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Bayesian Inference for The Left Truncated Exponential Distribution Based on Ordered Pooled Sample of Records

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Abstract: In this paper, the maximum likelihood and Bayesian estimations are developed based on an ordered pooled sample from two independent samples of record values from the left truncated exponential distribution. The Bayesian estimation for the unknown parameters is discussed using different loss functions. Also, the maximum likelihood and the Bayesian estimators of the corresponding reliability and p^{th} quantile functions are calculated. The problem of predicting the record values from a future sample from the sample population is also discussed from a Bayesian viewpoint. A Monte Carlo simulation study is conducted to compare the maximum likelihood estimator with the Bayesian estimators. Finally, an illustrative example is presented to demonstrate the different inference methods discussed here.

Keywords: Bayesian estimation, bayesian prediction, left truncated exponential distribution, maximum likelihood estimation, record values

1 Introduction

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed (iid) random variables. Then, an observation X_j is called an upper record value if it exceeds all previous observations, i.e., if $X_j > X_i$ for every $i < j$. Record values are defined as a model for successive extremes in a sequence of iid random variables such as successive largest insurance claims in non-life insurance, highest water levels or highest temperatures. Records are also used in reliability theory. Suppose that a technical system is subject to shocks, e.g. peaks of voltages. If the shocks are viewed as realizations of an iid sequence, then the model of record statistics (values of successive peak voltages) is adequate. Moreover, record values can also be applied in the analysis of a minimal-repair system data; see [1]. In a minimal repair experiment, the system is put back into operation, after a failure had occurred that necessitated a minimal repair of the system. Interestingly, in this case, the observed repair times possess the same joint distribution as upper record values. The theory of record values was introduced for the first time by Chandler in [2], and since then, many authors have studied record values and the associated statistics; see, for example, [3], [4], [5], [6], [7], [8] and [9].

The expected number of observed record values in a random sample of size n is approximately $\log n + \gamma$, where γ is the Eulers constant 0.5772. Thus, in a sequence of 1000 observations, we would expect to observe only 7 records. Hence, the precision of the statistical inference developed based on this data will be quite low. In such a situation, if it will be possible and convenient to take an additional independent sample of record values, it might be possible to use the ordered pooled sample from these two samples in order to increase the precision of the statistical inference.

Recently, Beutner and Cramer in [10] derived the joint distribution of the ordered pooled sample from two independent minimal-repair systems (two independent samples of record values) as a mixture of the joint distribution of particular generalized order statistics from the same population and then applied these results to construct nonparametric prediction intervals for the future repair times of an identically structured minimal-repair system. Amini and Balakrishnan in [11]

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discussed the same problem for the general case of pooling from k independent samples of record values. They derived the joint and marginal distributions of the combined ordered record sample for the general case and then used these distributional results to develop exact nonparametric confidence intervals for the quantiles of the population and also exact nonparametric prediction intervals for a future record value. In this paper, we use the joint distribution of the ordered pooled sample from two independent samples of record values derived by Beutner and Cramer in [10] to develop maximum likelihood and Bayesian estimation for the two parameters of the left truncated exponential distribution and also Bayesian prediction for record values from a future sample from the same population.

For the Bayesian estimation in this context, we consider here three types of loss functions. The first is the squared error SE loss function which is a symmetric function that gives equal importance to overestimation and underestimation in the parameter estimation. The second is the linear-exponential LINEX loss function, introduced by Varian in [12], which is asymmetric and gives differing weights to overestimation and underestimation. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. The third loss function is the generalization of the entropy GE loss used by several authors (see, for example, [13]). This more general version allows for different shapes of the loss function.

In many practical problems, one may wish to use past data to predict an observation from a future sample from the same population. As in the case of estimation, a predictor can be either a point or an interval predictor. Prediction of record values has potential environmental applications dealing, for example, predicting the flood level of a river that is greater than the previous ones is of importance to climatologists and hydrologists. Predicting the magnitude of an earthquake which has a greater magnitude than the previous ones, in a given region, is of importance to seismologists as well. For more examples, see [14]. Prediction for future records have been discussed by many authors, including [15], [16], [17], [18], [19], [20], [21] and [22].

The rest of this paper is organized as follows. In Section 2, the description of the model of the ordered pooled sample from two independent samples of record values is presented. The maximum likelihood ML estimator and the Bayesian estimators under SE, LINEX and GE loss functions for the unknown parameters and the corresponding reliability and p^{th} quantile functions are derived in Section 3. The problem of predicting record values from a future sample is then discussed in Section 4. Finally, in Section 5, some computational results are presented for illustrating all the inferential methods developed here.

2 The model description

Let $X_{(1)},\ldots,X_{(r)}$ and $Y_{(1)},\ldots,Y_{(s)}$ be two independent samples of record values from the same population with cumulative distribution function (CDF) F. In the following, the ordered pooled sample from $X_{(1)},\ldots,X_{(r)};Y_{(1)},\ldots,Y_{(s)}$ will be denoted by $\mathbf{Z}=(Z_{(1)},\ldots,Z_{(r+s)})$ where $Z_{(1)}\leq\ldots\leq Z_{(r+s)}$.

Beutner and Cramer in [10] derived the joint density function of the pooled sample $\mathbf{Z}=(Z_{(1)},\ldots,Z_{(r+s)})$ (the joint distribution of the ordered pooled sample from two independent minimal-repair systems) as a mixture of the joint distribution of particular generalized order statistics from the same population as follows:

$$f^{\mathbf{Z}}(\mathbf{z})=\sum_{i=0}^{r-1}\beta_i f^{\mathbf{W}^{(s+i)}}(\mathbf{z})+\sum_{j=0}^{s-1}\phi_j f^{\mathbf{V}^{(r+j)}}(\mathbf{z}), \tag{1}$$

where $\mathbf{z}=(z_1,\ldots,z_{r+s})$ is a vector of realizations, $\mathbf{W}^{(s+i)}=(W^{(s+i)}_{(*1)},\ldots,W^{(s+i)}_{(*r+s)})$ for $i=0,\ldots,r-1$, and $\mathbf{V}^{(r+j)}=(V^{(r+j)}_{(*1)},\ldots,V^{(r+j)}_{(*r+s)})$ for $j=0,\ldots,s-1$, are generalized order statistics from the same population based on parameters

$$\gamma_\ell^{(s+i)}=1+1_{[1,\ldots,s+i]}(\ell),\ \ 0\leq i\leq r-1,$$
$$\eta_\ell^{(r+j)}=1+1_{[1,\ldots,r+j]}(\ell),\ \ 0\leq j\leq s-1,\quad 1\leq\ell\leq r+s,$$

respectively ($1_A(\cdot)$ denotes the indicator function on A), and the mixture probabilities are given by

$$\beta_i=\binom{s+i-1}{s-1}2^{-(s+i)},\ \ 0\leq i\leq r-1,$$
$$\phi_j=\binom{r+j-1}{r-1}2^{-(r+j)},\ \ 0\leq j\leq s-1.$$

Using the concept of generalized order statistics given by Kamps in [23], ordered random variables $V_1,\ldots,V_n$ are called generalized order statistics based on continuous CDF F with probability density function (PDF) f and on positive parameters $\gamma_1,...,\gamma_n$ if they have the joint PDF

$$f^{V_1,...,V_n}(v_1,...,v_n) = \left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(v_i)]^{\gamma_i-\gamma_{i+1}-1}f(v_i)\right)[1-F(v_n)]^{\gamma_n-1}f(v_n), \quad (2)$$

for $F^{-1}(0) \leq v_1 \leq v_2 \leq ... \leq v_n \leq F^{-1}(1)$.

By using the joint density function of the generalized order statistics in (2), the joint density function of the ordered pooled sample $\mathbf{Z} = (Z_{(1)},...,Z_{(r+s)})$ in (1) becomes

$$f^{\mathbf{Z}}(\mathbf{z}) = \sum_{i=0}^{r-1}\beta_i^*\left(\prod_{\substack{q=1\\q\neq s+i}}^{r+s-1}\frac{f(z_q)}{1-F(z_q)}\right)f(z_{s+i})f(z_{r+s}) + \sum_{j=0}^{s-1}\phi_j^*\left(\prod_{\substack{q=1\\q\neq r+i}}^{r+s-1}\frac{f(z_q)}{1-F(z_q)}\right)f(z_{r+j})f(z_{r+s}), \quad (3)$$

where

$$\beta_i^* = 2^{s+i}\beta_i = \binom{s+i-1}{s-1},\ 0 \leq i \leq r-1,$$
$$\phi_j^* = 2^{r+j}\phi_j = \binom{r+j-1}{r-1},\ 0 \leq j \leq s-1.$$

In this paper, the underlying distribution is assumed to be the left truncated exponential with PDF and CDF as

$$f(x \mid \theta,\mu) = \theta\exp(-\theta(x-\mu)),\ x \geqslant \mu, \quad (4)$$

and

$$F(x \mid \mu,\theta) = 1-\exp(-\theta(x-\mu)),\ x \geqslant \mu, \quad (5)$$

with rate parameter $\theta > 0$, and location parameter $\mu > 0$. If μ is not restricted to be nonnegative then (5) is more appropriately referred to as the two-parameter exponential distribution. Introducing distinctive names for these two distributions is necessary since it is only the former (with $\mu \geqslant 0$) which is really appropriate as a lifetime distribution model.

The reliability function $R(t)$ and the p^{th} quantile ξ_p of the left truncated exponential distribution are given, respectively, by

$$R(t) = \exp(-\theta(t-\mu)), \quad t \geqslant \mu, \quad (6)$$

and

$$\xi_p = \mu - \frac{\log(1-p)}{\theta},\ 0 \leqslant p \leqslant 1. \quad (7)$$

3 ML and Bayesian estimation

In this section, we derive the ML estimator and the Bayesian estimators under SE, LINEX and GE loss functions for the unknown parameters θ and μ. Also, the ML and the Bayesian estimators of the corresponding reliability and p^{th} quantile functions are calculated.

Using (3), (4) and (5), the likelihood function of θ and μ based on the pooled sample $\mathbf{Z} = (Z_{(1)},...,Z_{(r+s)})$ can be written as

$$L(\theta,\mu \mid Z) = \sum_{i=0}^{r-1}\beta_i^*\theta^{r+s}\exp(-\theta[u_i+2(z_1-\mu)]) + \sum_{j=0}^{s-1}\phi_j^*\theta^{r+s}\exp\left(-\theta\left[u_j^*+2(z_1-\mu)\right]\right) \quad (8)$$

where

$$u_i = (z_{s+i}-z_1)+(z_{r+s}-z_1)\ \text{ for } i = 0,1,\ldots,r-1,$$

and

$$u_j^* = (z_{r+j}-z_1)+(z_{r+s}-z_1)\ \text{ for } j = 0,1,\ldots,s-1.$$

3.1 ML estimation

From (8), the log-likelihood function of (θ,μ) is given by

$$\log L(\theta,\mu \mid \mathbf{z}) = \log\left\{\sum_{i=0}^{r-1}\beta_i^*\theta^{r+s}\exp\left(-\theta\left[u_i+2(z_1-\mu)\right]\right)+\sum_{j=0}^{s-1}\phi_j^*\theta^{r+s}\exp\left(-\theta\left[u_j^*+2(z_1-\mu)\right]\right)\right\}. \tag{9}$$

Now, the likelihood function is maximized with respect to μ by taking $\hat{\mu}_{ML}=z_1$. To maximize relative to θ, we need to differentiate (9) with respect to θ and solve the likelihood equation

$$\frac{\partial \log L(\theta,\mu \mid Z)}{\partial\theta}=0$$

and so the ML estimator $\hat{\theta}_{ML}$ of θ is readily obtained by solving the following equation

$$\sum_{i=0}^{r-1}\beta_i^*\left(r+s-\theta u_i\right)\exp\left(-\theta u_i\right)+\sum_{j=0}^{s-1}\phi_j^*\left(r+s-\theta u_j^*\right)\exp\left(-\theta u_j^*\right)=0. \tag{10}$$

By using the invariance property, the ML estimators of the reliability function and the p^{th} quantile function can be obtained, respectively, as

$$\hat{R}_{ML}(t)=\exp\left(-\hat{\theta}_{ML}\left(t-\hat{\mu}_{ML}\right)\right) \tag{11}$$

and

$$\hat{\xi}_{p_{ML}}=\hat{\mu}_{ML}-\frac{\log(1-p)}{\hat{\theta}_{ML}}. \tag{12}$$

3.2 Bayesian estimation

For Bayesian estimation, we use here the natural conjugate prior density function for (θ,μ) given by

$$\pi(\theta,\mu)\propto\theta^g\exp\left(-\theta\left[h+c(b-\mu)\right]\right),\quad 0<\mu<b,\ \theta>0, \tag{13}$$

where $g>-1$, $h>0$ and $c>0$; see [24]. By taking $g\to-1$, $h\to0$, $c\to0$ and $b\to\infty$, the non-informative prior density function for (θ,μ) is given by

$$\pi(\theta,\mu)\propto\frac{1}{\theta},\qquad \theta>0. \tag{14}$$

It follows that the joint posterior density function of (θ,μ), given $\mathbf{Z}=\mathbf{z}$, is given by

$$\pi^*(\theta,\mu)=I^{-1}\left\{\sum_{i=0}^{r-1}\beta_i^*\theta^G\exp\left(-\theta\left[H_i+C(B-\mu)\right]\right)+\sum_{j=0}^{s-1}\phi_j^*\theta^G\exp\left(-\theta\left[H_j^*+C(B-\mu)\right]\right)\right\} \tag{15}$$

where I is the normalizing constant given by

$$\begin{aligned} I &= \int_0^\infty\int_0^B \pi^*(\theta,\mu)\,d\mu\,d\theta \\ &= \frac{\Gamma(G)}{C}\sum_{i=0}^{r-1}\beta_i^*\left[(H_i)^{-G}-(H_i+CB)^{-G}\right]+\sum_{j=0}^{s-1}\phi_j^*\left[\left(H_j^*\right)^{-G}-\left(H_j^*+CB\right)^{-G}\right], \end{aligned} \tag{16}$$

with $G=r+s+g$, $C=c+2$, $B=\min(b,z_1)$, $H_i=u_i+h+bc+2z_1-CB$, $H_j^*=u_j^*+h+bc+2z_1-CB$, and $\Gamma(\cdot)$ denotes the complete gamma function.

Hence, the Bayesian estimator of θ under the SE loss function is given by

$$\begin{aligned} \hat{\theta}_{BS} &= E[\theta] \\ &= \frac{\Gamma(G+1)I^{-1}}{C}\left\{\sum_{i=0}^{r-1}\beta_i^*\left[(H_i)^{-(G+1)}-(H_i+CB)^{-(G+1)}\right]+\sum_{j=0}^{s-1}\phi_j^*\left[\left(H_j^*\right)^{-(G+1)}-\left(H_j^*+CB\right)^{-(G+1)}\right]\right\}, \end{aligned} \tag{17}$$

and the Bayesian estimator of μ under the SE loss function is given by

$$\begin{aligned}\hat{\mu}_{BS} &= E[\mu] \\ &= \frac{\Gamma(G-1)I^{-1}}{C^2}\left\{\sum_{i=0}^{r-1}\beta_i^*\left[BC(G-1)(H_i)^{-G}+(H_i+CB)^{-G+1}-(H_i)^{-G+1}\right]\right. \\ &\qquad \left.+\sum_{j=0}^{s-1}\phi_j^*\left[BC(G-1)\left(H_j^*\right)^{-G}+\left(H_j^*+CB\right)^{-G+1}-\left(H_j^*\right)^{-G+1}\right]\right\}. \qquad (18)\end{aligned}$$

The Bayesian estimator of θ under the LINEX loss function is given by

$$\begin{aligned}\hat{\theta}_{BL} &= \frac{-1}{\upsilon}\log\left(E\left[\exp(-\upsilon\theta)\right]\right) \\ &= \frac{-1}{\upsilon}\log\left(\frac{\Gamma(G)I^{-1}}{C}\left\{\sum_{i=0}^{r-1}\beta_i^*\left[(H_i+\upsilon)^{-G}-(H_i+\upsilon+CB)^{-G}\right]+\sum_{j=0}^{s-1}\phi_j^*\left[\left(H_j^*+\upsilon\right)^{-G}-\left(H_j^*+\upsilon+CB\right)^{-G}\right]\right\}\right), \qquad (19)\end{aligned}$$

and the Bayesian estimator of μ under the LINEX loss function is given by

$$\begin{aligned}\hat{\mu}_{BL} &= \frac{-1}{\upsilon}\log\left(E\left[\exp(-\upsilon\mu)\right]\right) \\ &= \frac{-1}{\upsilon}\log\left(\Gamma(G+1)I^{-1}\left\{\sum_{i=0}^{r-1}\beta_i^*\int_0^B \exp(-\upsilon\mu)\left[H_i+C(B-\mu)\right]^{-(G+1)}d\mu\right.\right. \\ &\qquad \left.\left.+\sum_{j=0}^{s-1}\phi_j^*\int_0^B \exp(-\upsilon\mu)\left[H_j^*+C(B-\mu)\right]^{-(G+1)}d\mu\right\}\right). \qquad (20)\end{aligned}$$

The Bayesian estimator of θ under the GE loss function is given by

$$\begin{aligned}\hat{\theta}_{BE} &= \left(E\left[\theta^{-d}\right]\right)^{\frac{-1}{d}} \\ &= \left(\frac{\Gamma(G-d)I^{-1}}{C}\left\{\sum_{i=0}^{r-1}\beta_i^*\left[(H_i)^{(d-G)}-(H_i+CB)^{(d-G)}\right]+\sum_{j=0}^{s-1}\phi_j^*\left[\left(H_j^*\right)^{(d-G)}-\left(H_j^*+CB\right)^{(d-G)}\right]\right\}\right)^{\frac{-1}{d}}, \qquad (21)\end{aligned}$$

and the Bayesian estimator of μ under the GE loss function is given by

$$\begin{aligned}\hat{\mu}_{BE} &= \left(E\left[\mu^{-d}\right]\right)^{\frac{-1}{d}} \\ &= \left(\Gamma(G+1)I^{-1}\left\{\sum_{i=0}^{r-1}\beta_i^*\int_0^B \mu^{-d}\left[H_i+C(B-\mu)\right]^{-(G+1)}d\mu+\sum_{j=0}^{s-1}\phi_j^*\int_0^B \mu^{-d}\left[H_j^*+C(B-\mu)\right]^{-(G+1)}d\mu\right\}\right)^{\frac{-1}{d}}. \qquad (22)\end{aligned}$$

The Bayesian estimator of the reliability function under the SE loss function is given by

$$\begin{aligned}\hat{R}_{BS}(t) &= E\left[\exp\left(-\theta(t-\mu)\right)\right] \\ &= \frac{\Gamma(G)I^{-1}}{C+1}\left\{\sum_{i=0}^{r-1}\beta_i^*\left[(H_i+t-B)^{-(G)}-(H_i+t+CB)^{-(G)}\right]+\sum_{j=0}^{s-1}\phi_j^*\left[\left(H_j^*+t-B\right)^{-(G)}-\left(H_j^*+t+CB\right)^{-(G)}\right]\right\}, \qquad (23)\end{aligned}$$

and the Bayesian estimator of the p^{th} quantile function under the SE loss function is given by

$$\begin{aligned}\hat{\xi}_{p_{BS}} &= E[\mu]-\log(1-p)E[\frac{1}{\theta}] \\ &= \hat{\mu}_{BS}-\log(1-p)\frac{\Gamma(G-1)I^{-1}}{C}\left\{\sum_{i=0}^{r-1}\beta_i^*\left[(H_i)^{(1-G)}-(H_i+CB)^{(1-G)}\right]+\sum_{j=0}^{s-1}\phi_j^*\left[\left(H_j^*\right)^{(1-G)}-\left(H_j^*+CB\right)^{(1-G)}\right]\right\}. \qquad (24)\end{aligned}$$

4 Bayesian prediction of order statistics from a future sample

Let $W_{(1)}, W_{(2)}, W_{(3)}, \ldots$ be a sequence of record values from a future sample from the same population. We discuss here the Bayesian prediction of $W_{(k)}$, for $k = 1, 2, 3...$, based on the observed pooled sample $\mathbf{Z} = (Z_{(1)}, ..., Z_{(r+s)})$. We derive the Bayesian predictive distribution for $W_{(k)}$, and then find the Bayesian point predictor and prediction interval.

It is well known that the marginal density function of the k^{th} record value is given; see [6], by

$$f_{W_{(k)}}(w \mid \theta, \mu) = \frac{1}{\Gamma(k)} \left[-\log \bar{F}(w) \right]^{k-1} f(w) \ , w \geq 0. \tag{25}$$

Upon substituting (4) and (5) in (25), the marginal density function of $W_{(k)}$ becomes

$$f_{W_{(k)}}(w \mid \theta, \mu) = \frac{1}{\Gamma(k)} (w-\mu)^{k-1} \theta^k \exp(-\theta(w-\mu)). \tag{26}$$

By forming the product of (15) and (26), and integrating out (θ, μ) over the set $\{(\theta,\mu) : \theta > 0,\ 0 < \mu < \min(B, W_{(k)})\}$, the Bayesian predictive density function of $W_{(k)}$, given $\mathbf{Z} = \mathbf{z}$, is then

$$f^*_{W_{(k)}}(w|\mathbf{z}) = \begin{cases} f^*_{1,W_{(k)}}(w|\mathbf{z}), & 0 < w < B, \\ f^*_{2,W_{(k)}}(w|\mathbf{z}), & w > B, \end{cases} \tag{27}$$

where

$$\begin{aligned} & f^*_{1,W_{(k)}}(w|\mathbf{z}) \\ &= \int_0^\infty \int_0^w \pi^*(\theta,\mu) f_{W_{(k)}}(w \mid \theta,\mu) \, d\mu d\theta \\ &= \frac{\Gamma(G+k+1) I^{-1}}{\Gamma(k)} \left\{ \sum_{i=0}^{r-1} \sum_{h=0}^{k-1} \sum_{q=0}^{h} \frac{\beta_i^* C_h C_k w^{k-h-1} (H_i + CB + w)^{h-q}}{(q-G-k)(C+1)^{h+1}} \left[(H_i + CB + w)^{q-G-k} - (H_i + CB - Cw)^{q-G-k} \right] \right. \\ & \left. + \sum_{j=0}^{s-1} \sum_{h=0}^{k-1} \sum_{q=0}^{h} \frac{\phi_j^* C_h C_k w^{k-h-1} (H_j + CB + w)^{h-q}}{(q-G-k)(C+1)^{h+1}} \left[\left(H_j^* + CB + w\right)^{q-G-k} - \left(H_j^* + CB - Cw\right)^{q-G-k} \right] \right\} \end{aligned} \tag{28}$$

and

$$\begin{aligned} & f^*_{2,W_{(k)}}(w|\mathbf{z}) \\ &= \int_0^\infty \int_0^B \pi^*(\theta,\mu) f_{W_{(k)}}(w \mid \theta,\mu) \, d\mu d\theta \\ &= \frac{\Gamma(G+k+1) I^{-1}}{\Gamma(k)} \left\{ \sum_{i=0}^{r-1} \sum_{h=0}^{k-1} \sum_{q=0}^{h} \frac{\beta_i^* C_h C_k w^{k-h-1} (H_i + CB + w)^{h-q}}{(q-G-k)(C+1)^{h+1}} \left[(H_i + CB + w)^{q-G-k} - (H_i + w - B)^{q-G-k} \right] \right. \\ & \left. + \sum_{j=0}^{s-1} \sum_{h=0}^{k-1} \sum_{q=0}^{h} \frac{\phi_j^* C_h C_k w^{k-h-1} \left(H_j^* + CB + w\right)^{h-q}}{(q-G-k)(C+1)^{h+1}} \left[\left(H_j^* + CB + w\right)^{q-G-k} - \left(H_j^* + w - B\right)^{q-G-k} \right] \right\}, \end{aligned} \tag{29}$$

with $C_h = (-1)^h \frac{(k-1)!}{(k-h-1)!h!}$ and $C_q = (-1)^q \frac{h!}{(h-q)!q!}$.

From (27), we simply obtain the predictive survival function of $W_{(k)}$, given $\mathbf{Z} = \mathbf{z}$, as

$$\bar{F}^*_{W_{(k)}}(t|\mathbf{z}) = \begin{cases} \bar{F}^*_{1,W_{(k)}}(t|\mathbf{z}), & 0 < t < B, \\ \bar{F}^*_{2,W_{(k)}}(t|\mathbf{z}), & t > B, \end{cases} \tag{30}$$

where

$$\bar{F}^*_{1,W_{(k)}}(t|\mathbf{z}) = \int_t^B f^*_{1,W_{(k)}}(w|\mathbf{z}) \ dw + \int_B^\infty f^*_{2,W_{(k)}}(w|\mathbf{z}) \, dw, \tag{31}$$

and

$$\bar{F}^{*}_{2,W_{(k)}}(t|\mathbf{z}) = \int_{t}^{\infty} f^{*}_{2,W_{(k)}}(w|\mathbf{z})\,dw. \tag{32}$$

The Bayesian point predictor of $W_{(k)}$ under SE loss function is the mean of the predictive density, given by

$$\bar{W}_{(k)} = \int_{0}^{B} w f^{*}_{1,W_{(k)}}(w|\mathbf{z})\,dw + \int_{B}^{\infty} w f^{*}_{2,W_{(k)}}(w|\mathbf{z})\,dw \tag{33}$$

which would of course require numerical integration.

The Bayesian predictive bounds of a two-sided equi-tailed $100\,(1-\gamma)\,\%$ interval for $W_{(k)}$, can be obtained by solving the following two equations:

$$\bar{F}^{*}_{W_{(k)}}(L \mid z) = 1 - \frac{\gamma}{2} \quad \text{and} \quad \bar{F}^{*}_{W_{(k)}}(U \mid z) = \frac{\gamma}{2},$$

where $\bar{F}^{*}_{W_{(k)}}(t \mid z)$ is as in (30), and L and U denote the lower and upper bounds, respectively.

5 Numerical results and an illustrative example

In this section, the ML and Bayesian estimates using the SE, LINEX and GE loss functions are all compared by means of a Monte Carlo simulation study. A numerical example is finally presented to illustrate all the inferential results established in the preceding sections.

5.1 Monte Carlo simulation

A simulation study is carried out for evaluating the performance of the ML estimate and all the Bayesian estimates discussed in Section 3. We choose the parameter θ to be 0.5, 1 and 3 with $\mu = 1$ and different choices of r and s. For these cases, we computed the ML estimate and Bayesian estimates of θ and μ under the SE, LINEX (with $\upsilon = 0.5$) and GE (with $d = 0.5$) loss functions using informative priors (IP) and non-informative prior (NIP). We also computed the ML estimate and Bayesian estimate under the SE loss function for the corresponding reliability (with $t = 3$) and p^{th} quantile (with $p = 0.5$) functions. We repeated this process 1000 times and computed, for each estimate, the estimated bias (EB) and the estimated risk (ER) by using the root mean square error. The EB and ER of all the estimates of θ and μ are summarized in Tables 1 and 2, respectively. The EB and ER of all the estimates of the reliability and p^{th} quantile functions are summarized in Table 3.

From Tables 1-3, we observe that, for the different choices of θ, the estimated bias and risk of the Bayesian estimates based on the SE, LINEX and GE loss functions are smaller than those of the ML estimates. We also observe that the estimated bias and risk of all the estimates decrease with increasing r and s. Moreover, a comparison of the results for the informative priors with the corresponding ones for non-informative priors reveals that the former produce more precise results, as we would expect. Finally, we observe that the estimated bias and risk of the ML estimates are close to the corresponding ones of the Bayesian estimates based on the SE loss function under non-informative priors.

From Table 1, we observe that the estimated bias and risk of all the estimates of θ increase with increasing θ. But, from Tables 2 and 3, we observe that the estimated bias and risk of all the estimates of μ, $R(3)$ and $\xi_{0.5}$ decrease with increasing θ.

5.2 Illustrative example

In order to illustrate all the inferential results established in the preceding sections, we consider two simulated samples of record values with sizes $r = 4$ and $s = 4$ from the left truncated exponential distribution with $\theta = 3$ and $\mu = 1$. The simulated samples are as follows:

The first simulated sample	1.3090	1.8571	3.1230	3.1973
The second simulated sample	1.2832	1.3403	1.6357	1.6368

Table 1: The values of EB and ER of the ML and Bayesian estimates of θ for different choices of θ, r and s with $\mu = 1$.

				$\hat{\theta}_{ML}$		$\hat{\theta}_{BS}$		$\hat{\theta}_{BL}$		$\hat{\theta}_{BE}$	
θ	r	s		EB	ER	EB	ER	EB	ER	EB	ER
0.5	4	4	IP	0.1969	0.3870	0.0930	0.2326	0.0809	0.2190	0.0384	0.1966
			NIP	–	–	0.1347	0.3196	0.1178	0.2949	0.0679	0.2662
	6	4	IP	0.1579	0.3023	0.0832	0.2062	0.0734	0.1962	0.0376	0.1775
			NIP	–	–	0.1113	0.2561	0.0989	0.2413	0.0579	0.2174
	6	6	IP	0.1198	0.2454	0.0650	0.1791	0.0576	0.1721	0.0290	0.1587
			NIP	–	–	0.0848	0.2130	0.0760	0.2036	0.0441	0.1866
	8	6	IP	0.0995	0.2122	0.0560	0.1630	0.0497	0.1574	0.0244	0.1461
			NIP	–	–	0.0710	0.1873	0.0636	0.1802	0.0358	0.1662
	8	8	IP	0.0831	0.1940	0.0482	0.1543	0.0429	0.1498	0.0213	0.1407
			NIP	–	–	0.0587	0.1738	0.0527	0.1682	0.0294	0.1574
1	4	4	IP	0.3939	0.7740	0.0614	0.3739	0.0239	0.3411	0.0362	0.3352
			NIP	–	–	0.2453	0.6256	0.1813	0.5352	0.1110	0.5212
	6	4	IP	0.3158	0.6046	0.0574	0.3373	0.0264	0.3128	0.0245	0.3068
			NIP	–	–	0.2030	0.5006	0.1553	0.4455	0.0958	0.4254
	6	6	IP	0.2396	0.4907	0.0490	0.3037	0.0242	0.2859	0.0178	0.2808
			NIP	–	–	0.1535	0.4173	0.1194	0.3820	0.0718	0.3665
	8	6	IP	0.1989	0.4244	0.0400	0.2809	0.0184	0.2668	0.0189	0.2627
			NIP	–	–	0.1277	0.3672	0.0992	0.3405	0.0572	0.3269
	8	8	IP	0.1662	0.3880	0.0326	0.2691	0.0143	0.2574	0.0179	0.2543
			NIP	–	–	0.1055	0.3421	0.0823	0.3211	0.0468	0.3107
3	4	4	IP	1.1816	2.3221	0.5146	1.5259	0.1197	1.1052	0.1698	1.3071
			NIP	–	–	0.6622	1.8592	0.1671	1.2493	0.2491	1.5553
	6	4	IP	0.9473	1.8138	0.4478	1.2824	0.1088	0.9909	0.1645	1.1154
			NIP	–	–	0.5495	1.4874	0.1480	1.0952	0.2213	1.2699
	6	6	IP	0.7188	1.4722	0.3474	1.1025	0.0919	0.8983	0.1247	0.9856
			NIP	–	–	0.4113	1.2415	0.1296	0.9822	0.1618	1.0963
	8	6	IP	0.5968	1.2732	0.2967	1.0036	0.0841	0.8396	0.1017	0.9080
			NIP	–	–	0.3399	1.0931	0.1020	0.8957	0.1251	0.9786
	8	8	IP	0.4985	1.1639	0.2521	0.9442	0.0733	0.8100	0.0872	0.8676
			NIP	–	–	0.2802	1.0212	0.0838	0.8629	0.1014	0.9319

These samples are now assumed to have come from the left truncated exponential distribution, with both parameters θ and μ being unknown. Based on the ordered pooled sample $\mathbf{Z} = (1.2832, 1.3090, 1.3403, 1.6357, 1.6368, 1.8571, 3.1230, 3.1973)$ from these two samples, we computed the ML estimate and the Bayesian estimates of θ and μ based on the SE, LINEX (with $v = 0.5$) and GE (with $d = 0.5$) loss functions using informative prior with $(g,h,c,b) = (1, 0.1, 0.1, 1.5)$ and non-informative prior with $(g,h,c,b) \to (-1, 0, 0, \infty)$. Also, we computed the ML estimate and Bayesian estimates of the reliability (with $t = 3$) and p^{th} quantile (with $p = 0.5$) functions. Moreover, we computed the point predictors as well as the bounds of the equi-tailed prediction intervals for the future record values $W_{(k)}$, for $k = 1, 2, ..., 7$, from a future sample from the same population. All these results are summarized in Tables 4 and 5.

Table 2: The values of EB and ER of the ML and Bayesian estimates of μ for different choices of θ, r and s with $\mu = 1$.

				$\hat{\mu}_{ML}$		$\hat{\mu}_{BS}$		$\hat{\mu}_{BL}$		$\hat{\mu}_{BE}$	
θ	r	s		EB	ER	EB	ER	EB	ER	EB	ER
0.5	4	4	*IP*	1.0056	1.4154	0.2835	0.5388	0.2327	0.4955	0.0044	0.4451
			NIP	–	–	0.3926	0.9408	0.3162	0.8537	0.0434	0.8133
	6	4	*IP*	1.0355	1.4465	0.3021	0.5523	0.2511	0.5085	0.0254	0.4537
			NIP	–	–	0.4168	0.9667	0.3392	0.8755	0.0681	0.8375
	6	6	*IP*	0.9926	1.4119	0.2657	0.5332	0.2159	0.4914	0.0106	0.4492
			NIP	–	–	0.3764	0.9472	0.3018	0.8638	0.0307	0.8371
	8	6	*IP*	0.9999	1.4199	0.2683	0.5332	0.2182	0.4911	0.0085	0.4489
			NIP	–	–	0.3779	0.9467	0.3022	0.8617	0.0293	0.8308
	8	8	*IP*	0.9930	1.3606	0.2837	0.5392	0.2329	0.4961	0.0066	0.4475
			NIP	–	–	0.3658	0.8757	0.2910	0.7952	0.0140	0.7579
1	4	4	*IP*	0.4577	0.5583	0.0600	0.0742	0.0326	0.0823	0.1154	0.1366
			NIP	–	–	0.0650	0.2641	0.0676	0.2471	0.1563	0.2890
	6	4	*IP*	0.4630	0.5468	0.0562	0.0734	0.0353	0.0812	0.1082	0.1335
			NIP	–	–	0.0678	0.2317	0.0635	0.2131	0.1526	0.2558
	6	6	*IP*	0.5589	0.6813	0.0596	0.0718	0.0666	0.0792	0.1075	0.1271
			NIP	–	–	0.1784	0.4527	0.1486	0.4413	0.1725	0.4737
	8	6	*IP*	0.3169	0.4741	0.0715	0.0917	0.0782	0.0984	0.1178	0.1431
			NIP	–	–	0.0577	0.3055	0.0850	0.3002	0.2662	0.3956
	8	8	*IP*	0.3108	0.3964	0.0465	0.0582	0.0527	0.0647	0.0852	0.1029
			NIP	–	–	0.0688	0.2172	0.0958	0.2204	0.2817	0.3476
3	4	4	*IP*	0.1676	0.2359	0.0007	0.0410	0.0014	0.0418	0.0094	0.0475
			NIP	–	–	0.0183	0.0952	0.0273	0.0971	0.0817	0.1329
	6	4	*IP*	0.1726	0.2411	0.0041	0.0400	0.0023	0.0405	0.0043	0.0445
			NIP	–	–	0.0153	0.0834	0.0241	0.0853	0.0752	0.1218
	6	6	*IP*	0.1654	0.2353	0.0024	0.0392	0.0006	0.0396	0.0049	0.0426
			NIP	–	–	0.0218	0.1623	0.0138	0.1640	0.0337	0.1925
	8	6	*IP*	0.1625	0.2270	0.0032	0.0380	0.0015	0.0384	0.0037	0.0409
			NIP	–	–	0.0633	0.1332	0.0715	0.1374	0.1213	0.1801
	8	8	*IP*	0.1683	0.2372	0.0041	0.0390	0.0025	0.0393	0.0023	0.0416
			NIP	–	–	0.0616	0.0990	0.0692	0.1042	0.1131	0.1448

Table 3: The values of EB and ER of the ML and Bayesian estimates for R(3) and $\xi_{0.5}$ for different choices of θ, r and s with $\mu = 1$.

				$\hat{R}_{ML}$		$\hat{R}_{BS}$		$\hat{\xi}_{0.5ML}$		$\hat{\xi}_{0.5BS}$	
θ	r	s		EB	ER	EB	ER	EB	ER	EB	ER
0.5	4	4	IP	0.5166	3.4632	0.0592	0.1534	0.7951	1.3638	0.3853	0.7431
			NIP	–	–	0.7571	2.4993	–	–	0.4857	1.1109
	6	4	IP	0.4162	1.8468	0.0633	0.1503	0.8507	1.3852	0.3746	7080
			NIP	–	–	0.2497	1.9744	–	–	0.4784	1.0871
	6	6	IP	0.3902	1.2743	0.0598	0.1459	0.8463	1.3659	0.3295	0.6693
			NIP	–	–	0.2035	0.9183	–	–	0.4292	1.0390
	8	6	IP	0.4277	2.1525	0.0626	0.1444	0.8776	1.3898	0.3299	0.6597
			NIP	–	–	0.2696	2.8214	–	–	0.4298	1.0372
	8	8	IP	0.4051	2.8984	0.0681	0.1430	0.8937	1.3284	0.3405	0.6398
			NIP	–	–	0.2452	3.2733	–	–	0.4175	0.9486
1	4	4	IP	0.0975	0.6007	0.0161	0.0730	0.3976	0.6819	0.0751	0.2240
			NIP	–	–	0.0768	0.5706	–	–	0.1760	0.5363
	6	4	IP	0.0999	0.4721	0.0096	0.0668	0.4254	0.6926	0.0985	0.2128
			NIP	–	–	0.0659	0.3236	–	–	0.1704	0.5270
	6	6	IP	0.0978	0.3160	0.0041	0.0610	0.4231	0.6829	0.1191	0.2049
			NIP	–	–	0.0575	0.2039	–	–	0.1406	0.5074
	8	6	IP	0.1086	0.4081	0.0023	0.0587	0.4388	0.6949	0.1260	0.2031
			NIP	–	–	0.0619	0.2847	–	–	0.1393	0.5075
	8	8	IP	0.1001	0.3254	0.0003	0.0548	0.4468	0.6642	0.1317	0.1968
			NIP	–	–	0.0528	0.2136	–	–	0.1337	0.4619
3	4	4	IP	0.0030	0.0117	0.0113	0.0186	0.1325	0.2273	0.0236	0.0808
			NIP	–	–	0.0130	0.0215	–	–	0.0285	0.1717
	6	4	IP	0.0030	0.0120	0.0090	0.0155	0.1418	0.2309	0.0228	0.0725
			NIP	–	–	0.0103	0.0183	–	–	0.0247	0.1713
	6	6	IP	0.0028	0.0098	0.0072	0.0124	0.1410	0.2276	0.0123	0.0655
			NIP	–	–	0.0081	0.0144	–	–	0.0193	0.1682
	8	6	IP	0.0031	0.0105	0.0065	0.0115	0.1463	0.2316	0.0120	0.0635
			NIP	–	–	0.0073	0.0136	–	–	0.0190	0.1691
	8	8	IP	0.0029	0.0087	0.0056	0.0098	0.1489	0.2214	0.0096	0.0587
			NIP	–	–	0.0062	0.0112	–	–	0.0192	0.1548

Table 4: The ML and Bayesian estimates for θ, μ, $R(3)$ and $\xi_{0.5}$.

	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{BS}$	$\widehat{\theta}_{BL}$	$\widehat{\theta}_{BE}$	$\widehat{\mu}_{ML}$	$\widehat{\mu}_{BS}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BE}$	$\widehat{R}_{ML}(3)$	$\widehat{R}_{BS}(3)$	$\widehat{\xi}_{0.5ML}$	$\widehat{\xi}_{0.5BS}$
IP	3.3407	2.7985	2.5689	2.5242	1.2832	1.0180	1.0157	1.0078	0.0032	0.0148	1.4907	1.3031
NIP	–	2.8996	2.6221	2.5748	—	1.0895	1.0782	1.0200	—	0.0167	—	1.3708

Table 5: Bayesian prediction of $W_{(k)}$ for $k = 1, \ldots, 7$.

	Point predictor		Equi-tailed interval	
k	IP	NIP	IP	NIP
1	1.5155	1.4953	(0.7799, 2.7608)	(0.7875, 2.8435)
2	1.9414	1.9011	(1.0219, 3.7634)	(1.0349, 3.8976)
3	2.3674	2.3069	(1.2297, 4.7053)	(1.2465, 4.8873)
4	2.7933	2.7127	(1.0291, 5.6236)	(1.0173, 5.8519)
5	3.2193	3.1184	(1.5818, 6.5300)	(1.6087, 6.8038)
6	3.6452	3.5242	(1.7543, 7.4293)	(1.7883, 7.7482)
7	4.0712	3.9299	(1.9281, 8.3240)	(1.9698, 8.6876)

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