmatrix}$

11 $X = \begin{bmatrix} \frac{7}{3} \\ \frac{5}{11} \\ \frac{29}{33} \end{bmatrix}$

12 $X = \begin{bmatrix} \frac{13}{5} \\ -\frac{43}{10} \\ \frac{29}{10} \end{bmatrix}$

13 $X = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

14 $X = \begin{bmatrix} \frac{25}{2} \\ 2 \\ -\frac{11}{2} \end{bmatrix}$

15 $X = \begin{bmatrix} \frac{79}{5} \\ -10 \\ \frac{96}{5} \end{bmatrix}$

16 $X = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$

17 $X = \begin{bmatrix} \frac{7}{2} \\ -\frac{7}{2} \\ 7 \end{bmatrix}$

18 $X = \begin{bmatrix} \frac{3}{10} \\ \frac{4}{5} \\ \frac{7}{10} \end{bmatrix}$

19 $X = \begin{bmatrix} \frac{7}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$

20 $X = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

21 $X = \begin{bmatrix} -\frac{16}{3} \\ \frac{17}{3} \\ \frac{10}{3} \end{bmatrix}$

22 $X = \begin{bmatrix} -1 \\ \frac{13}{4} \\ -\frac{5}{4} \end{bmatrix}$

23 $X = \begin{bmatrix} \frac{19}{10} \\ \frac{7}{5} \\ -\frac{21}{10} \end{bmatrix}$

24 $X = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$

Chapter 4:

1 $\frac{1}{3}\tan(3x-5)+c$

2 $\sin^{-1}\frac{x}{4}+c$

3 $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$

4 $\frac{x}{2}\sin 2x+\frac{1}{4}\cos 2x+c$

5 $x\sin^{-1}x+\sqrt{1-x^2}+c$

6 $-\frac{1}{3}\ln|x+1|+\frac{1}{3}\ln|x-2|+c$

7 $\frac{1}{36}(2x^2-3)^9+c$

8 $3\sin\sqrt[3]{x}+c$

9 $\frac{21}{2}$

10 $\frac{8}{3}$

11 1

12 1

13 $1+\frac{1}{\sqrt{3}}-\sqrt{2}$

14 12

15 $\frac{5}{2}$

16 $\frac{275}{6}$

17 0

18 $-\frac{11}{20}$

19 $\frac{1}{3}\left(5\sqrt{5}-2\sqrt{2}\right)$

20 $\frac{17}{2}$

21 $\frac{1}{4}x^2\ln|x|-\frac{1}{8}x^2+c$

22 $\frac{32}{3}$

23 $\ln(3)-\ln(e-2)$

24 $2\ln(7)-\ln(4)$

25 $\frac{17}{6}$

26 $40\sqrt{10}+10$

27 $4(\sqrt{2}-1)$

28 1

29 $\frac{10}{3}$

30 $\frac{1}{\sqrt{3}}$

31 $\frac{9}{2}$

32 $\frac{2}{3}$

33 2

34 $\frac{1}{2}$

35 -2

36 $\frac{1}{2}\sin x^2 + c$

37 $-2\cot\sqrt{x} + c$

38 $\tan x + \sec x + c$

39 $\frac{2}{3}x^{\frac{3}{2}}\ln|x| - \frac{4}{9}x^{\frac{3}{2}} + c$

40 $x\tan x + \ln|\cos x| + c$

41 $-\frac{x}{4} - \frac{1}{16}e^{-4x} + c$

42 $x(\ln x)^2 - 2x\ln|x| + 2x + c$

43 $-\frac{1}{4}\ln|x-1| + \frac{1}{2(x-1)} + \frac{1}{4}\ln|x-3| + c$

44 $\frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x+4| + c$

45 $\frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| + c$

46 $-\frac{1}{4}\left(\ln|\frac{x+1}{2}+1| - \ln|\frac{x+1}{2}-1|\right) + c$

47 $4 + \frac{2}{3}\ln(50) - \ln(2)$

48 $-\frac{3x^2-10}{x-2} + 6(x-2+2\ln|x-2|) + c$

49 $\frac{(x-1)^2}{2} + 2(x-1) - 8\ln|x-1| + c$

50 $x^2 - x - \frac{3}{2}\ln|x+1| + \frac{1}{x+1} + \frac{1}{2}\ln|x-3| + c$

51 $x - \ln(e^x+1) + c$

52 $\frac{53}{27}\ln|x+1| + \frac{1}{27}\ln|x-2| + \frac{55}{9(x-2)} + \frac{1}{3(x-2)^2} + c$

53 -2

54 4

55 7

56 -1

57 $\frac{65}{3}$

58 0

65 (a)

66 (c)

67 (d)

68 (a)

69 (d)

70 (c)

71 (b)

Chapter 5:

1 $\frac{9}{2}$

2 4

3 $\frac{32}{3}$

4 4

5 $\frac{2}{3}$

6 44

7 1

8 $\frac{11}{4}$

9 1

10 $\frac{3}{2}$

11 $\frac{32}{3}$

12 12

13 $\frac{1}{3}$

14 $\frac{11}{6}$

15 $\frac{1}{2}$

16 $\frac{23-8\sqrt{2}}{6}$

17 $\frac{1}{6}$

18 $\frac{4}{3}$

19 $\frac{9}{2}$

20 $\frac{4}{3}$

21 $\frac{1}{6}$

22 $\frac{1}{3}$

23 $\frac{1}{3}$

24 $2\sqrt{2} - 2$

25 3

26 $\frac{3}{2}$

27 $\frac{e^3-1}{e^2}$

28 $\frac{\sqrt{3}-1}{2}$

29 $2\ln(2) - 1$

30 1

31 $\frac{1}{2}$

32 $\frac{\sqrt{2}-1}{\sqrt{2}}$

33 2

34 $\ln(\sqrt{2})$

35 $\frac{4}{3}$

36 $e^2 + 1$

37 $\frac{4}{3}\pi$

38 16π

39 $\frac{64\sqrt{2}}{3}\pi$

40 $\frac{2}{3}\pi$

41 $\frac{3}{10}\pi$

42 $\frac{2\sqrt{2}}{3}\pi$

43 $\frac{2}{35}\pi$

44 $\frac{373}{14}\pi$

45 10π

46 $\frac{2}{15}\pi$

47 $\frac{4387}{4480}\pi$

48 $\frac{512}{15}\pi$

49 $\frac{(e^4-1)}{2}\pi$

50 $\frac{1944}{5}\pi$

51 $\frac{2}{15}\pi$

52 $\left(16\ln^2(2) - 16\ln(2) + 6\right)\pi$

53 $\frac{\pi}{2}$

54 $\frac{\pi^2}{4}$

55 3π

56 $\frac{64}{15}\pi$

57 $\frac{3}{5}\pi$

58 8π

59 $\frac{72}{5}\pi$

60 $(\pi - 2)\pi$

61 $\left(\frac{\pi}{\sqrt{2}} - 2\right)\pi$

62 $\frac{16}{15}\pi$

63 $\frac{10}{3}\pi$

64 6π

65 $\frac{9}{2}\pi$

66 $\frac{4}{21}\pi$

67 $\frac{16}{3}\pi$

68 $\frac{\pi}{6}$

69 $\frac{3}{2}\pi$

70 8π

71 $\frac{\pi}{2}$

72 $\frac{4}{3}\pi$

73 (d)

74 (a)

75 (a)

76 (a)

77 (a)

78 (d)

Chapter 6:

1 $f_x = 8x^3y^3 - y^2$
$f_y = 6x^4y^2 - 2xy + 3$
$f_{xx} = 24x^2y^3$
$f_{yy} = 12x^4y - 2x$

2 $f_x = 8xy^3e^{x^2y^3}$
$f_y = 12x^2y^2e^{x^2y^3}$
$f_{xx} = 8y^3e^{x^2y^3} + 16x^2y^6e^{x^2y^3}$
$f_{yy} = 24x^2ye^{x^2y^3} + +36x^4y^4e^{x^2y^3}$

3 $f_x = 3$
$f_y = 4$
$f_{xx} = 0$
$f_{yy} = 0$

4 $f_x = y^3 + 2xy^2$
$f_y = 3xy^2 + 2x^2y$
$f_{xx} = 2y^2$
$f_{yy} = 6xy + 2x^2$

5 $f_x = 3x^2y + e^x$
$f_y = x^3$
$f_{xx} = 6xy + e^x$
$f_{yy} = 0$

6 $f_x = e^{2x+3y} + 2xe^{2x+3y}$
$f_y = 3xe^{2x+3y}$
$f_{xx} = 4e^{2x+3y} + 4xe^{2x+3y}$
$f_{yy} = 9xe^{2x+3y}$

7 $f_x = \frac{2y}{(x+y)^2}$
$f_y = \frac{-2x}{(x+y)^2}$
$f_{xx} = \frac{-4y}{(x+y)^3}$
$f_{yy} = \frac{4x}{(x+y)^3}$

8 $f_x = 2\sin(x^2y) + 4x^2y\cos(x^2y)$
$f_y = 2x^3\cos(x^2y)$
$f_{xx} = 12xy\cos(x^2y) - 8x^3y^2\sin(x^2y)$
$f_{yy} = -2x^5\sin(x^2y)$

9 $f_x = 2x\sin y - y^2\sin x$
$f_y = x^2\cos y + 2y\cos x$
$f_{xx} = 2\sin y - y^2\cos x$
$f_{yy} = x^2\sin y + 2\cos x$

10 $f_x = 3x^2 + y^2$
$f_y = 2xy + 1$
$f_{xx} = 6x$
$f_{yy} = 2x$

11 $f_x = 2xy^2 + y^2$
$f_y = 2x^2y + 2xy$
$f_{xx} = 2y^2$
$f_{yy} = 2x(x+1)$

12 $f_x = 3x^2 + 1$
$f_y = 4y + 1$
$f_{xx} = 6x$
$f_{yy} = 4$

13 $f_x = 3yx^2 + y^4 - 3$
$f_y = x^3 + 4xy^3 - 3$
$f_{xx} = 6yx$
$f_{yy} = 12xy^2$

14 $f_x = -\frac{y}{x^2}\ln x + \frac{y}{x^2}$
$f_y = \frac{\ln x}{x}$
$f_{xx} = \frac{2y}{x^3}\ln x - \frac{3y}{x^3}$
$f_{yy} = 0$

15 $f_x = \frac{-2x}{(x^2+y^2)^2}$
$f_y = \frac{-2y}{(x^2+y^2)^2}$
$f_{xx} = \frac{6x^2-2y^2}{(x^2+y^2)^3}$
$f_{yy} = \frac{6y^2-2x^2}{(x^2+y^2)^3}$

16 $f_x = 2x + y$
$f_y = x - 2y$
$f_{xx} = 2$
$f_{yy} = -2$

17 $f_x = \frac{2x}{x^2-y}$
$f_y = -\frac{1}{x^2-y}$
$f_{xx} = \frac{-2(x^2+y)}{(x^2-y)^2}$
$f_{yy} = \frac{-1}{(x^2-y)^2}$

18 $f_x = \cos y + ye^x$
$f_y = -x\sin y + e^x$
$f_{xx} = ye^x$
$f_{yy} = -x\cos y$

19 $f_x = y^2\cos(xy)$
$f_y = \sin(xy) + xy\cos(xy)$
$f_{xx} = -y^3\sin(xy)$
$f_{yy} = 2x\cos(xy) - x^2y\sin(xy)$

20 $f_x = 8x - 8y^4$
$f_y = -32xy^3 + 21y^2$
$f_{xx} = 8$
$f_{yy} = -96xy^2 + 42y$

21 $f_x = y\cos(xy)$
$f_y = x\cos(xy)$
$f_{xx} = -y^2\sin(xy)$
$f_{yy} = -x^2\sin(xy)$

22 $f_x = 3x^2 + 6xy + 4$
$f_y = 3x^2 + 2y$
$f_{xx} = 6x + 6y$
$f_{yy} = 2$

23 $f_x = 2xy + 4y^3$
$f_y = x^2 + 12xy^2$
$f_{xx} = 2y$
$f_{yy} = 24xy$

24 $f_x = 2x\tan y$
$f_y = x^2\sec^2 y + 2y$
$f_{xx} = 2\tan y$
$f_{yy} = 2x^2\sec^2 y\tan y + 2$

25 $f_x = 3x^2\ln y + y^4$
$f_y = \frac{x^3}{y} + 4xy^3$
$f_{xx} = 6x\ln y$
$f_{yy} = -\frac{x^3}{y^2} + 12xy^2$

26 $f_x = 3x^2y + y^3$
$f_y = x^3 - 3xy^2$
$f_{xx} = 6xy$
$f_{yy} = -6xy$

27 6, 2, 3, 1

28 0, -1, 0, 0

29 $\frac{3}{4}$, $-\frac{1}{4}$, $-\frac{1}{2}$, $-\frac{1}{4}$

30 1, 0, -6, 6

31 1, 0, 1, 0

32 0, 3, -4, 0

33 -6, 3, 0, -18

34 0, 0, 0, 0

35 $0,\ 0,\ 1,\ 0$

36 $0,\ 0,\ 0,\ 4$

37 $12,\ 1,\ 0,\ 0$

38 $3,\ 0,\ 0,\ 2$

39 $2,\ 0,\ 6,\ 6$

40 $16,\ 0,\ 0,\ 6$

41 $2,\ -1,\ 1,\ 0$

42 $12,\ 12$

43 $3,\ -7$

44 $-2,\ 2$

45 $0,\ 0$

46 $8\sin(1),\ 4\cos(1)+8\sin(1)$

47 $-8\sin(4)+3,\ 4\sin(4)+3$

48 $0,\ 18e^9$

49 $6(\ln(4)+1),\ 6(\ln(4)+1)$

50 $-3,\ -2$

51 $-3\sin(2)+3\cos(2)+1,\ -3\sin(2)+4\cos(2)+1$

52 $1,\ 0$

53 $3,\ 10$

54 $0,\ 4$

55 $3-3\sin(3),\ 6-6\sin(3)+3\cos(3)$

56 $3e^6,\ 5e^6$

57 $10\sin^2(1),\ 4\sin(1)\big(2\cos(1)+\sin(1)\big)$

58 $\frac{y^2-x^2}{2xy+y^2}$

59 $\frac{y-2\sqrt{xy}}{6\sqrt{xy}-x}$

60 $-\frac{2}{3}$

61 $-\frac{x}{y}$

62 $-\frac{1}{2}$

63 $-\frac{x}{y}$

64 $-\frac{4x^4}{3y^2}$

65 $-\frac{2x}{3y^2}$

66 $\frac{3x^2-2x^3}{y}$

67 $-\frac{1}{\sin y+y\cos y}$

68 $-\frac{x}{y}$

69 $\frac{y+4\sqrt{xy}}{4y\sqrt{xy}-x}$

70 $\frac{8x}{y^{-\frac{1}{2}}+10}$

71 $-\frac{2xy^3+1}{3x^2y^2}$

72 $-\frac{y}{x}$

73 $\frac{1}{\sqrt{1-x^2}}$

74 $\frac{2y\sqrt{1+x^2y^2}-xy^2}{x^2y-2x\sqrt{1+x^2y^2}}$

75 $-\frac{6x+2xy}{x^2+3y^2}$

Chapter 7:

1 $y=-\frac{x}{1+cx}$

2 $y=-\frac{1}{\tan x+c}$

3 $y=x^2(\sec x+c)$

4 $y=e^x(-e^{-x}+c)$

5 $y=e^{-3x}(e^x+c)$

6 $y=\tan(-\cos x+c)$

7 $y=e^{-x}\big(\frac{1}{3}e^{3x}+c\big)$

8 $y=x(xe^x-e^x+c)$

9 $y=x(-e^{-x}+c)$

10 $y=e^{\frac{x}{2}}\left(4e^{-\frac{x}{2}}+c\right)$

11 $y=\sqrt{x^3+x^2-x+c}$

12 $y=e^{\sin x+c}$

13 $y=\frac{1}{x^2}\big(\frac{4}{5}x^5+c\big)$

14 $y=e^x\big(-\frac{1}{2}e^{-2x}+c\big)$

15 $y=\frac{1}{1+x^2}\big(x+\frac{x^3}{3}+c\big)$

16 $y=e^{-x}\big(\ln(e^x+1)+c\big)$

17 $y=e^{-2x}\big(\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}+\frac{5}{4}\big)$

18 $y=e^{-2x}(e^x-\frac{1}{4})$

19 $y=e^{x^2}\big(-\frac{1}{2}e^{-x^2}+\frac{1}{2}\big)$

20 $y=\frac{1}{1+x^2}\big(\frac{1}{2}\ln(1+x^2)+1\big)$

21 $y=\frac{1}{x}\big(-\cos\,x+\frac{4\pi+3}{6}\big)$

22 $y=e^{\frac{x}{3}}\left(-\frac{3}{4}e^{-\frac{4}{3}x}+a+\frac{3}{4}\right)$

23 $y=\frac{1}{x^2}\left(x^4+1\right)$

24 $y=\sqrt{x^3+2x^2+2x+1}$

25 $y=e^{x^3}\big(-e^{-x^3}+1\big)$

26 $y=\frac{1}{x}\big(\frac{x^4}{4}-\frac{13}{4}\big)$

27 $y=x(x+1)$

28 $y=-\frac{1}{x-1}$

29 $y=\sqrt[3]{x^3+x+1}$

30 $y=\cos x\big(\sin^2 x+\frac{11}{2}\big)$

31 $y=-\frac{1}{\frac{5}{2}(x^2-11)}$

32 $y=\sqrt{x^3+2x^2-4x+10}$

33 $y=-\frac{1}{\sqrt{1+x^2}}$

34 $y=\ln|x^2-4x-4|$

35 $y=-\frac{1}{\ln|x|-\frac{1}{2}}$

36 $y=\sin^{-1}x$

37 $y=\frac{1}{x}\big(\frac{x^4}{4}-\frac{13}{4}\big)$

38 $y=\frac{2}{\pi}\cos^{-1}\left(e^{-(x^2+x)}\right)$

39 $y=\tan\big(\frac{2}{3}x^3\big)$

40 $y=\frac{1}{x^2}\big(\frac{x^4}{4}-\frac{x^3}{3}+\frac{x^2}{2}+\frac{1}{12}\big)$

Chapter 8:

1 $(1, \sqrt{3})$

2 $(0, 1)$

3 $(-\sqrt{3}, -1)$

4 $(-3, 0)$

5 $(0, -\frac{1}{2})$

6 $(3, 0)$

7 $(\frac{7}{\sqrt{2}}, -\frac{7}{\sqrt{2}})$

8 $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$

9 $(\sqrt{2}, \frac{\pi}{4})$

10 $(2, \frac{\pi}{3})$

11 $(\sqrt{2}, \frac{3\pi}{4})$

12 $(2\sqrt{3}, \frac{\pi}{3})$

13 $(2, \frac{\pi}{6})$

14 $(\frac{1}{\sqrt{2}}, \frac{\pi}{4})$

15 $(2, \frac{5\pi}{6})$

16 $(3, 0)$

17 $r = 4\sec\theta$

18 $r = \sqrt{5}$

19 $y = 1$

20 $x^2 + y^2 - 6x = 0$

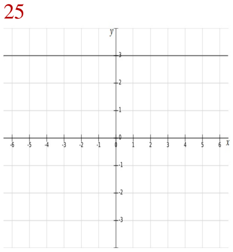
21 $r = 2\tan\theta\sec\theta$

22 $r = \frac{9\cos\theta}{\cos^2\theta - \sin^2\theta}$

23 $\sqrt{x^2+y^2} - y - 3 = 0$

24 $x^2 + y^2 - 2\sqrt{x^2+y^2} + 3y = 0$

25



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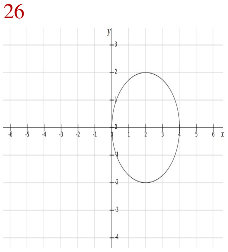

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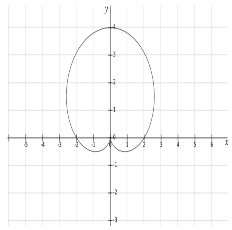

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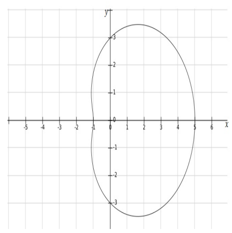


29 $\frac{9\pi}{4}$

30 $\frac{3\pi}{2}$

31 4π

32 4π

33 $36(\frac{3\pi}{4} + 2)$

34 6π

35 4π

36 5π

37 2π

38 $\frac{9\sqrt{3}}{2} - \pi$

39 10π

40 $\frac{5\pi-8}{4}$

41 $\frac{\pi-2}{2}$

Basic Mathematical Concepts

In this part of the book, we prepared some mathematical concepts that hopefully help students to understand the main ideas of the book. By taking into account the different scientific levels of the students, it is necessary to present these concepts with some examples and figures. The necessity of this part is not limited to this course, but it is for other courses. I personally recommend the students to give this additional part a primary attention before starting the course, where the necessity of it is not limited to this course, but it is for other courses.

Mathematical Expressions

1. $\Rightarrow$ is the symbol for implying.
2. $\Leftrightarrow$ is the symbol for "$\Rightarrow$ and $\Leftarrow$". Also, the expression "iff" means if and only if .
3. $b > a$ means b is greater than a and $a < b$ means a is less than b.
4. $b \geq a$ means b is greater than or equal to a.

Sets of Numbers & Notations

1. Natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.
2. Whole numbers $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.
3. Integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
4. Rational numbers $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.
5. Irrational numbers $\mathbb{I} = \{x \mid x \text{ is a real number that is not rational}\}$.
6. Real numbers $\mathbb{R}$ contains all the previous sets.

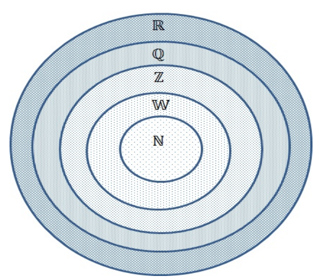


Figure A.1: Sets of Numbers.

Fractions Operations

• Adding or subtracting two fractions

To add or subtract two fractions, we do the following steps:

1. Find the least common denominator.
2. Write both original fractions as equivalent fractions with the least common denominator.
3. Add (or subtract) the numerators.
4. Write the result with the denominator.

Example A.14

(1) $\frac{2}{3} + \frac{4}{5} = \frac{10}{15} + \frac{12}{15} = \frac{10+12}{15} = \frac{22}{15}$

(2) $\frac{3}{7} + \frac{5}{7} = \frac{3+5}{7} = \frac{8}{7}$

(3) $\frac{4}{7} - \frac{1}{6} = \frac{24}{42} - \frac{7}{42} = \frac{24-7}{42} = \frac{14}{42} = \frac{7}{21}$

(4) $\frac{3}{7} - \frac{5}{7} = \frac{3-5}{7} = -\frac{2}{7}$

• Multiplying two fractions

To multiple two fractions, we do the following steps:

1. Multiply the numerator by the numerator.
2. Multiply the denominator by the denominator.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad \text{where} \;\; b \neq 0 \;\; \text{and} \;\; d \neq 0\,.$$

Example A.15

(1) $\frac{3}{4} \times \frac{2}{9} = \frac{3 \times 2}{4 \times 9} = \frac{6}{36} = \frac{1}{6}$

(2) $\frac{2}{5} \times \frac{-3}{7} = \frac{2 \times (-3)}{5 \times 7} = -\frac{6}{35}$

• Dividing two fractions

To divide two fractions, we do the following steps:

1. Find the multiplicative inverse of the second fraction.
2. Multiply the two fractions.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \quad \text{where } b \neq 0 \text{ and } d \neq 0\,.$$

■ Example A.16

(1) $\frac{2}{5} \div \frac{4}{9} = \frac{2}{5} \times \frac{9}{4} = \frac{2\times 9}{5\times 4} = \frac{18}{20}$

(2) $\frac{3}{7} \div \frac{-2}{5} = \frac{3}{7} \times \frac{5}{-2} = \frac{3\times 5}{7\times(-2)} = -\frac{15}{14}$

Logarithmic and Exponential Functions

■ The Natural Logarithmic Function

- The natural logarithmic function is defined as follows:

$$\ln : (0,\infty) \to \mathbb{R}\,,$$

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

for every $x > 0$.

- Some properties:

If $a, b > 0$ and $r \in \mathbb{Q}$, then

1. $\ln ab = \ln a + \ln b$.
2. $\ln \frac{a}{b} = \ln a - \ln b$.
3. $\ln a^r = r \ln a$.

- Differentiating the natural logarithmic function:

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\big(\ln|u|\big) = \frac{1}{u}\, u'\,.$$

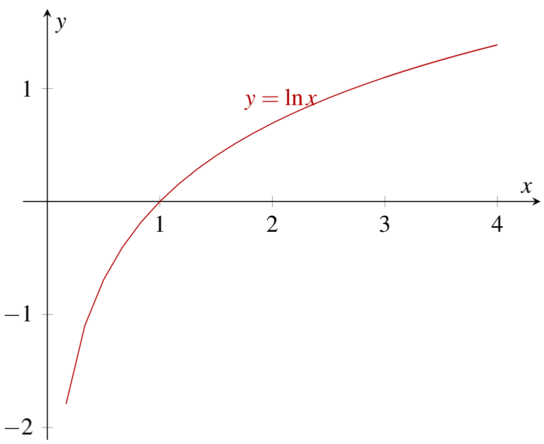


Figure A.2: The graph of the function $y = \ln x$.

■ Example A.17 Find the derivative of the function.

(1) $y = \ln(x^2+1)$

(2) $y = \ln\sqrt{x}$

Solution:

(1) $y' = \frac{2x}{x^2+1}$

(2) $y' = \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}} = \frac{1}{2x}$

■ Exponents

Assume n is a positive integer and a is a real number. The n^{th} power of a is

$$a^n = a\,.\,a\,.\,.\,.\,a\,.$$

■ The Natural Exponential Function

The natural exponential function is defined as follows:

$$exp : \mathbb{R} \longrightarrow (0,\infty)\,,$$

$$y = exp\; x \Leftrightarrow \ln y = x$$

- Some properties: If $a, b > 0$ and $r \in \mathbb{Q}$, then

1. $e^a e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$
3. $(e^a)^r = e^{ar}$

- Note that e^x and $\ln x$ are inverse functions, so

$$\ln e^x = x,\ \forall x \in \mathbb{R}\,, \text{ and } e^{\ln x} = x,\ \forall x \in (0,\infty).$$

- Differentiating the natural exponential function:

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx} e^u = e^u u'.$$

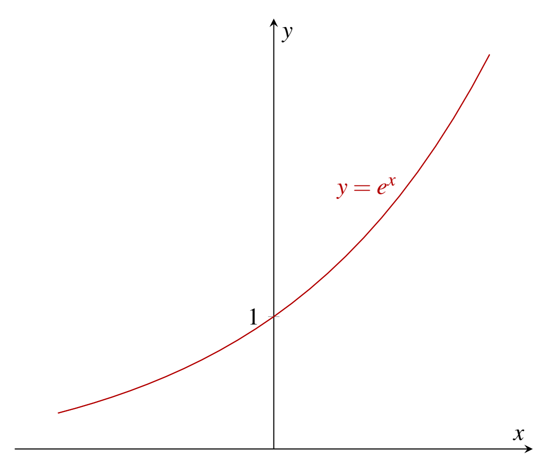


Figure A.3: The graph of the function $y = e^x$.

■ **Example A.18** Find the derivative of the function.

(1) $y = e^{\sqrt{x}}$

(2) $y = e^{\cos\ x}$

Solution:

(1) $y' = e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)$

(2) $y' = e^{\sin\ x} \cos\ x$

■ **General Exponential Function**

- The general exponential function is defined as follows:

$$a^x : \mathbb{R} \to (0,\infty)\ ,$$
$$a^x \ = \ e^{x \ln a} \ \text{ for every } \ a > 0.$$

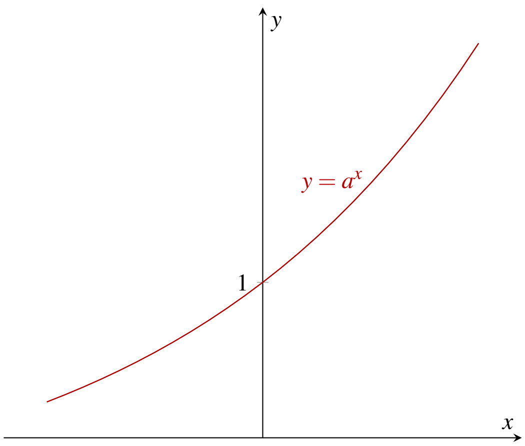


Figure A.4: The function $y = a^x$ for $a > 1$.

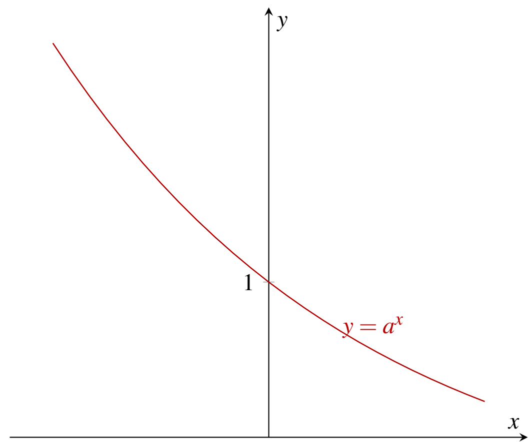


Figure A.5: The function $y = a^x$ for $a < 1$.

- Properties of the general exponential function:
For every $x, y > 0$ and $a, b \in \mathbb{R}$,

1. $x^0 = 1$

2. $x^a x^b = x^{a+b}$

3. $\frac{x^a}{x^b} = x^{a-b}$

4. $(x^a)^b = x^{ab}$

5. $(xy)^a = x^a y^a$

6. $x^{-a} = \frac{1}{x^a}$

■ **Example A.19**

(1) $2^4 2^{-7} = 2^{3-7} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

(2) $\frac{3^2}{3^{-2}} = 3^{2-(-2)} = 3^4 = 81$

(3) $(5x)^2 = 25x^2$

(4) $\frac{x^6 y^3}{(xyz)^5} = \frac{x^6 y^3}{x^5 y^5 z^5} = \frac{x^6}{x^5}\frac{y^3}{y^5}\frac{1}{z^5} = x^{6-5} y^{3-5}\frac{1}{z^5} = \frac{x}{y^2 z^5}$

- Differentiating the general exponential function:
If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(a^u\right) = a^u \ \ln a\ u'$$

■ **Example A.20** Find the derivative of the function.

(1) $y = 2^{\sqrt{x}}$

(2) $y = 3^{\tan\ x}$

Solution:

(1) $y' = 2^{\sqrt{x}}\ \ln 2\ \left(\frac{1}{2\sqrt{x}}\right)$

(2) $y' = 3^{\sin\ x}\ \ln 3\ \cos\ x$

■ **General Logarithmic Function**

- The general logarithmic function is defined as follows:

$$\log_a : (0,\infty) \to \mathbb{R}\ ,$$

$$x = a^y \Leftrightarrow y = \log_a x.$$

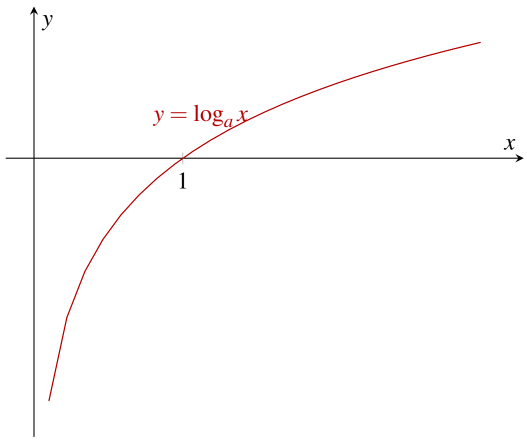


Figure A.6: The function $y = \log_a x$ for $a > 1$.

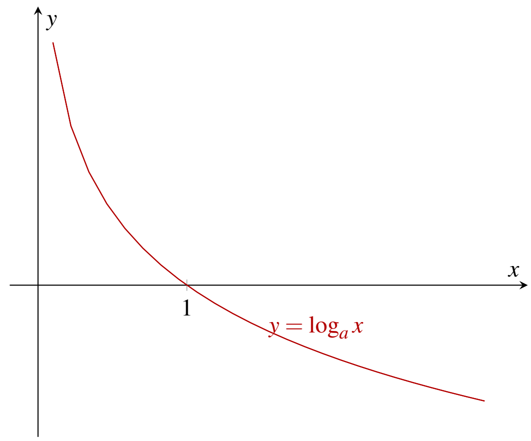


Figure A.7: The function $y = \log_a x$ for $a < 1$.

- Properties of general logarithmic function:
  If $x, y > 0$ and $r \in \mathbb{R}$, then
  1. $\log_a xy = \log_a x + \log_a y$
  2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
  3. $\log_a x^r = r \log_a x$
- Differentiating the general logarithmic function:
  If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

■ **Example A.21** Find the derivative of the function.

(1) $y = \log_2(x^2+1)$

(2) $y = \log_3 \sqrt{x}$

Solution:

(1) $y' = \frac{2x}{(x^2+1)\ln 2}$

(2) $y' = \frac{1}{\sqrt{x}\ln 3}\frac{1}{2\sqrt{x}} = \frac{1}{2x\ln 3}$

■ Algebraic Expressions

Let a and b be real numbers. Then,

1. $(a+b)^2 = a^2 + 2ab + b^2$

2. $(a-b)^2 = a^2 - 2ab + b^2$

3. $(a+b)(a-b) = a^2 - b^2$

4. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

5. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

6. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

7. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

8. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$

■ **Example A.22**

(1) $(x \pm 2)^2 = x^2 \pm 4x + 4$

(2) $x^2 - 25 = (x-5)(x+5)$

(3) $(x \pm 2)^3 = x^3 \pm 6x^2 + 12x \pm 8$

(4) $x^3 \pm 27 = (x \pm 3)(x^2 \mp 3x + 9)$

■ Absolute Value

The absolute value of x is defined as follows:

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

■ **Example A.23** $|2| = 2$, $|-2| = 2$, $|0| = 0$.

Equations and Inequalities

If $b > 0$,

1. $|x-a| = b \Leftrightarrow x = a-b$ or $x = a+b$.
2. $|x-a| < b \Leftrightarrow a-b < x < a+b$.
3. $|x-a| > b \Leftrightarrow x < a-b$ or $x > a+b$.

Example A.24 Solve for x.

(1) $|3x-4| = 7$ **(2)** $|2x+1| < 1$

Solution:

(1) $|3x-4| = 7 \Leftrightarrow 3x-4 = 7$ or $3x-4 = -7$. Thus, $x = \frac{11}{3}$ or $x = -1$.
(2) $|2x+1| < 1 \Leftrightarrow -1 < 2x+1 < 1$. By subtracting 1 and then dividing by 2, we have $-1 < x < 0$.

Functions

A function $f : D \to S$ is a mapping that assigns each element in D to an element in S. The set D is called the domain of the function f. All values of $f(x)$ belong to a set $R \subseteq S$ called the range.

• Domains and Ranges

In the following, we show the domain and range of some functions:

1. Polynomials $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.
Domain: $\mathbb{R}$ Range: $\mathbb{R}$
2. Square Roots $f(x) = \sqrt{g(x)}$.
Domain: $\forall x \in \mathbb{R}$ such that $g(x) \geq 0$ Range: $\mathbb{R}^+$
3. Rational Functions $q(x) = \frac{f(x)}{g(x)}$.
To determine the domain, we need to find the intersection of the domains of f and g. Then, we remove zeros of the function g.

Example A.25 Find the domain of the function.

(1) $f(x) = \sqrt{x-1}$
(2) $q(x) = \frac{x+1}{2x-1}$
(3) $q(x) = \frac{3x^2+x+2}{\sqrt{x+2}}$

Solution:

(1) We need to find all $x \in \mathbb{R}$ such that $x-1 \geq 0$. By solving the inequality, we have $x-1 \geq 0 \Rightarrow x \geq 1$. Thus, the domain is $[1,\infty)$. Now, $\forall x \in D(f)$, $f(x) = \sqrt{g(x)} \geq 0$ i.e., the range is $[0,\infty)$.
(2) The domain of the numerator and the denominator is $\mathbb{R}$. The denominator $g(x) = 0$ if $x = \frac{1}{2}$. Thus, the domain is $\mathbb{R} \setminus \{\frac{1}{2}\}$.
(3) The domain of the numerator is $\mathbb{R}$, but the domain of the denominator is $[-2,\infty)$. Also, the denominator $g(x) = 0$ if $x = -2$. Thus, the domain is $(-2,\infty)$.

• Operations on Functions

Let f and g be two functions, then

1. $(f \pm g)(x) = f(x) \pm g(x)$.
2. $(fg)(x) = f(x)g(x)$.
3. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$.

Example A.26 If $f(x) = x^2 - 1$ and $g(x) = x-1$, find the following:

(1) $(f+g)(x)$ **(2)** $(fg)(x)$ **(3)** $(\frac{f}{g})(x)$

Solution:

(1) $(f+g)(x) = f(x)+g(x) = (x^2-1)+(x-1) = x^2+x-2$.
(2) $(fg)(x) = f(x)g(x) = (x^2-1)(x-1) = x^3-x^2-x+1$.
(3) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1$.

• Composition of Functions

Let f and g be two functions. The composition of the two functions is $(f \circ g)(x) = f(g(x))$ where $D(f \circ g) = \{g(x) \in D(f) \forall x \in D(g)\}$.

Example A.27 If $f(x) = x^2$ and $g(x) = x+2$, find $(f \circ g)(x)$.

Solution:

$(f \circ g)(x) = f(g(x)) = \left(x+2\right)^2 = x^2+4x+4$.

• Inverse Functions

A function f has an inverse function f^{-1} if it is one to one: $y = f^{-1}(x) \Leftrightarrow x = f(y)$.[1]

[1] The -1 in f^{-1} is not exponent where $\frac{1}{f(x)}$ is written as $\left(f(x)\right)^{-1}$.

Properties of inverse functions:

1. $D(f^{-1})$ is the range of f.

2. The range of f^{-1} is the domain of f.

3. $f^{-1}\big(f(x)\big) = x, \forall x \in D(f)$.

4. $f\big(f^{-1}(x)\big) = x, \forall x \in D(f^{-1})$.

5. $(f^{-1})^{-1}(x) = f(x), \forall x \in D(f)$.

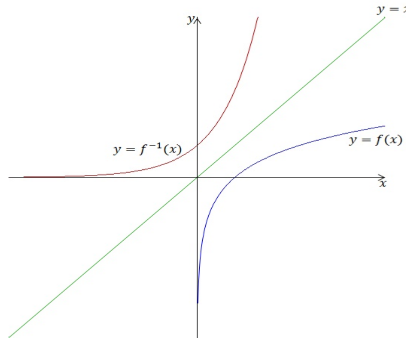


Figure A.8: Inverse functions.

• Even and Odd Functions

Let f be a function and $-x \in D(f)$.

1. If $f(-x) = -f(x)\ \forall x \in D(f)$, the function f is odd.
2. If $f(-x) = f(x)\ \forall x \in D(f)$, the function f is even.

■ Example A.28

(1) The function $f(x) = 2x^3 + x$ is odd because $f(-x) = 2(-x)^3 + (-x) = -2x^3 - x = -(2x^3 + x) = -f(x)$.
(2) The function $f(x) = x^4 + 3x^2$ is even because $f(-x) = (-x)^4 + 3(-x)^2 = x^4 + 3x^2 = f(x)$.

Roots of Linear and Quadratic Equations

• Linear Equations

A linear equation can be written in the form $ax+b=0$ where x is the unknown, and $a, b \in \mathbb{R}$ and $a \neq 0$. To solve the equation, subtract b from both sides and then divide the result by a:

$$ax+b = 0 \Rightarrow ax+b-b = 0-b \Rightarrow ax = -b \Rightarrow x = \frac{-b}{a} .$$

■ Example A.29 Solve for x the equation $x+2=5$.

Solution:

$$3x+2 = 5 \Rightarrow 3x = 5-2 \Rightarrow 3x = 3 \Rightarrow x = \frac{3}{3} = 1 .$$

• Quadratic Equations

A quadratic equation can be written in the form $ax^2+bx+c=0$ where a, b, and c are constants and $a \neq 0$. The quadratic equations can be solved by using the factorization method, the quadratic formula, or the completing the square.

Factorization Method

The factorization method depends on finding factors of c that add up to b. Then, we use the fact that if $x, y \in \mathbb{R}$, then

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0 .$$

■ Example A.30 Solve for x the following quadratic equations:

(1) $x^2+2x-8=0$
(2) $x^2+5x+6=0$

Solution:

(1) $a=1, b=2$ and $c=-8$. by factoring c, we have $c = 2\times(-4)$ or $c = -2\times 4$. However, $b \neq (-2)+4$, so we consider 2 and -4. Thus,

$$x^2+2x-8 = (x-2)(x+4) = 0 \Rightarrow x-2 = 0 \ \text{ or } \ x+4 = 0 \Rightarrow x = 2 \ \text{ or } \ x = -4 .$$

(2) By factoring the left side, we have

$$(x+2)(x+3) = 0 \Rightarrow x+2 = 0 \ \text{ or } \ x+3 = 0 \Rightarrow x = -2 \ \text{ or } \ x = -3 .$$

Quadratic Formula Solutions

We can solve the quadratic equations by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} .$$

Remark: The expression b^2-4ac is called the discriminant of the quadratic equation $ax^2+bx+c=0$.

1. If $b^2-4ac > 0$, the quadratic equation has two distinct real solutions.

2. If $b^2-4ac = 0$, the quadratic equation has one distinct real solution.

3. If $b^2-4ac<0$, the quadratic equation has no real solutions.

■ **Example A.31** Solve for x the following quadratic equations:

(1) $x^2+2x-8=0$
(2) $x^2+2x+1=0$
(3) $x^2+2x+8=0$

Solution:

(1) $a=1,\ b=2,\ c=-8$. Since $b^2-4ac=2^2-4(1)(-8)>0$, then there are two solutions $x=2$ and $x=-4$.
(2) $a=1,\ b=2,\ c=1$. Since $b^2-4ac=2^2-4(1)(1)=0$, then there is one solution $x=-1$.
(3) $a=1,\ b=2,\ c=8$. Since $b^2-4ac=2^2-4(1)(8)<0$, then there are no real solutions.

Completing the Square Method
To solve the quadratic equation by the completing the square method, we need to do the following steps:
Step 1: Divide all terms by a (the coefficient of x^2).
Step 2: Move the term $(\frac{c}{a})$ to the right side of the equation.
Step 3: Complete the square on the left side of the equation and balance this by adding the same value to the right side.
Step 4: Take the square root of both sides and subtract the number that remains on the left side.

■ **Example A.32** Solve for x the quadratic equation $x^2+2x-8=0$.
Solution: $a=1,\ b=2,\ c=-8$.
Step 1 can be skipped in this example since $a=1$.
Step 2: $x^2+2x=8$.
Step 3: To complete the square, we need to add $(\frac{b}{2})^2$ since $a=1$.

$$x^2+2x+1=8+1 \Rightarrow (x+1)^2=9\ .$$

Step 4: $x+1=\pm 3 \Rightarrow x=\pm 3-1 \Rightarrow x=2$ or $x=-4$.

Systems of Equations

A system of equations consists of two or more equations with the same set of unknowns. The equations in the system can be linear or non-linear, but for the purpose of this book, we only consider the linear ones.
Consider a system of two equations in two unknowns x and y

$$ax+by=c$$

$$dx+ey=f\ .$$

To solve the system, we try to find values of the unknowns that will satisfy each equation in the system. To do this, we can use elimination or substitution.

■ **Example A.33** Solve the following system of equations:

$$x-3y=4 \rightarrow (1)$$

$$2x+y=6 \rightarrow (2)$$

Solution:
• By using the elimination method.
Multiply equation (2) by 3, then add the result to equation (1). This implies $7x=22 \Rightarrow x=\frac{22}{7}$. Substitute the value of x into the first or the second equation to obtain $y=-\frac{2}{7}$.
• By using the substitution method.
From the first equation, we have $x=4+3y$. By substituting that into the second equation, we obtain

$$2(4+3y)+y=6 \Rightarrow 7y+8=6 \Rightarrow y=-\frac{2}{7}$$

Substitute value of y into $x=4+3y$ to have $x=\frac{22}{7}$.

Pythagorean Theorem

If c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, the Pythagorean theorem can be expressed as follows:

$$a^2+b^2=c^2 \text{ or } c=\sqrt{a^2+b^2}.$$

If a and c are known and b is unknown, then

$$b=\sqrt{c^2-a^2}.$$

Similarly, if b and c are known and a is unknown, then

$$a=\sqrt{c^2-b^2}$$

The trigonometric functions for a right triangle are

$$\cos\theta=\frac{a}{c} \quad \sin\theta=\frac{b}{c} \quad \tan\theta=\frac{b}{a}$$

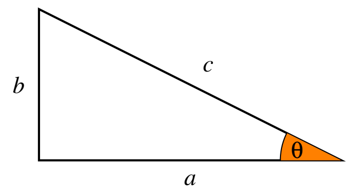


Figure A.9
a is adjacent to the angle θ
b is opposite
c is hypotenuse

■ **Example A.34** Find value of x. Then find $\cos\theta$, and $\sin\theta$.

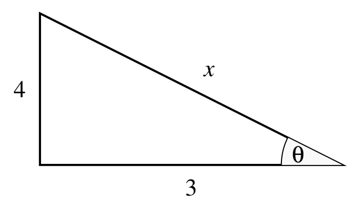


Figure A.10

Solution:
$a=3,\ b=4 \Rightarrow x^2=4^2+3^2=25 \Rightarrow x=5$
$\cos\theta=\frac{3}{5}$
$\sin\theta=\frac{4}{5}$

Trigonometric Functions

• If (x,y) is a point on the unit circle, and if the ray from the origin $(0,0)$ to that point (x,y) makes an angle θ with the positive x-axis, then

$$\cos\theta=x, \quad \sin\theta=y,$$

• Each point (x,y) on the unit circle can be written as $(\cos\theta,\sin\theta)$.
• Since $x^2+y^2=1$, then $\cos^2\theta+\sin^2\theta=1$. Therefore,

$$1+\tan^2\theta=\sec^2\theta \text{ and } \cot^2\theta+1=\csc^2\theta.$$

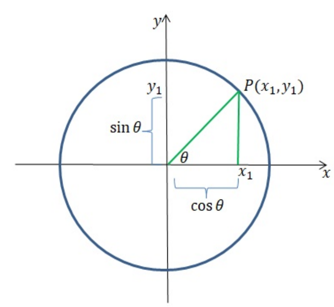


Figure A.11: Trigonometric functions.

Also,

$$\tan\theta=\frac{\sin\theta}{\cos\theta} \quad \cot\theta=\frac{\cos\theta}{\sin\theta} \quad \sec\theta=\frac{1}{\cos\theta} \quad \csc\theta=\frac{1}{\sin\theta}$$

• Trigonometric functions of negative angles

$$\cos(-\theta)=\cos(\theta), \quad \sin(-\theta)=-\sin(\theta), \quad \tan(-\theta)=-\tan(\theta)$$

• Double and half angle formulas

$$\sin 2\theta=2\sin\theta\cos\theta, \quad \cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}, \quad \cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2}$$

• Angle addition formulas

$$\sin(\theta_1\pm\theta_2)=\sin\theta_1\cos\theta_2\pm\cos\theta_1\sin\theta_2$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}$$

• Values of trigonometric functions of most commonly used angles

Degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Table A.1

• Graphs of trigonometric functions

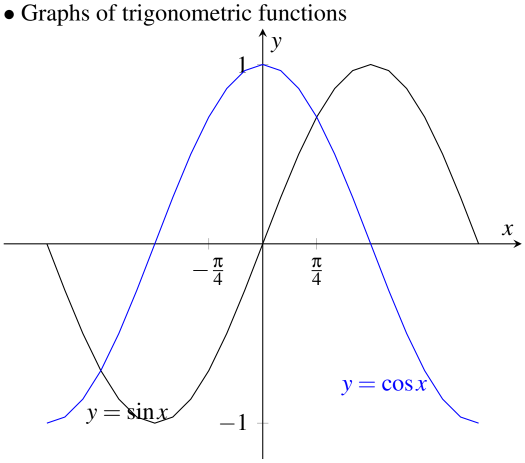


Figure A.12: The graphs of $\sin x$ and $\cos x$.

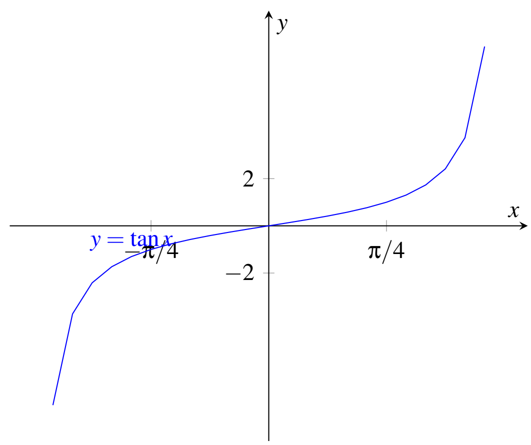


Figure A.13: The graph of $\tan x$.

Distance Formula

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the Cartesian plane. The distance between P_1 and P_2 is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \,.$$

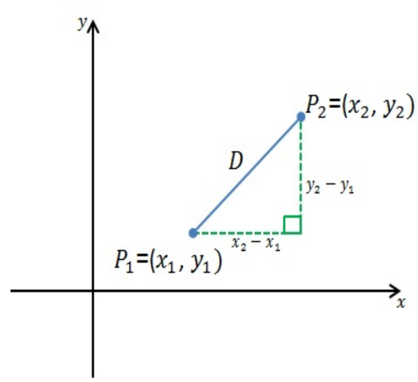


Figure A.14: The distance between two points.

Example A.35 Find the distance between the two points $P_1(1,1)$ and $P_2(-3,4)$.

Solution:

$$D = \sqrt{(-3-1)^2 + (4-1)^2} = \sqrt{16+9} = \sqrt{25} = 5 \,.$$

Differentiation of Functions

Differentiation Rules

$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$

$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

Elementary Derivatives

$\frac{d}{dx}x^r = rx^{r-1}$ $\qquad$ $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$ $\qquad$ $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Derivative of Composite Functions (Chain Rule)

Let $y = f(u)$ and $u = g(x)$ such that dy/du and du/dx exist. Then, the derivative of the composite function $(f \circ g)(x)$ exists and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)g'(x) = f'\big(g(x)\big)g'(x) \; .$$

■ Derivative of Inverse Functions

If a function f has an inverse function f^{-1}, then $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$.

■ Graphs of Functions

• The First and Second Derivative Test

1. Let f be continuous on $[a,b]$ and f' exists on (a,b).
- If $f'(x) > 0, \forall x \in (a,b)$, then f is increasing on $[a,b]$.
- If $f'(x) < 0, \forall x \in (a,b)$, then f is decreasing on $[a,b]$.

2. Let f be continuous at a critical number c and differentiable on an open interval (a,b), except possibly at c.
- $f(c)$ is a local maximum of f if f' changes from positive to negative at c.
- $f(c)$ is a local minimum of f if f' changes from negative to positive at c.

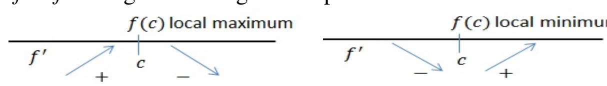


Figure A.15: The local maximum and minimum value of the function f.

3. If f'' exists on an open interval I,
- the graph of f is concave upward on I if $f''(x) > 0$ on I.
- the graph of f is concave downward on I if $f''(x) < 0$ on I.

• Shifting Graphs

Let $y = f(x)$ is be function.

1. Replacing each x in the function with $x - c$ shifts the graph c units horizontally.
- If $c > 0$, the shift will be to the right.
- If $c < 0$, the shift will be to the left.

2. Replacing y in the function with $y - c$ shifts the graph c units vertically.
- If $c > 0$, the shift will be upward.
- If $c < 0$, the shift will be downward.

• Symmetry about the y-axis and the origin

1. If the function f is odd, the graph of f is symmetric about the origin.

2. If the function f is even, the graph of f is symmetric about the y-axis.

• Lines

The general linear equation in two variables x and y can be written in the form:

$$ax + by + c = 0 \; ,$$

where a, b and c are constants with a and b not both 0.

■ Example A.36

$2x + y = 4$

$a = 2,\; b = -1,\; c = -4$

To plot the line, we rewrite the equation to become $y = -2x + 4$. Then, we use the following table to make points on the plane:

x	0	2
y	4	0

The line $2x + y = 4$ passes through the points $(0,4)$ and $(2,0)$.

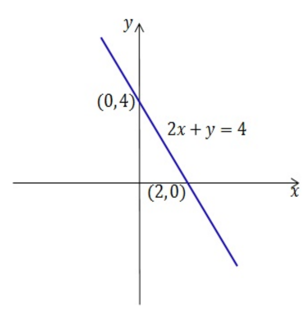


Figure A.16: The line $2x + y = 4$.

Slope

1. The slope of a line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2. Point-Slope form: $y - y_1 = m(x - x_1)$.

3. Slope-Intercept form:
If $b \neq 0$, the general linear equation can be rewritten as

$$ax+by+c=0 \Rightarrow by=-ax-c \Rightarrow y=-\frac{a}{b}x-\frac{c}{b} \Rightarrow y=mx+d\ ,$$

where m is the slope.

■ **Example A.37** Find the slope of the line $2x-5y+9=0$.
Solution: $2x-5y+9=0 \Rightarrow -5y=-2x-9 \Rightarrow y=\frac{2}{5}x+\frac{9}{5}$.
Thus, the slope is $\frac{2}{5}$. Alternatively, take any two points on that line say $(-2,1)$ and $(3,3)$. Then,

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{3-(-2)}=\frac{2}{5}\ .$$

Special cases of lines in a plane

1. If m is undefined, the line is vertical.

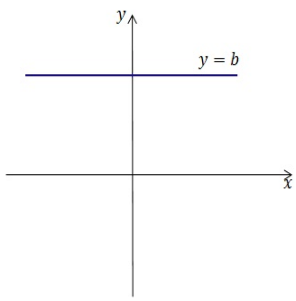


Figure A.17

2. If $m=0$, the line is horizontal.

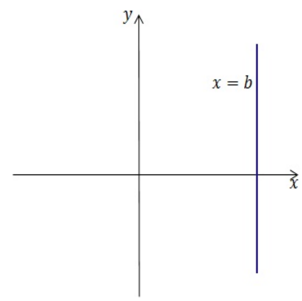


Figure A.18

3. Let L_1 and L_2 be two lines in a plane, and let m_1 and m_2 be the corresponding slopes, respectively.

• If L_1 and L_2 are parallel, then $m_1=m_2$.

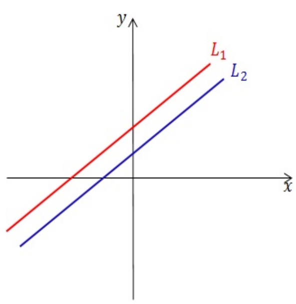


Figure A.19

• If L_1 and L_2 are vertical, then $m_1=\frac{-1}{m_2}$.

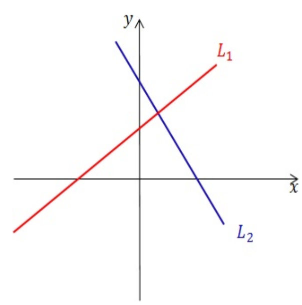
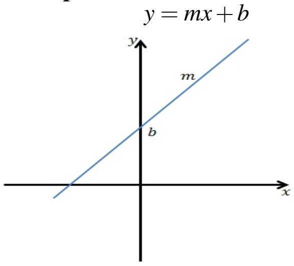


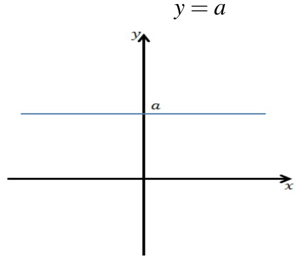
Figure A.20

• **Graph of Some Functions**

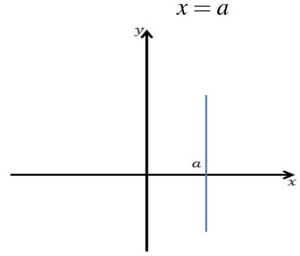
$y=mx+b$



$y=a$



$x=a$



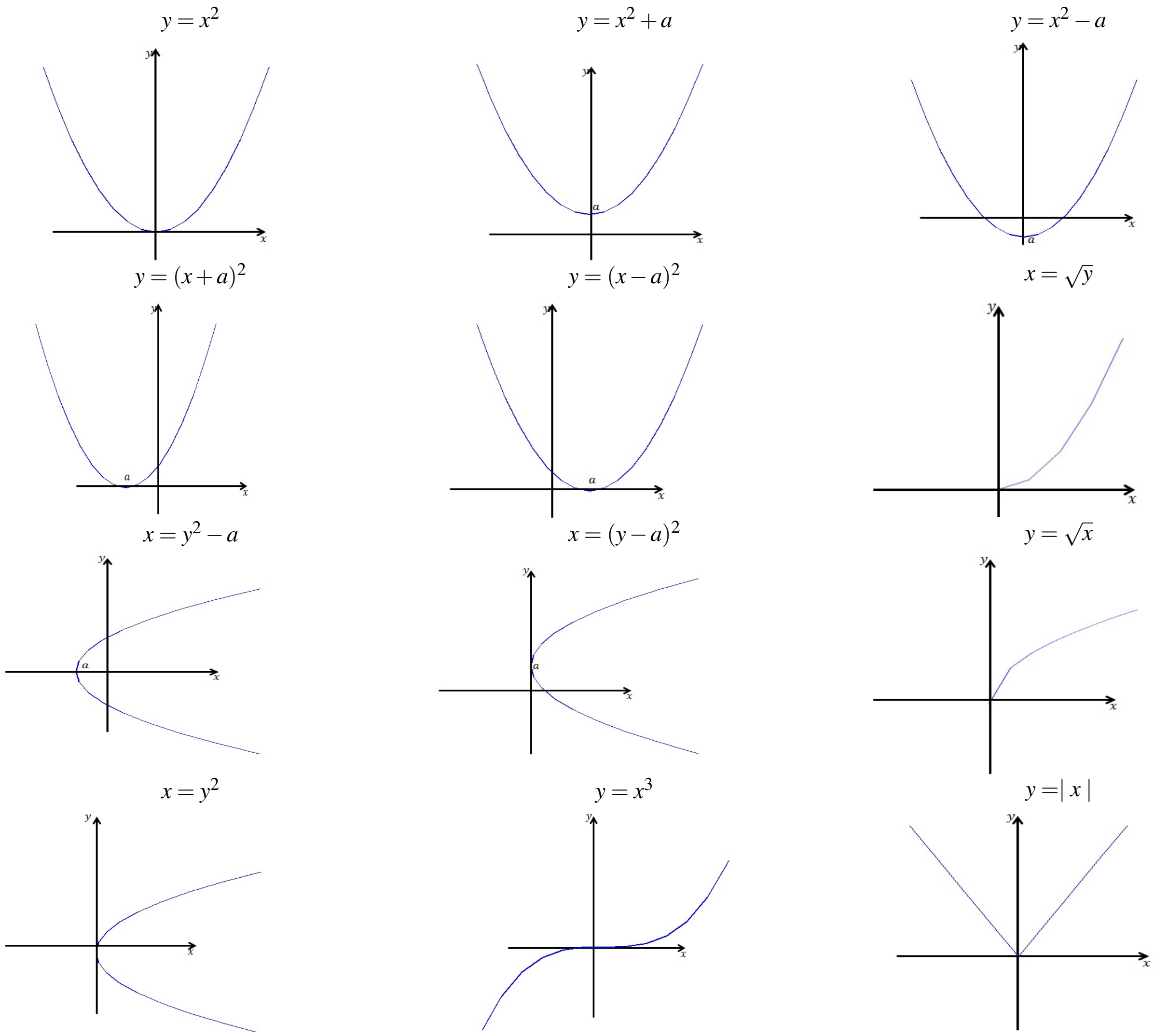
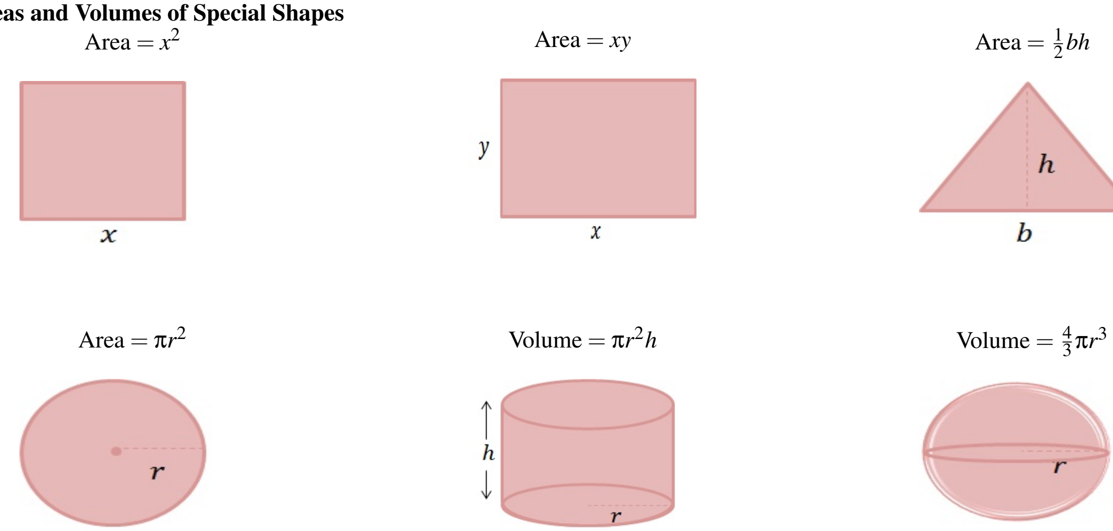


Figure A.21

Areas and Volumes of Special Shapes



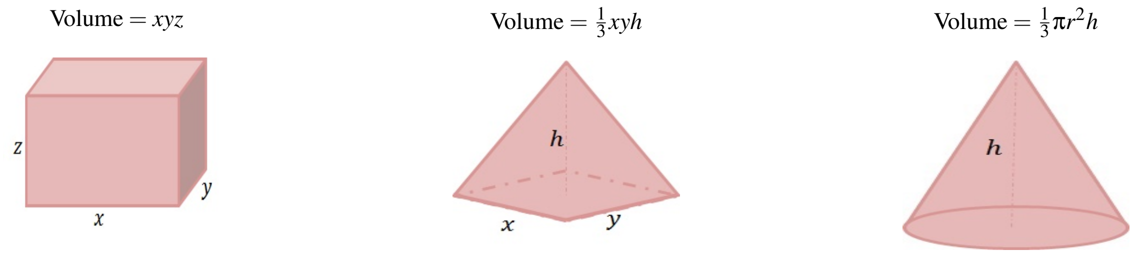


Figure A.22