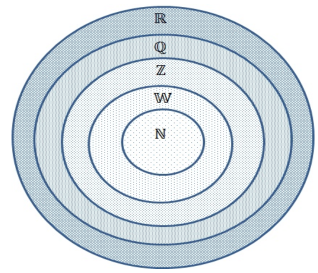
➤ Mathematical Expressions

1. $\Rightarrow$ is the symbol for implying.
2. $\Leftrightarrow$ is the symbol for "$\Rightarrow$ and $\Leftarrow$. Also, the expression "iff" means if and only if .
3. $b > a$ means b is greater than a and $a < b$ means a is less than b.
4. $b \geq a$ to denote that b is greater than or equal to a.

➤ Set of Numbers & Notation



1. Natural numbers: $\mathbb{N} = \{1,2,3,...\}$
2. Whole numbers: $\mathbb{W} = \{0,1,2,3,...\}$
3. Integers: $\mathbb{Z} = \{...,-3,-2,-1,0,1,2,3,...\}$
4. Rational numbers: $\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z} \text{ and } b \neq 0\}$
5. Irrational numbers: $\mathbb{I} = \{x \mid x \text{ is a real number that is not rational}\}$
6. Real numbers: $\mathbb{R}$ contains all the previous sets.

➤ Fraction Operations

• Adding (or subtracting) two fractions:

1. Find the least common denominator.
2. Write both original fractions as equivalent fractions with the least common denominator.
3. Add (or subtract) the numerators.
4. Write the result with the denominator.

• Multiplying two fractions:

1. Multiply the numerator by the numerator.
2. Multiply the denominator by the denominator.

$$\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd} \quad textwhere \ b \neq 0 \text{ and } d \neq 0 \, .$$

• Dividing two fractions:

1. Change the division sign to multiplication.
2. Invert the second fraction and multiply the fractions.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}.\frac{d}{c} = \frac{ac}{bd} \quad textwhere \ b \neq 0 \text{ and } d \neq 0 \, .$$

Example 1

1. $\frac{3}{7} + \frac{2}{5} = \frac{15+14}{35} = \frac{29}{35}$
2. $\frac{4}{9} - \frac{3}{7} = \frac{28-27}{63} = \frac{1}{63}$
3. $\frac{2}{5} \times \frac{4}{9} = \frac{2\times4}{5\times9} = \frac{8}{45}$
4. $\frac{2}{5} \div \frac{4}{9} = \frac{2}{5} \times \frac{9}{4} = \frac{2\times9}{5\times4} = \frac{18}{20}$

➤ Exponents

Assume n is a positive integer and a is a real number. The expression a^n is given by

$$a^n = a.a...a \, .$$

Basic Rules:

For every $x,y > 0$ and $a,b \in \mathbb{R}$, then

1. $x^0 = 1$.
2. $x^a x^b = x^{a+b}$.
3. $\frac{x^a}{x^b} = x^{a-b}$.
4. $(x^a)^b = x^{ab}$.
5. $(xy)^a = x^a y^a$.
6. $x^{-a} = \frac{1}{x^a}$.

Example 2

1. $2^3 2^{-5} = 2^{3-5} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
2. $\frac{3^2}{3^{-2}} = 3^{2-(-2)} = 81$
3. $(5x)^2 = 25x^2$
4. $\frac{x^2y^3}{(yz)^5} = \frac{x^2y^3}{y^5z^5} = \frac{x^2y^{3-5}}{z^5} = \frac{x^2}{y^2z^5}$

➤ Algebraic Expressions

Let a and b be real numbers. Then,

1. $(a+b)^2 = a^2+2ab+b^2$
2. $(a-b)^2 = a^2-2ab+b^2$
3. $(a+b)(a-b) = a^2-b^2$
4. $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
5. $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$
6. $a^3+b^3 = (a+b)(a^2-ab+b^2)$
7. $a^3-b^3 = (a-b)(a^2+ab+b^2)$
8. $a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1})$

Example 3

1. $(x\pm 2)^2 = x^2 \pm 4x+4$
2. $x^{-}25 = (x-5)(x+5)$
3. $(x\pm 2)^3 = x^3 \mp 6x^2+12x\pm 8$
4. $x^3 \pm 27 = (x\pm 3)(x^2 \pm 3x+9)$

➤ Intervals

Let $a, b \in \mathbb{R}$ and $a < b$.

• Open interval (a,b).
It contains on all real numbers between a and b, i.e.,

$$x \in (a,b) \Leftrightarrow a < x < b$$

• Closed interval $[a,b]$.
It contains on all real numbers between a and b including a and b, i.e.,

$$x \in [a,b] \Leftrightarrow a \le x \le b$$

• Half-open interval $(a,b]$.
It contains on all real numbers between a and b including b, i.e.,

$$x \in (a,b] \Leftrightarrow a < x \le b$$

• Half-open interval $[a,b)$.
It contains on all real numbers between a and b including a, i.e.,

$$x \in [a,b) \Leftrightarrow a \le x < b$$

• $[a,\infty)$
It contains on all real numbers larger or equal a, i.e.,

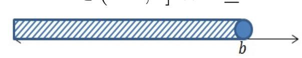
$$x \in [a,\infty) \Leftrightarrow a \le x$$

• (a,∞)
It contains on all real numbers larger than a, i.e.,

$$x \in (a,\infty) \Leftrightarrow a < x$$

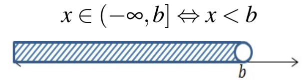
• $(-\infty,b]$
It contains on all real numbers less or equal b, i.e.,

$$x \in (-\infty,b] \Leftrightarrow x \le b$$



• $(-\infty,b)$
It contains on all real numbers less b, i.e.,

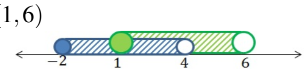
$$x \in (-\infty,b] \Leftrightarrow x < b$$



Example 4

1. $(2,5]$

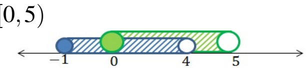
2. $[-2,4) \cap [1,6)$



$(-2,6)$

3. $[-1,\infty)$

4. $[-1,4) \cup [0,5)$



$[-1,5)$

➤ Absolute Value

The absolute value of x is defined as follows:

$$|x| = \begin{cases} x & : x \ge 0 \\ -x & : x < 0 \end{cases}$$

Example 5 $|2| = 2$, $|-2| = 2$, $|0| = 0$

Equations and Inequalities

If $b > 0$,

1. $|x-a| = b \Leftrightarrow x = a-b$ or $x = a+b$
2. $|x-a| < b \Leftrightarrow a-b < x < a+b$
3. $|x-a| > b \Leftrightarrow x < a-b$ or $x > a+b$

Example 6 Solve the following:

1. $|3x-4| = 7$
2. $|2x+1| < 1$

Solution:

1. $|3x-4| = 7 \Leftrightarrow 3x-4 = 7$ or $3x-4 = -7$. Thus, $x = \frac{11}{3}$ or $x = -1$.
2. $|2x+1| < 1 \Leftrightarrow -1 < 2x+1 < 1$. By subtracting 1 and then dividing by 2, we have $-1 < x < 0$.

Functions

A function $f : D \to S$ is a mapping that assigns each element in D to an element in S. The set D is called the domain of the function f. All values of $f(x)$ belong to a set $R \subseteq S$ called the range.

• Determine Domains and Ranges

In the following, we show how to determine the domain and range of some functions.

1. Polynomials $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

   Domain: $\mathbb{R}$ Range: $\mathbb{R}$

2. Square Roots $f(x) = \sqrt{g(x)}$

   Domain: $in\mathbb{R}$ such that $g(x) \geq 0$ Range: $\mathbb{R}^+$

   Example 7 $f(x) = \sqrt{x-1}$. We need to find all x such that $x-1 \geq 0$. By solving the inequality, we have $x-1 \geq 0 \Rightarrow x \geq 1$. Thus, the domain is $[1,\infty)$. Since the function $f(x) = \sqrt{g(x)} \geq 0$, then the range is $[0,\infty)$.

3. Rational Functions $q(x) = \frac{f(x)}{g(x)}$

   For the domain, we need to find the intersection the domains of $f(x)$ and $g(x)$ and avoid zeros of the function $g(x)$.

   Example 8 Find the domain of the following functions:

   (a) $f(x) = \frac{x+1}{2x-1}$.

   (b) $f(x) = \frac{3x^2+x+2}{\sqrt{x+2}}$.

   Solution:

   (a) The domain of the numerator is $\mathbb{R}$ and the denominator is $\mathbb{R}\setminus\left\{\frac{1}{2}\right\}$. The domain is the intersection of the two domains $\mathbb{R}\setminus\left\{\frac{1}{2}\right\}$.

   (b) The domain of the numerator is $\mathbb{R}$ and the denominator is $[-2,\infty)$. The domain is the intersection of the two domains $[-2,\infty)$.

• Functions Operations

Let f and g be functions such that x belongs to their domains. Then

1. $(f \pm g)(x) = f(x) \pm g(x)$
2. $(fg)(x) = f(x)g(x)$

3. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Example 9 If $f(x) = x^2 - 1$ and $g(x) = x - 1$, find the following

1. $(f+g)(x)$
2. $(fg)(x)$
3. $(\frac{f}{g})(x)$

Solution:

1. $(f+g)(x) = f(x)+g(x) = (x^2-1)+(x-1) = x^2+x-2$
2. $(fg)(x) = f(x)g(x) = (x^2-1)(x-1) = x^3-x^2-x+1$
3. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1$

• Composite Functions

If f and g are two functions, the composite function $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is $\{\forall x in D(g) : g(x) \in D(f)\}$.

Example 10 If $f(x) = x^2$ and $g(x) = x+2$, find $(f \circ g)(x)$.

Solution:

$(f \circ g)(x) = f(g(x)) = \big(x+2\big)^2 = x^2+4x+4.$

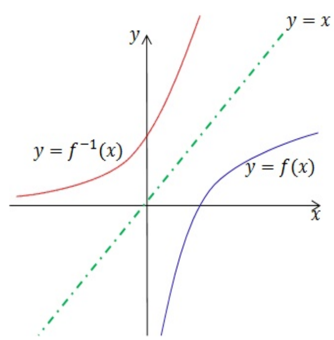
• Inverse Functions

A function f has an inverse function f^{-1} if it is one to one:

$$y = f^{-1}(x) \Leftrightarrow x = f(y)$$

Properties of inverse functions:

1. $D(f^{-1})$ is the range of f.
2. The range of $D(f^{-1})$ is the domain of f.
3. $f^{-1}\big(f(x)\big) = f(x), \forall x \in D(f)$.
4. $f\big(f^{-1}(x)\big) = f(x), \forall x \in D(f^{-1})$.
5. $(f^{-1})^{-1}(x) = f(x) \forall x in D(f)$.



• **Even and Odd Functions**

Let f be a function and $-x \in D(f)$.

1. If $f(-x) = -f(x)$ $\forall x \in D(f)$, the function f is odd.
2. If $f(-x) = f(x)$ $\forall x \in D(f)$, the function f is even.

Example 11

1. The function $f(x) = 2x^3+x$ is odd because $f(-x) = 2(-x)^3+(-x) = -2x^3-x = -(2x^3+x) = -f(x)$.
2. The function $f(x) = x^4+3x^2$ is even because $f(-x) = (-x)^4+3(-x)^2 = x^4+3x^2 = f(x)$.

Roots of Linear and Quadratic Equations

• Linear Equations

A linear equation is an equation that can be written in the form

$$ax+b = 0\ .$$

x is unknown variable and $a, b \in \mathbb{R}$ where $a \neq 0$. To solve the equation, we subtract b from both sides and then divide the result by a:

$$ax+b = 0 \Rightarrow ax+b-b = 0-b \Rightarrow ax = -b \Rightarrow x = \frac{-b}{a}\ .$$

Example 12 Solve the following equation $x+2=5$.

Solution:

$$3x+2=5 \Rightarrow 3x=5-2 \Rightarrow 3x=3 \Rightarrow x=\frac{3}{3}=1\ .$$

• Quadratic Equations

A quadratic equation is an equation that can be written in the form

$$ax^2+bx+c=0\ ,$$

where a, b, and c are constants and $a \neq 0$. The quadratic equations are solved either by factorization method, by the quadratic formula, or by completing the square.

Factorization Method

The factorization method depends on finding factors of c that add up to b. Then, we use the fact that if $x,y \in \mathbb{R}$, then $xy=0 \Rightarrow x=0$ or $y=0$.

Example 13 Solve the following equations

1. $x^2+2x-8=0$
2. $x^2+5x+6=0$

Solution:

1. Note that, $2\times(-4)=-8=c$, but $2+(-4)=-2\neq b$. Since $-2\times 4=-8=c$ and $-2+4=2=b$, then

$$(x-2)(x+4)=0 \Rightarrow x-2=0 \ \text{ or } \ x+4=0 \Rightarrow x=2 \ \text{ or } \ x=-4\ .$$

2. By factoring the left side, we have

$$(x+2)(x+3)=0 \Rightarrow x+2=0 \ \text{ or } \ x+3=0 \Rightarrow x=-2 \ \text{ or } \ x=-3\ .$$

Quadratic Formula Solutions

We can solve the quadratic equations by the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\ .$$

Remark: The expression b^2-4ac is called the discriminant of the quadratic equation.

1. If $b^2-4ac>0$, then the equation has two distinct solutions.
2. If $b^2-4ac=0$, then the equation has one distinct solution.
3. If $b^2-4ac<0$, then the equation has no real solutions.

Example 14 Solve the following quadratic equations:

1. $x^2+2x-8=0$
2. $x^2+2x+1=0$
3. $x^2+2x+8=0$

Solution:

1. $a=1,\ b=2,\ c=-8 \Rightarrow x=\frac{-2\pm\sqrt{4+32}}{2}=\frac{-2\pm 6}{2}$. The solution are $x=2$ and $x=-4$.
2. $a=1,\ b=2,\ c=1$. Since $b^2-4ac=2^2-4(1)(1)=0$, then there is one solution $x=-1$.
3. Since $b^2-4ac=2^2-4(1)(8)<0$, then there is no real solutions.

Completing the Square Method

To solve a quadratic equation by the completing the square method, you need to do the following steps:

Step 1: Divide all terms by a (the coefficient of x^2).

Step 2: Move the term $(\frac{c}{a})$ to the right side of the equation.

Step 3: Complete the square on the left side of the equation and balance this by adding the same value to the right side.

Step 4: Take the square root on both sides and subtract the number that remains on the left side.

Example 15 Solve the following quadratic equation $x^2+2x-8=0$.

Solution: Step 1 can be skipped in this example since $a=1$.

Step 2: $x^2+2x=8$.

Step 3: To complete the square, we need to add $(\frac{b}{2})^2$ since $a=1$.

$$x^2+2x+1=8+1 \Rightarrow (x+1)^2=9$$

Step 4: $x+1=\pm 3 \Rightarrow x=\pm 3-1 \Rightarrow x=2$ or $x=-4$.

Systems of Equations

A system of equations consists of two or more equations with a same set of unknowns. The equations in the system can be linear or non-linear, but for the purpose of this book, we consider the linear ones.

Consider systems consist of two equations where a system of two equations in two unknowns x and y can be written as

$$ax+by=c$$

$$dx+ey=f\ .$$

To solve the system, we try to find values for each of the unknowns that will satisfy every equation in the system. Students can use elimination or substitution.

Example 16 Solve the following system of equations:

$$x-3y=4$$
$$2x+y=6\ .$$

Solution:

• Use elimination:

By multiplying the second equation by 3

$$x-3y=4$$
$$6x+3y=18\ .$$

By adding the two equations, we have

$$7x=22 \Rightarrow x=\frac{22}{7}$$

Substituting the value of x into the first or second equation yields $y=-\frac{2}{7}$.

• Use substitution:

From the first equation, we can have $x=4+3y$. By substituting that into the second equation , we have

$$2(4+3y)+y=6$$

$$\Rightarrow 7y+8=6$$

$$\Rightarrow y=-\frac{2}{7}$$

Thus, by substituting into $x=4+3y$, we have $\frac{22}{7}$.

Pythagoras Theorem

If c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, the Pythagorean theorem can be expressed as the

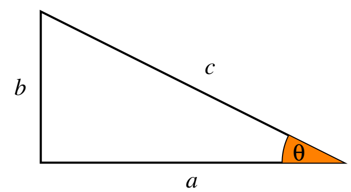
$$a^2+b^2=c^2 \Rightarrow c=\sqrt{a^2+b^2}\,.$$

If a and c are known and b is unknown, then

$$b=\sqrt{c^2-a^2}\,.$$

Similarly, if b and c are known and a is unknown, then

$$a=\sqrt{c^2-b^2}$$

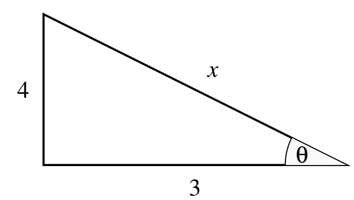


a is adjacent
b is opposite
c is hypotenuse

The trigonometric functions for a right triangle:

$$\cos\theta=\frac{a}{c}\qquad \sin\theta=\frac{b}{c}\qquad \tan\theta=\frac{b}{a}$$

Example 17 Find value of x. Then find $\cos\theta$, $\sin\theta$



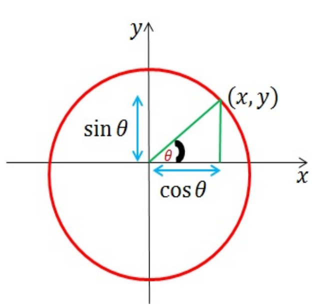
Solution:
$a=3,\ b=4 \Rightarrow c^2=4^2+3^2=25 \Rightarrow c=5$
$\cos\theta=\frac{3}{5}$ $\sin\theta=\frac{4}{5}$

Trigonometric Functions

• If (x,y) is a point on the unit circle, and if the ray from the origin $(0,0)$ to that point (x,y) makes an angle θ with the positive x-axis, then

$$\cos\theta=x\,,\quad \sin\theta=y\,,$$

• Each point (x,y) on the unit circle can be written as $(\cos\theta,\sin\theta)$.
• From the equation $x^2+y^2=1$, we have $\cos^2\theta+\sin^2\theta=1$. From this, $1+\tan^2\theta=\sec^2\theta$ and $\cot^2\theta+1=\csc^2\theta$.



$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\qquad\cot\theta=\frac{\cos\theta}{\sin\theta}$

$\sec\theta=\frac{1}{\cos\theta}$ $\qquad\csc\theta=\frac{1}{\sin\theta}$

• Trigonometric functions of negative angles:

$\cos(-\theta)=\cos(\theta),\ \sin(-\theta)=-\sin(\theta),\ \tan(-\theta)=-\tan(\theta)$

• Double and half angle formulas

$$\sin 2\theta=2\sin\theta\cos\theta$$
$$\cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$
$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$
$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}},\ \cos\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

• Angle addition formulas

$$\sin(\theta_1\pm\theta_2)=\sin\theta_1\cos\theta_2\pm\cos\theta_1\sin\theta_2$$
$$\cos(\theta_1\pm\theta_2)=\cos\theta_1\cos\theta_2\mp\sin\theta_1\sin\theta_2$$
$$\tan(\theta_1\pm\theta_2)=\frac{\tan\theta_1\pm\tan\theta_2}{1\mp\tan\theta_1\tan\theta_2}$$

• Values of trigonometric functions of most commonly used angles:

Degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1

• Graphs of trigonometric functions:

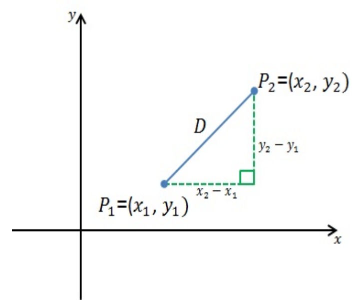
——- ——-

Distance Formula

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points in the Cartesian plane. Then the distance between P_1 and P_2 is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 18 Find the distance between the two points $P_1(1,1)$ and $P_2(-3,4)$.
Solution: $D = \sqrt{(1-(-3))^2 + (1-4)^2} = \sqrt{16+9} = \sqrt{25} = 5$.



Differentiation Rules:

$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$

$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x)+f(x)g'(x)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x))^2}$

$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-g'(x)}{g(x))^2}$

$\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

Elementary Derivatives:

$\frac{d}{dx}x^r = rx^{r-1}$ $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$ $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Derivative of Composite Functions (Chain Rule)

If $y = f(u), u = g(x)$ such that dy/du and du/dx exist, then the derivative of the composite function $(f \circ g)(x)$ exists and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)g'(x) = f'\big(g(x)\big)g'(x)\ .$$

Derivative of Inverse Functions:

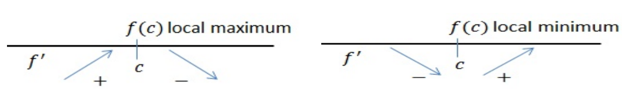
If a function f has an inverse function f^{-1}, then $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$.

•

Graphs of Functions

The First and Second Derivative Tests

1. Let f be continuous on $[a,b]$ and f' exists on (a,b).
   - if $f'(x) > 0, \forall x \in (a,b)$, then f is increasing on $[a,b]$.
   - if $f'(x) < 0, \forall x \in (a,b)$, then f is decreasing on $[a,b]$.
2. Let f be continuous at a critical number c and differentiable on an open interval (a,b), except possibly at c.
   - $f(c)$ is a local maximum of f if f' changes from positive to negative at c.
   - $f(c)$ is a local minimum of f if f' changes from negative to positive at c.



3. If f'' exists on an open interval I,
   - the graph of f is concave upward on I if $f''(x) > 0$ on I.
   - the graph of f is concave downward on I if $f''(x) < 0$ on I.

• Shifting Graphs

Let $y = f(x)$ is a function.

1. Replacing each x in the function with $x - c$ shifts the graph c units horizontally.
   - If $c > 0$, the shift will be to the right.
   - If $c < 0$, the shift will be to the left.
2. Replacing y in the function with $y - c$ shifts the graph c units vertically.
   - If $c > 0$, the shift will be upward.
   - If $c < 0$, the shift will be downward.

• Symmetry about the y-axis and the Origin

1. If a function f is odd, the graph of f is symmetric about the origin.
2. If a function f is even, the graph of f is symmetric about the y-axis.

• Lines

The general linear equation in two variables x and y can be written in the form:

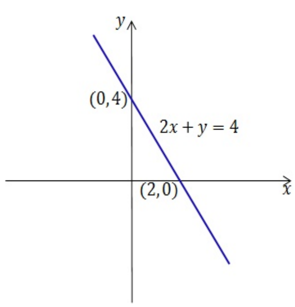
$$ax + by + c = 0 \ ,$$

where a, b and c are constants with a and b not both 0.

Example 19

$2x + y = 4$
$a = 2,\ b = -1,\ c = -4$
To plot the lines, we rewrite the equation to the form $y = -2x + 4$. Then we use the following table to make points on the plane.

x	0	2
y	4	0



Slope

1. By knowing $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on a straight line:
$$m = \frac{y_2 - y_1}{x_2 - x_1} \ .$$
2. Point-Slope form:
$$y - y_1 = m(x - x_1) \ .$$
3. Slope-Intercept form: The general linear equation can be rewritten as
$$ax + by + c = 0 \Rightarrow by = -ax + c \Rightarrow y = -\frac{a}{b}x + \frac{c}{b} \Rightarrow y = mx + d \ ,$$
   where m is the slope.

Example 20 Find the slope of the line $2x - 5y + 9 = 0$.

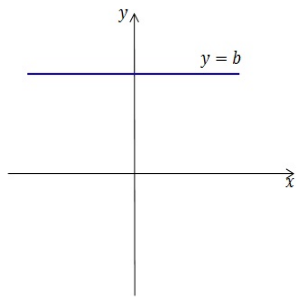
Solution: $2x - 5y + 9 = 0 \Rightarrow -5y = -2x - 9 \Rightarrow y = \frac{2}{5}x + \frac{9}{5}$.

Thus, the slope is $\frac{2}{5}$. Alternatively, take any two points lie on that line like $(-2,1)$ and $(3,3)$. Then,

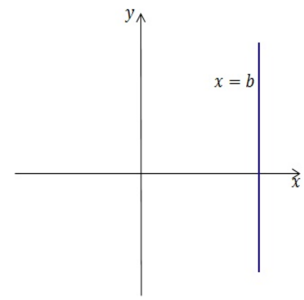
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{3-(-2)} = \frac{2}{5} \ .$$

Special Cases of lines in a Plane:

1. If m is undefined, the line is vertical.

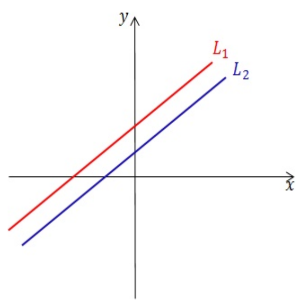


2. If $m = 0$, the line is horizontal.

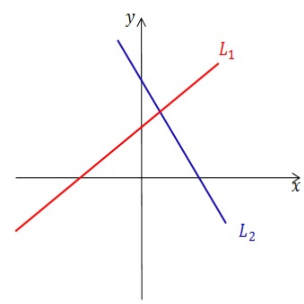


3. Let L_1 and L_2 be two lines in a plane, and let m_1 and m_2 be their slopes, respectively.

- If L_1 and L_2 are parallel, $m_1 = m_2$.



- If L_1 and L_2 are vertical, $m_1 = \frac{-1}{m_2}$.



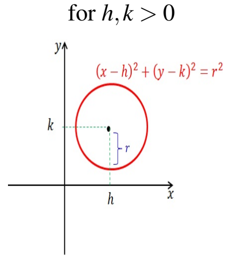
- **Quadratic Functions**

Circles

Let $C(h,k)$ be the center of a circle and r be the radius. Then, the equation of the circle is

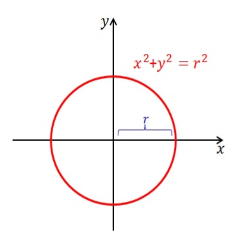
$$(x-h)^2 + (y-k)^2 = r^2$$

for $h,k > 0$



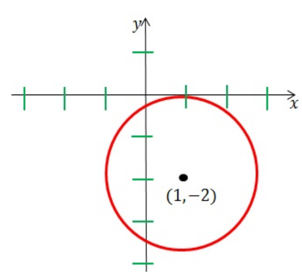
If $h = k = 0$, the center of the circle is at the origin $(0,0)$ and the equation of the circle becomes

$$x^2 + y^2 = r^2 \ .$$



Example 21 Find an equation of the circle that has center $(1,-2)$ and radius $r = 2$.
Solution:

$$(x-1)^2 + (y+2)^2 = 4$$
$$x^2 + y^2 - 2x + 4y = -1 \ .$$



• Conic Sections

Parabola:

A parabola is the set of all points in the plane equidistant from a fixed point F (called the focus) and a fixed line D (called the directrix) in the same plane.

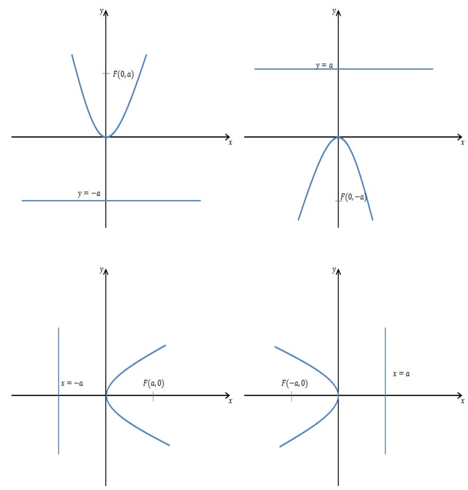
(1) The vertex of the parabola is the origin $(0,0)$

(A) $x^2 = 4ay$, where $a > 0$.
- The parabola opens upward.
- Focus: $F(0,a)$.
- Directrix equation: $y = -a$.
- Parabola axis: the y-axis.

(B) $x^2 = -4ay$, where $a > 0$.
- The parabola opens downward.
- Focus: $F(0,-a)$.
- Directrix equation: $y = a$.
- Parabola axis: the y-axis.

(C) $y^2 = 4ax$, where $a > 0$.
- The parabola opens to the right.
- Focus: $F(a,0)$.
- Directrix equation: $x = -a$.
- Parabola axis: the x-axis.

(D) $y^2 = -4ax$, where $a > 0$.
- The parabola opens to the left.
- Focus: $F(-a,0)$.
- Directrix equation: $x = a$.
- Parabola axis: the x-axis.



(2) The general formula of a parabola:

(A) $(x-h)^2 = 4a(y-k)$, where $a > 0$.
- Vertex: $V(h,k)$.
- The parabola opens upwards.
- Focus: $F(h,k+a)$.
- Directrix equation: $y = k-a$.
- Parabola axis: parallel to the y-axis.

(B) $(x-h)^2 = -4a(y-k)$, where $a > 0$.
- Vertex: $V(h,k)$.
- The parabola open downwards.
- Focus: $F(h,k-a)$.
- Directrix equation: $y = k+a$.
- Parabola axis: parallel to the y-axis.

(C) $(y-k)^2 = 4a(x-h)$, where $a > 0$
- Vertex: $V(h,k)$.
- The parabola opens to the right.
- Focus: $F(h+a,k)$.
- Directrix equation: $x = h-a$.
- Parabola axis: parallel to the x-axis.

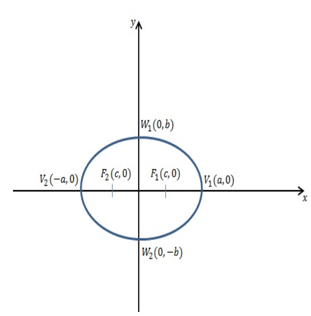
(D) $(y-k)^2 = -4a(x-h)$, where $a > 0$
- Vertex: $V(h,k)$.
- The parabola opens to the left.
- Focus: $F(h-a,k)$.
- Directrix equation: $x = h+a$.
- Parabola axis: parallel to the x-axis.

Ellipse:

An ellipse is the set of all points in the plane for which the sum of the distances to two fixed points is constant.

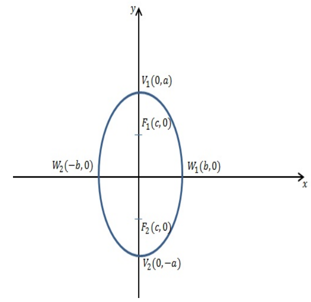
(1) The center of the ellipse is the origin $(0,0)$:

(A) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$ and $c = \sqrt{a^2 - b^2}$.
- Foci: $F_1(-c,0)$ and $F_2(c,0)$.
- Vertices: $V_1(-a,0)$ and $V_2(a,0)$.
- Major axis: the x-axis, its length is $2a$.
- Minor axis: the y-axis, its length is $2b$.
- Minor axis endpoints: $W_1(0,b)$ and $W_2(0,-b)$.



(B) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a<b$ and $c=\sqrt{b^2-a^2}$.

- Foci: $F_1(0,c)$ and $F_2(0,-c)$.
- Vertices: $V_1(0,b)$ and $V_2(0,-b)$.
- Major axis: the y-axis, its length is $2b$.
- Minor axis: the x-axis, its length is $2a$.
- Minor axis endpoints: $W_1(-a,0)$ and $W_2(a,0)$.



(2) The general formula of the ellipse:

(A) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $a>b$ and $c=\sqrt{a^2-b^2}$.

- Center: $P(h,k)$.
- Foci: $F_1(h-c,k)$ and $F_2(h+c,k)$.
- Vertices: $V_1(h-a,k)$ and $V_2(h+a,k)$.
- Major axis: parallel to the x-axis, its length is $2a$.
- Minor axis: parallel to the y-axis, its length is $2b$.
- Minor endpoints: $W_1(h,k+b)$ and $W_2(h,k-b)$.

(B) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $a<b$ and $c=\sqrt{b^2-a^2}$.

- Center: $P(h,k)$.
- Foci: $F_1(h,k+c)$ and $F_2(h,k-c)$.
- Vertices: $V_1(h,k+b)$ and $V_2(h,k-b)$.
- Major axis: parallel to the y-axis, its length is $2b$.
- Minor axis: parallel to the x-axis, its length is $2a$.
- Minor endpoints: $W_1(h-a,k)$ and $W_2(h+a,k)$.

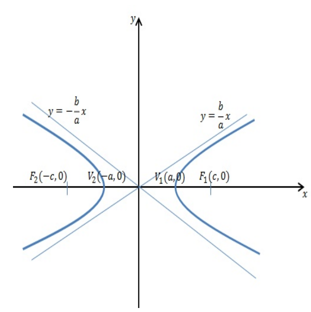
Hyperbola:

A hyperbola is the set of all points in the plane for which the absolute difference of the distances between two fixed points is constant.

(1) The center of the hyperbola is the origin $(0,0)$:

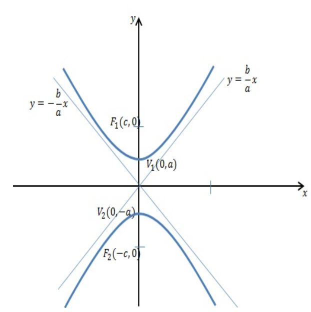
(A) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a>b>0$ and $c=\sqrt{a^2+b^2}$.

- Foci: $F_1(-c,0)$ and $F_2(c,0)$.
- Vertices: $V_1(-a,0)$ and $V_2(a,0)$.
- Transverse axis: the x-axis, its length is $2a$.
- Asymptotes: $y=\pm\frac{b}{a}x$.



(B) $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ where $0<a<b$ and $c=\sqrt{a^2+b^2}$.

- Foci: $F_1(0,c)$ and $F_2(0,-c)$.
- Vertices: $V_1(0,b)$ and $V_2(0,-b)$.
- Transverse axis: the y-axis, its length is $2b$.
- Asymptotes: $y=\pm\frac{b}{a}x$.



(2) The general formula of the hyperbola:

(A) $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $a>b>0$ and $c=\sqrt{a^2+b^2}$.

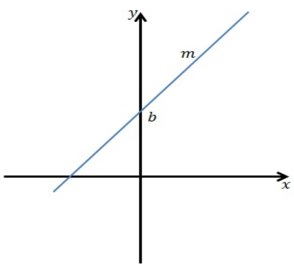
- Center: $P(h,k)$.
- Foci: $F_1(h-c,k)$ and $F_2(h+c,k)$.
- Vertices: $V_1(h-a,k)$ and $V_2(h+a,k)$.
- Transverse axis: parallels to the x-axis, its length is $2a$.
- Asymptotes: $(y-k)=\pm\frac{b}{a}(x-h)$.

(B) $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ where $0<a<b$ and $c=\sqrt{a^2+b^2}$.

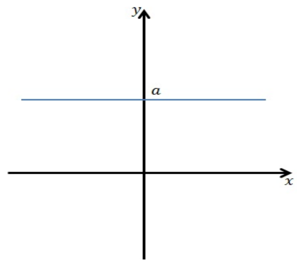
- Center: $P(h,k)$.
- Foci: $F_1(h,k+c)$ and $F_2(h,k-c)$.
- Vertices: $V_1(h,k+b)$ and $V_2(h,k-b)$.
- Transverse axis: parallels to the y-axis, its length is $2b$.
- Asymptotes: $(y-k)=\pm\frac{b}{a}(x-h)$.

Sketch of Some Functions:

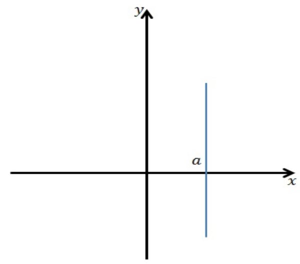
$y = mx + b$



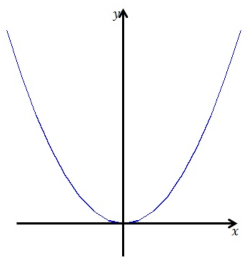
$y = a$



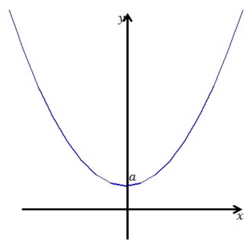
$x = a$



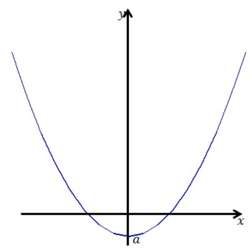
$y = x^2$



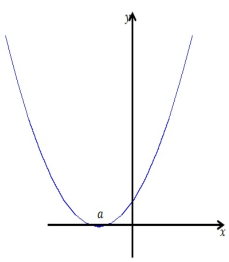
$y = x^2 + a$



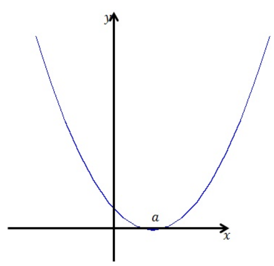
$y = x^2 - a$



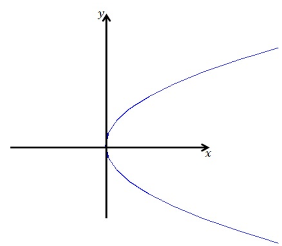
$y = (x + a)^2$



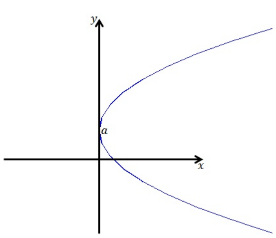
$y = (x - a)^2$



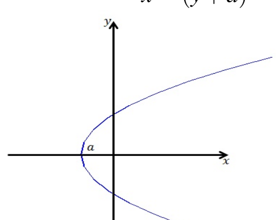
$x = y^2$



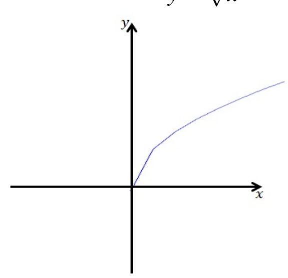
$x = y^2 + a$



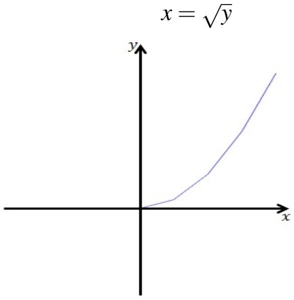
$x = (y + a)^2$



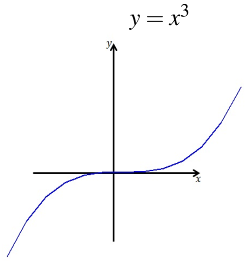
$y = \sqrt{x}$



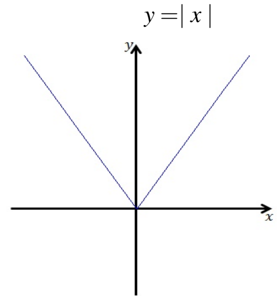
$x = \sqrt{y}$



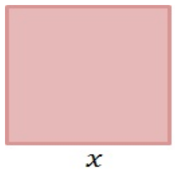
$y = x^3$



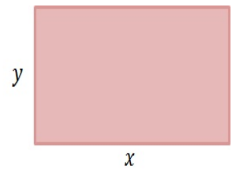
$y = |x|$



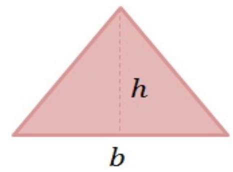
Area and Volume of Special Shapes:



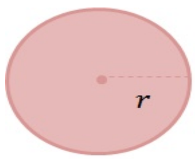
Area $= x^2$



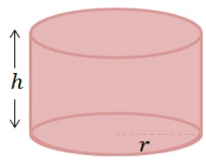
Area $= xy$



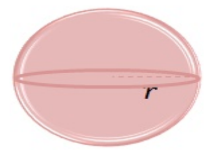
Area $= \frac{1}{2}bh$



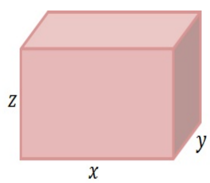
Area $= \pi r^2$



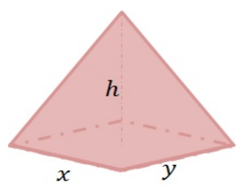
Volume $= \pi r^2 h$



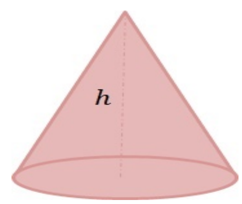
Volume $= \frac{4}{3}\pi r^3$



Volume $= xyz$



Volume $= \frac{1}{3}xyh$



Volume $= \frac{1}{3}\pi r^2 h$