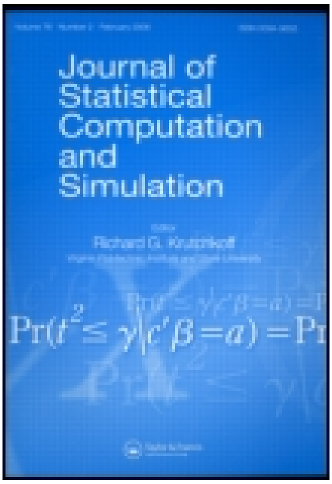

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Bayesian inference based on a jointly type-II censored sample from two exponential populations

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In this paper, based on a jointly type-II censored sample from two exponential populations, the Bayesian inference for the two unknown parameters are developed with the use of squared-error, linear-exponential and general entropy loss functions. The problem of predicting the future failure times, both point and interval prediction, based on the observed joint type-II censored data, is also addressed from a Bayesian viewpoint. A Monte Carlo simulation study is conducted to compare the Bayesian estimators with the maximum likelihood estimator developed by Balakrishnan and Rasouli [Exact likelihood inference for two exponential populations under joint type-II censoring. Comput Stat Data Anal. 2008;52:2725–2738]. Finally, a numerical example is utilized for the purpose of illustration.

Keywords: exponential distribution; joint type-II censoring; maximum likelihood estimation; Bayesian estimation; Bayesian prediction; squared-error loss; linear-exponential loss; general entropy loss

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1. Introduction

The joint type-II censoring may occur while conducting comparative life-tests of products from different lines of production, for example. To be more precise, suppose products are being manufactured by two different lines under the same conditions and that two independent samples of sizes m and n are selected from these two lines, respectively, and are placed simultaneously on a life-testing experiment. Then, due to cost and time considerations, the experimenter may choose to terminate the life-testing experiment as soon as a certain number of failures occur. The successive failure times and the corresponding product types will be recorded, and the life-testing experiment will get terminated as soon as a pre-specified number of failures is observed. The described joint type-II censoring and inferential methods based on such a scheme have been discussed earlier in the literature; see, for example, Rao et al.,[1] Basu,[2] Johnson and Mehrotra,[3] Mehrotra and Johnson,[4] Bhattacharyya and Mehrotra,[5] Mehrotra and Bhattacharyya,[6] and Bhattacharyya.[7] Recently, Balakrishnan and Rasouli [8] developed exact likelihood inference for the parameters of two exponential populations under joint type-II censoring. They developed

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exact inferential methods based on maximum likelihood (ML) estimates and compared their performance with those based on some other approaches such as bootstrap; see also Rasouli and Balakrishnan [9] for a generalization of their results to progressive type-II censoring.

For Bayesian estimation, we consider in this paper three types of loss functions. The first is the squared-error (SE) loss function (quadratic loss) which is a symmetric function that gives equal importance to overestimation and underestimation in the parameter estimation. The second is the linear-exponential (LINEX) loss function, introduced by Varian,[10] which is asymmetric and gives differing weights to overestimation and underestimation. These loss functions have been used by many authors; see, for example, Rojo,[11] Basu and Ebrahimi,[12] Pandey,[13] Soliman,[14,15] Soliman et al.,[16] and Nassar and Eissa.[17] This function rises approximately exponentially on one side of zero and approximately linearly on the other side. The third loss function is the generalization of the entropy (GE) loss used by several authors (see, for example, Dey et al. [18] and Dey and Liao [19]) in which the shape parameter c is taken to be 1. This more general version allows different shapes of the loss function.

The SE loss function is given by

$$L_{BS}(\phi^*, \phi) \propto (\phi^* - \phi)^2, \tag{1}$$

where ϕ^* is an estimate of the parameter ϕ. Under the assumption that the minimal loss occurs at $\phi^* = \phi$, the LINEX loss function can be expressed as

$$L_{BL}(\phi^*, \phi) \propto \mathrm{e}^{v(\phi^* - \phi)} - v(\phi^* - \phi) - 1, \tag{2}$$

where $v \neq 0$. The sign and magnitude of the shape parameter v represent the direction and degree of asymmetry, respectively. If $v > 0$, the overestimation is considered to be more serious than the underestimation, and vice-versa. For v close to zero, the LINEX loss function is approximately SE loss function and, therefore, almost symmetric. It is easily seen that the (unique) Bayesian estimator of ϕ, denoted by ϕ^*_{BL} under the LINEX loss function, is the value ϕ^* which minimizes $E_\phi(L_{BL}(\phi^*, \phi))$ in Equation (2) and is given by

$$\phi^*_{BL} = -\frac{1}{v} \ln\{E_\phi[\mathrm{e}^{-v\phi}]\}, \tag{3}$$

provided that the involved expectation $E_\phi[\mathrm{e}^{-v\phi}]$ is finite. The problem of choosing the value of the parameter v has been discussed by Calabria and Pulcini.[20] Finally, the modified LINEX loss, i.e. the GE loss function, is given by

$$L_{BE}(\Delta) \propto \left(\frac{\phi^*}{\phi}\right)^c - c \ln\left(\frac{\phi^*}{\phi}\right) - 1. \tag{4}$$

It may be noted that when $c > 0$, a positive error is regarded as more serious than a negative error; on the other hand, when $c < 0$, a negative error is regarded as more serious than a positive error. The Bayes estimate ϕ^*_{BE} relative to the GE loss function is given by

$$\phi^*_{BE} = \{E_\phi[\phi^{-c}]\}^{-1/c}, \tag{5}$$

provided that the involved expectation $E_\phi[\phi^{-c}]$ is finite. It can be shown that, when $c = 1$, the Bayesian estimate in Equation (5) coincides with the Bayesian estimate under the weighted SE loss function. Similarly, when $c = -1$, the Bayesian estimate in Equation (5) coincides with the Bayesian estimate under the SE loss function.

Prediction of future events on the basis of past and present knowledge is a problem of practical interest and it arises naturally in many situations, and possesses varied solutions. As in the case of estimation, a predictor can be either a point or an interval predictor. Bayesian prediction for future observations from certain distributions based on type-II censored samples has been discussed by several authors, including Dunsmore,[21] Nigm and Hamdy,[22] Nigm,[23,24] AL-Hussaini,[25] and Kundu and Howlader.[26]

In this paper, we consider independent samples from two exponential populations when the censoring is implemented on the two samples in a combined manner, and develop Bayesian inference with the use of the SE, LINEX and GE loss functions. In Section 2, the Bayesian estimators based on the SE, LINEX and GE loss functions for the two unknown parameters are derived, and the admissibility of these estimators is then discussed. The problem of predicting the future failure times, both point and interval prediction, based on the observed joint type-II censored data, is then treated from a Bayesian viewpoint in Section 3. Finally, in Section 4, the ML estimates, developed by Balakrishnan and Rasouli,[8] and Bayesian estimates are compared by means of a Monte Carlo simulation study and then a numerical example is presented to illustrate all the inferential results developed here.

2. The posterior distribution and the Bayesian estimation

Suppose $X_1, \dots, X_m$, the lifetimes of m units of Product A, are independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$, and $Y_1, \dots, Y_n$ are i.i.d. random variables with cdf $G(x)$ and pdf $g(x)$, representing the lifetimes of n units of Product B. Furthermore, suppose $W_1 \le \cdots \le W_N$ denote the order statistics of the $N = m + n$ random variables $\{X_1, \dots, X_m; Y_1, \dots, Y_n\}$. Then, under the joint type-II censoring scheme described above, the observable data consist of $(\mathbf{Z}, \mathbf{W})$, where $\mathbf{W} = (W_1, \dots, W_r)$ with r $(1 \le r < N)$ being a pre-fixed integer, and $\mathbf{Z} = (Z_1, \dots, Z_r)$ with $Z_i = 1$ or 0 depending on whether W_i is an X- or Y-failure. Letting $M_j = \sum_{i=1}^{j} Z_i$ denote the number of X-failures among $(W_1, \dots, W_j)$ and $N_j = \sum_{i=1}^{j}(1 - Z_i)$ denote the number of Y-failures among $(W_1, \dots, W_j)$, where $1 \le j \le N$, the joint density function of $(\mathbf{Z}, \mathbf{W})$ is given by

$$f(\mathbf{z}, \mathbf{w}) = \frac{m!n!}{(m - m_r)!(n - n_r)!} \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{1-z_i} \{\bar{F}(w_r)\}^{m-m_r} \{\bar{G}(w_r)\}^{n-n_r}, \tag{6}$$

where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ are the survival functions of the two populations; see, for example, Bhattacharyya.[7]

Here, we suppose that the two populations are exponential with survival functions

$$\bar{F}(x) = \mathrm{e}^{-\theta_1 x} \quad \text{and} \quad \bar{G}(x) = \mathrm{e}^{-\theta_2 x}, x > 0, \theta_1 > 0, \theta_2 > 0. \tag{7}$$

In this case, the likelihood function in Equation (6) readily becomes

$$L(\theta_1, \theta_2, \mathbf{z}, \mathbf{w}) = \frac{m!n!\theta_1^{m_r}\theta_2^{n_r}}{(m - m_r)!(n - n_r)!} \exp[-\{\theta_1 u_1 + \theta_2 u_2\}], \quad 0 < w_1 < \cdots < w_r < \infty, \tag{8}$$

where

$$u_1 = \sum_{i=1}^{r} z_i w_i + (m - m_r) w_r$$

and

$$u_2 = \sum_{i=1}^{r} (1 - z_i) w_i + (n - n_r) w_r.$$

Balakrishnan and Rasouli [8] derived the ML estimators of the two parameters as

$$\hat{\theta}_{1M} = \frac{m_r}{u_1} \quad \text{and} \quad \hat{\theta}_{2M} = \frac{n_r}{u_2}. \tag{9}$$

From a Bayesian viewpoint, we may consider the prior distributions of θ_1 and θ_2 as independent gamma prior distributions, $\mathrm{G}(a_1, b_1)$ and $\mathrm{G}(a_2, b_2)$, respectively. Then, the joint prior distribution of θ_1 and θ_2 is

$$\pi(\theta_1, \theta_2) = \pi_1(\theta_1)\pi_2(\theta_2), \tag{10}$$

where

$$\pi_1(\theta_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1-1} \mathrm{e}^{-b_1\theta_1}, \quad \pi_2(\theta_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} \alpha^{a_2-1} \mathrm{e}^{-b_2\theta_2}, \tag{11}$$

and $\Gamma(\cdot)$ denotes the complete gamma function.

Combining Equations (8) and (10), the posterior joint density function of θ_1 and θ_2 is given by

$$\pi(\theta_1, \theta_2|\mathbf{z}, \mathbf{w}) = \frac{(u_1+b_1)^{m_r+a_1}(u_2+b_2)^{n_r+a_2}\theta_1^{m_r+a_1-1}\theta_2^{n_r+a_2-1}}{\Gamma(m_r+a_1)\Gamma(n_r+a_2)} \exp[-\{\theta_1(u_1+b_1)+\theta_2(u_2+b_2)\}]. \tag{12}$$

From Equation (12), we see that the joint posterior density function of (θ_1, θ_2) is a product of two density functions, and so the posterior density functions of θ_1 and θ_2, given the data, are $\mathrm{G}(m_r + a_1, u_1 + b_1)$ and $\mathrm{G}(n_r + a_2, u_2 + b_2)$, respectively. Therefore, the Bayesian estimators of θ_1 and θ_2 under the SE loss function are simply

$$\hat{\theta}_{1BS} = \frac{m_r + a_1}{u_1 + b_1} \quad \text{and} \quad \hat{\theta}_{2BS} = \frac{n_r + a_2}{u_2 + b_2}. \tag{13}$$

Clearly, $\hat{\theta}_{1BS}$ and $\hat{\theta}_{2BS}$ are the unique Bayesian estimators of θ_1 and θ_2 under the SE loss function, and so are admissible.

Under the LINEX loss function, the Bayesian estimators of θ_1 and θ_2 are given by

$$\hat{\theta}_{1BL} = \frac{m_r + a_1}{\upsilon} \log\left(1 + \frac{\upsilon}{u_1 + b_1}\right) \quad \text{and} \quad \hat{\theta}_{2BL} = \frac{n_r + a_2}{\upsilon} \log\left(1 + \frac{\upsilon}{u_2 + b_2}\right). \tag{14}$$

Further, $\hat{\theta}_{1BL}$ and $\hat{\theta}_{2BL}$ are the unique Bayesian estimators of θ_1 and θ_2 under the LINEX loss function, and so are admissible.

Under the GE loss function, the Bayesian estimators of θ_1 and θ_2 are given by

$$\hat{\theta}_{1BE} = \left(\frac{\Gamma(m_r + a_1 - c)}{\Gamma(m_r + a_1)}\right)^{-1/c} \frac{1}{u_1 + b_1} \quad \text{and} \quad \hat{\theta}_{2BE} = \left(\frac{\Gamma(n_r + a_2 - c)}{\Gamma(n_r + a_2)}\right)^{-1/c} \frac{1}{u_2 + b_2}. \tag{15}$$

Once again, $\hat{\theta}_{1BE}$ and $\hat{\theta}_{2BE}$, being the unique Bayesian estimators of θ_1 and θ_2 under the GE loss function, are admissible.

Remark 2.1 (1) It may be noted that $\hat{\theta}_{1BL}$ and $\hat{\theta}_{2BL}$ tend to $\hat{\theta}_{1BS}$ and $\hat{\theta}_{2BS}$, respectively, as $\upsilon \to 0$;

(2) It can be shown that, when $c = 1$, the Bayesian estimators $\hat{\theta}_{1BE}$ and $\hat{\theta}_{2BE}$ coincide with the corresponding Bayesian estimators under the weighted SE loss function. Similarly, when $c = -1$, the Bayesian estimators $\hat{\theta}_{1BE}$ and $\hat{\theta}_{2BE}$ coincide with the corresponding Bayesian estimators under the SE loss function;

(3) If we use Jeffreys' non-informative priors $I(\theta_1) \propto (1/\theta_1)$ and $I(\theta_2) \propto (1/\theta_2)$ corresponding to the special case when $a_1 = a_2 = b_1 = b_2 = 0$, the Bayesian estimators $\hat{\theta}_{1BS}$ and $\hat{\theta}_{2BS}$ in Equation (13) coincide with the ML estimators $\hat{\theta}_{1L}$ and $\hat{\theta}_{2L}$ in Equation (9).

3. Bayesian prediction

In this section, we discuss the Bayesian point and interval predictions of the future failure W_s, where $r < s \le N$, based on the joint type-II censored data $(\mathbf{Z}, \mathbf{W})$. Here, we consider three different cases for the future failure W_s. The first case is when $m_r < m$ and $n_r = n$. Hence, the future failure W_s is surely an X-failure. The second case is when $m_r = m$ and $n_r < n$, which means the future failure W_s is surely a Y-failure. The third case is when $m_r < m$ and $n_r < n$, which corresponds to the case when we do not know whether the future failure W_s is going to be an X- or Y-failure.

The joint density function of $(\mathbf{Z}, Z_s, \mathbf{W}, W_s)$, where $r < s \le N$, is given by

$$f(\mathbf{z}, z_s, \mathbf{w}, w_s) = \begin{cases} f_1(\mathbf{z}, z_s, \mathbf{w}, w_s), & m_r < m,\ n_r = n, \\ f_2(\mathbf{z}, z_s, \mathbf{w}, w_s), & m_r = m,\ n_r < n, \\ f_3(\mathbf{z}, z_s, \mathbf{w}, w_s), & m_r < m,\ n_r < n, \end{cases} \tag{16}$$

where

$$f_1(\mathbf{z}, z_s, \mathbf{w}, w_s) = \frac{m!n!}{(s-r-1)!(m-m_r-s+r)!} \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{(1-z_i)} \{\bar{F}(w_r) - \bar{F}(w_s)\}^{s-r-1} \times \{\bar{F}(w_s)\}^{m-m_r-s+r} f(w_s), \tag{17}$$

$$f_2(\mathbf{z}, z_s, \mathbf{w}, w_s) = \frac{m!n!}{(s-r-1)!(n-n_r-s+r)!} \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{(1-z_i)} \{\bar{G}(w_r) - \bar{G}(w_s)\}^{s-r-1} \times \{\bar{G}(w_s)\}^{n-n_r-s+r} g(w_s) \tag{18}$$

and

$$f_3(\mathbf{z}, z_s, \mathbf{w}, w_s) = \sum_{z_{r+1}=0}^{1} \cdots \sum_{z_{s-1}=0}^{1} \frac{m!n!}{(m_{s-1}-m_r)!(n_{s-1}-n_r)!(m-m_s)!(n-n_s)!} \times \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{(1-z_i)} \{\bar{F}(w_r) - \bar{F}(w_s)\}^{m_{s-1}-m_r} \{\bar{G}(w_r) - \bar{G}(w_s)\}^{n_{s-1}-n_r} \times \{\bar{F}(w_s)\}^{m-m_s} \{\bar{G}(w_s)\}^{n-n_s} f(w_s)^{z_s} g(w_s)^{(1-z_s)}. \tag{19}$$

Then, from Equations (6) and (16), the conditional density function of W_s, given $(\mathbf{Z}, \mathbf{W}) = (\mathbf{z}, \mathbf{w})$, becomes

$$f(w_s|\mathbf{z}, \mathbf{w}) = \begin{cases} f_1(w_s|\mathbf{z}, \mathbf{w}), & m_r < m,\ n_r = n, \\ f_2(w_s|\mathbf{z}, \mathbf{w}), & m_r = m,\ n_r < n, \\ f_3(w_s|\mathbf{z}, \mathbf{w}) & m_r < m,\ n_r < n, \end{cases} \tag{20}$$

where

$$f_1(w_s|\mathbf{z}, \mathbf{w}) = \frac{(m-m_r)!}{(s-r-1)!(m-m_r-s+r)!} \{\bar{F}(w_r) - \bar{F}(w_s)\}^{s-r-1} \{\bar{F}(w_s)\}^{m-m_r-s+r} \times \frac{f(w_s)}{\{\bar{F}(w_r)\}^{m-m_r}}, \tag{21}$$

$$f_2(w_s|\mathbf{z}, \mathbf{w}) = \frac{(n-n_r)!}{(s-r-1)!(n-n_r-s+r)!} \{\bar{G}(w_r) - \bar{G}(w_s)\}^{s-r-1} \{\bar{G}(w_s)\}^{n-n_r-s+r} \times \frac{g(w_s)}{\{\bar{G}(w_r)\}^{n-n_r}} \tag{22}$$

and

$$f_3(w_s|\mathbf{z},\mathbf{w}) = \sum_{z_{r+1}=0}^{1} \cdots \sum_{z_{s-1}=0}^{1} \sum_{z_s=0}^{1} \frac{(m-m_r)!(n-n_r)!}{(m_{s-1}-m_r)!(n_{s-1}-n_r)!(m-m_s)!(n-n_s)!}$$
$$\times \{\bar{F}(w_r)-\bar{F}(w_s)\}^{m_{s-1}-m_r}\{\bar{G}(w_r)-\bar{G}(w_s)\}^{n_{s-1}-n_r}\{\bar{F}(w_s)\}^{m-m_s}\{\bar{G}(w_s)\}^{n-n_s}$$
$$\times \frac{\{f(w_s)^{z_s} g(w_s)^{(1-z_s)}\}}{\{\bar{F}(w_r)\}^{m-m_r}\{\bar{G}(w_r)\}^{n-n_r}}. \tag{23}$$

Upon substituting Equation (7) in Equations (21)–(23), the conditional density functions of W_s, given $(\mathbf{Z},\mathbf{W}) = (\mathbf{z},\mathbf{w})$, for the three cases become

$$f_1(w_s|\mathbf{z},\mathbf{w}) = \frac{(m-m_r)!}{(m-m_r-s+r)!} \sum_{j_1=0}^{s-r-1} c_{j_1}(s-r-1)$$
$$\times\, \theta_1 \exp[-\theta_1(m-m_r-s+r+j_1+1)(w_s-w_r)], \tag{24}$$
$$f_2(w_s|\mathbf{z},\mathbf{w}) = \frac{(n-n_r)!}{(n-n_r-s+r)!} \sum_{j_2=0}^{s-r-1} c_{j_2}(s-r-1)$$
$$\times\, \theta_2 \exp[-\theta_2(n-n_r-s+r+j_2+1)(w_s-w_r)] \tag{25}$$

and

$$f_3(w_s|\mathbf{z},\mathbf{w}) = (m-m_r)!(n-n_r)! \sum_{z_{r+1}=0}^{1} \cdots \sum_{z_{s-1}=0}^{1} \sum_{z_s=0}^{1} \sum_{j_1=0}^{m_{s-1}-m_r} \sum_{j_2=0}^{n_{s-1}-n_r} \frac{c_{j_1}(m_{s-1}-m_r)c_{j_2}(n_{s-1}-n_r)}{(m-m_s)!(n-n_s)!}$$
$$\times \exp[-\{\theta_1(m-m_{s-1}+j_1)+\theta_2(n-n_{s-1}+j_2)\}(w_s-w_r)], \tag{26}$$

where $c_j(i) = (-1)^j/j!(i-j)!$ for $j = 0, 1, \ldots, i$.

From Equation (12) and Equations (24)–(26), the Bayesian predictive density function of W_s, given $(\mathbf{Z},\mathbf{W}) = (\mathbf{z},\mathbf{w})$, is given by

$$f^*(w_s|\mathbf{z},\mathbf{w}) = \begin{cases} f_1^*(w_s|\mathbf{z},\mathbf{w}), & m_r < m,\ n_r = n, \\ f_2^*(w_s|\mathbf{z},\mathbf{w}), & m_r = m,\ n_r < n, \\ f_3^*(w_s|\mathbf{z},\mathbf{w}), & m_r < m,\ n_r < n, \end{cases} \tag{27}$$

where

$$f_1^*(w_s|\mathbf{z},\mathbf{w}) = \int_0^\infty \int_0^\infty f_1(w_s|\mathbf{z},\mathbf{w})\pi^*(\theta_1,\theta_2|\mathbf{z},\mathbf{w})\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$
$$= \frac{(m-m_r)!(m_r+a_1)}{(m-m_r-s+r)!(u_1+b_1)} \sum_{j_1=0}^{s-r-1} c_{j_1}(s-r-1)$$
$$\times \left[1+\frac{(m-m_r-s+r+j_1+1)(w_s-w_r)}{u_1+b_1}\right]^{-(m_r+a_1+1)}, \tag{28}$$

$$f_2^*(w_s|\mathbf{z},\mathbf{w}) = \int_0^\infty \int_0^\infty f_2(w_s|\mathbf{z},\mathbf{w})\pi^*(\theta_1,\theta_2|\mathbf{z},\mathbf{w})\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$
$$= \frac{(n-n_r)!(n_r+a_2)}{(n-n_r-s+r)!(u_2+b_2)} \sum_{j_2=0}^{s-r-1} c_{j_2}(s-r-1)$$
$$\times \left[1 + \frac{(n-n_r-s+r+j_2+1)(w_s-w_r)}{u_2+b_2}\right]^{-(n_r+a_2+1)} \tag{29}$$

and

$$f_3^*(w_s|\mathbf{z},\mathbf{w}) = \int_0^\infty \int_0^\infty f_3(w_s|\mathbf{z},\mathbf{w})\pi^*(\theta_1,\theta_2|\mathbf{z},\mathbf{w})\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$
$$= (m-m_r)!(n-n_r)! \sum_{z_{r+1}=0}^{1} \cdots \sum_{z_{s-1}=0}^{1} \sum_{j_1=0}^{m_{s-1}-m_r} \sum_{j_2=0}^{n_{s-1}-n_r} c_{j_1}(m_{s-1}-m_r)c_{j_2}(n_{s-1}-n_r)$$
$$\times \left\{ \frac{(n_r+a_2)(u_2+b_2)^{-1}}{(m-m_{s-1})!(n-n_{s-1}-1)!} \left[1 + \frac{(m-m_{s-1}+j_1)(w_s-w_r)}{(u_1+b_1)}\right]^{-(m_r+a_1)} \right.$$
$$\times \left[1 + \frac{(n-n_{s-1}+j_2)(w_s-w_r)}{(u_2+b_2)}\right]^{-(n_r+a_2+1)}$$
$$+ \frac{(m_r+a_1)(u_1+b_1)^{-1}}{(m-m_{s-1}-1)!(n-n_{s-1})!} \left[1 + \frac{(m-m_{s-1}+j_1)(w_s-w_r)}{(u_1+b_1)}\right]^{-(m_r+a_1+1)}$$
$$\left. \times \left[1 + \frac{(n-n_{s-1}+j_2)(w_s-w_r)}{(u_2+b_2)}\right]^{-(n_r+a_2)} \right\}. \tag{30}$$

From Equations (27)– (30), we simply obtain the predictive survival function of W_s, $\bar{F}^*(t|\mathbf{z},\mathbf{w})$, as

$$\bar{F}^*(t|\mathbf{z},\mathbf{w}) = \begin{cases} \bar{F}_1^*(t|\mathbf{z},\mathbf{w}), & m_r < m,\ n_r = n, \\ \bar{F}_2^*(t|\mathbf{z},\mathbf{w}), & m_r = m,\ n_r < n, \\ \bar{F}_3^*(t|\mathbf{z},\mathbf{w}) & m_r < m,\ n_r < n, \end{cases} \tag{31}$$

where

$$\bar{F}_1^*(w_s|\mathbf{z},\mathbf{w}) = \int_t^\infty f_1^*(w_s|\mathbf{z},\mathbf{w})\,\mathrm{d}w_s = \frac{(m-m_r)!}{(m-m_r-s+r)!} \sum_{j_1=0}^{s-r-1} \frac{c_{j_1}(s-r-1)}{m-m_r-s+r+j_1+1}$$
$$\times \left[1 + \frac{(m-m_r-s+r+j_1+1)(t-w_r)}{u_1+b_1}\right]^{-(m_r+a_1)}, \tag{32}$$

$$\bar{F}_2^*(w_s|\mathbf{z},\mathbf{w}) = \int_t^\infty f_2^*(w_s|\mathbf{z},\mathbf{w})\,\mathrm{d}w_s = \frac{(n-n_r)!}{(n-n_r-s+r)!} \sum_{j_2=0}^{s-r-1} \frac{c_{j_2}(s-r-1)}{n-n_r-s+r+j_2+1}$$
$$\times \left[1 + \frac{(n-n_r-s+r+j_2+1)(w_s-w_r)}{u_2+b_2}\right]^{-(n_r+a_2)} \tag{33}$$

and

$$
\begin{aligned}
\bar{F}_3^*(w_s|\mathbf{z},\mathbf{w}) &= \int_t^\infty f_3^*(w_s|\mathbf{z},\mathbf{w})\,\mathrm{d}w_s \\
&= (m-m_r)!(n-n_r)! \sum_{z_{r+1}=0}^{1} \cdots \sum_{z_{s-1}=0}^{1} \sum_{j_1=0}^{m_{s-1}-m_r} \sum_{j_2=0}^{n_{s-1}-n_r} c_{j_1}(m_{s-1}-m_r)c_{j_2}(n_{s-1}-n_r) \\
&\quad \times \left\{ \frac{(n_r+a_2)(u_2+b_2)^{-1}}{(m-m_{s-1})!(n-n_{s-1}-1)!} \int_t^\infty \left[1+\frac{(m-m_{s-1}+j_1)(w_s-w_r)}{(u_1+b_1)}\right]^{-(m_r+a_1)} \right. \\
&\quad \times \left[1+\frac{(n-n_{s-1}+j_2)(w_s-w_r)}{(u_2+b_2)}\right]^{-(n_r+a_2+1)} \mathrm{d}w_s \\
&\quad + \frac{(m_r+a_1)(u_1+b_1)^{-1}}{(m-m_{s-1}-1)!(n-n_{s-1})!} \int_t^\infty \left[1+\frac{(m-m_{s-1}+j_1)(w_s-w_r)}{(u_1+b_1)}\right]^{-(m_r+a_1+1)} \\
&\quad \left. \times \left[1+\frac{(n-n_{s-1}+j_2)(w_s-w_r)}{(u_2+b_2)}\right]^{-(n_r+a_2)} \mathrm{d}w_s \right\}. \qquad (34)
\end{aligned}
$$

It does not seem possible to obtain the predictive survival function in Equation (34) in an explicit form unless a_1 and a_2 are integers, and would of course require numerical integration in general. However, we can use the partial fractions method to derive an explicit expression for the predictive survival function in Equation (34) if both a_1 and a_2 are integers.

The Bayesian point predictors of W_s, $r < s \le N$, under the SE loss function, are obtained simply as the means of the predictive densities in Equations (28)–(30), and this would of course require numerical integration.

The Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for W_s, $r < s \le N$, can be obtained by solving the following two equations

$$\bar{F}_i^*(L|\mathbf{z},\mathbf{w}) = 1-\frac{\gamma}{2} \quad \text{and} \quad \bar{F}_i^*(U|\mathbf{z},\mathbf{w}) = \frac{\gamma}{2}, \ \ i=1,2,3,$$

where $\bar{F}_1^*(t|\mathbf{z},\mathbf{w})$, $\bar{F}_2^*(t|\mathbf{z},\mathbf{w})$ and $\bar{F}_3^*(t|\mathbf{z},\mathbf{w})$ are given, respectively, by Equations (32), (33) and (34), and L and U denote the lower and upper bounds, respectively.

For the highest posterior density (HPD) method, we need to solve the following two equations:

$$\bar{F}_i^*(L_{W_s}|\mathbf{z},\mathbf{w}) - \bar{F}_i^*(U_{W_s}|\mathbf{z},\mathbf{w}) = 1-\gamma$$

and

$$f_i^*(L_{W_s}|\mathbf{z},\mathbf{w}) = f_i^*(U_{W_s}|\mathbf{z},\mathbf{w}), \quad i=1,2,3,$$

where $f_1^*(w_s|\mathbf{z},\mathbf{w})$, $f_2^*(w_s|\mathbf{z},\mathbf{w})$ and $f_3^*(w_s|\mathbf{z},\mathbf{w})$ are given, respectively, by Equations (28), (29) and (30), and L_{W_s} and U_{W_s} denote the HPD lower and upper bounds, respectively.

4. Numerical results and an illustrative example

In this section, the ML estimates, developed by Balakrishnan and Rasouli,[8] and the Bayesian estimates based on the SE, LINEX and GE loss functions are all compared by means of a Monte Carlo simulation study, and a numerical example is finally presented to illustrate all the inferential results established in the preceding sections.

Table 1. The average values of the ML and Bayesian estimates of θ_1 for different choices of m, n, r, a_1, b_1, a_2 and b_2.

(m,n)	r	a_1	b_1	a_2	b_2	$\hat{\theta}_{1M}$	$\hat{\theta}_{1BS}$	$\hat{\theta}_{1BL}$ $v=0.1$	$\hat{\theta}_{1BL}$ $v=0.5$	$\hat{\theta}_{1BL}$ $v=1$	$\hat{\theta}_{1BE}$ $c=-0.5$	$\hat{\theta}_{1BE}$ $c=0.1$	$\hat{\theta}_{1BE}$ $c=0.5$
(10,10)	8	2	1	5	1	2.4335	2.1303	2.0813	1.9124	1.7457	2.0205	1.8862	1.7948
		1.9	1	4.8	1	–	2.0851	2.0371	1.8717	1.7085	1.9754	1.8411	1.7497
		2	1.1	5	1.2	–	2.0339	1.9893	1.8345	1.6803	1.9290	1.8007	1.7135
		0	0	0	0	–	2.4335	1.9893	1.8345	1.6803	1.9290	1.8007	1.7135
	12	2	1	5	1	2.2964	2.1259	2.0894	1.9593	1.8241	2.0449	1.9463	1.8796
		1.9	1	4.8	1	–	2.0928	2.0569	1.9288	1.7957	2.0118	1.9132	1.8465
		2	1.1	5	1.2	–	2.0539	2.0200	1.8982	1.7710	1.9756	1.8803	1.8158
		0	0	0	0	–	2.2964	2.2310	2.0151	1.8134	2.1718	2.0193	1.9154
	16	2	1	5	1	2.2555	2.1395	2.1120	2.0111	1.9019	2.0792	2.0060	1.9567
		1.9	1	4.8	1	–	2.1150	2.0878	1.9880	1.8801	2.0547	1.9815	1.9323
		2	1.1	5	1.2	–	2.0850	2.0589	1.9630	1.8589	2.0262	1.9549	1.9068
		0	0	0	0	–	2.2555	2.2138	2.0674	1.9184	2.1737	2.0742	2.0070
(15,15)	12	2	1	5	1	2.2851	2.1179	2.0798	1.9443	1.8046	2.0322	1.9277	1.8569
		1.9	1	4.8	1	–	2.0828	2.0453	1.9121	1.7746	1.9971	1.8926	1.8219
		2	1.1	5	1.2	–	2.0426	2.0072	1.8809	1.7497	1.9599	1.8591	1.7908
		0	0	0	0	–	2.2851	2.2151	1.9869	1.7771	2.1502	1.9844	1.8709
	18	2	1	5	1	2.2070	2.1168	2.0900	1.9915	1.8845	2.0566	1.9836	1.9344
		1.9	1	4.8	1	–	2.0923	2.0658	1.9684	1.8627	2.0321	1.9592	1.9100
		2	1.1	5	1.2	–	2.0636	2.0382	1.9445	1.8424	2.0049	1.9338	1.8858
		0	0	0	0	–	2.2070	2.1679	2.0292	1.8864	2.1262	2.0279	1.9614
	24	2	1	5	1	2.1795	2.1211	2.1017	2.0287	1.9466	2.0777	2.0252	1.9900
		1.9	1	4.8	1	–	2.1036	2.0843	2.0119	1.9305	2.0602	2.0077	1.9725
		2	1.1	5	1.2	–	2.0825	2.0639	1.9935	1.9142	2.0399	1.9884	1.9538
		0	0	0	0	–	2.1795	2.1543	2.0615	1.9598	2.1263	2.0618	2.0185
(20,20)	16	2	1	5	1	2.1861	2.0912	2.0606	1.9496	1.8313	2.0212	1.9362	1.8788
		1.9	1	4.8	1	–	2.0627	2.0325	1.9230	1.8063	1.9927	1.9077	1.8503
		2	1.1	5	1.2	–	2.0307	2.0019	1.8970	1.7847	1.9627	1.8801	1.8243
		0	0	0	0	–	2.1861	2.1387	1.9744	1.8113	2.0870	1.9659	1.8836
	24	2	1	5	1	2.1406	2.0894	2.0686	1.9908	1.9039	2.0418	1.9841	1.9454
		1.9	1	4.8	1	–	2.0701	2.0495	1.9724	1.8863	2.0225	1.9648	1.9261
		2	1.1	5	1.2	–	2.0481	2.0281	1.9533	1.8696	2.0014	1.9449	1.9069
		0	0	0	0	–	2.1406	2.1133	2.0132	1.9048	2.0811	2.0089	1.9603
	32	2	1	5	1	2.1268	2.0937	2.0790	2.0228	1.9581	2.0601	2.0194	1.9922
		1.9	1	4.8	1	–	2.0802	2.0655	2.0097	1.9454	2.0465	2.0059	1.9786
		2	1.1	5	1.2	–	2.0644	2.0501	1.9955	1.9325	2.0312	1.9911	1.9642
		0	0	0	0	–	2.1268	2.1091	2.0422	1.9663	2.0876	2.0402	2.0085

4.1. *Monte Carlo simulation*

A simulation study was carried out for evaluating the performance of the ML estimates, developed by Balakrishnan and Rasouli,[8] and all the Bayesian estimates discussed in the preceding sections. We chose the two sample sizes as $(m,n) = (10,10), (15,15), (20,20)$, the choices of $r = 8, 12, 16, 18, 24, 32$, and the parameters (θ_1, θ_2) to be $(2,5)$. We also obtained results for some other choices of (θ_1, θ_2), but as the findings were quite similar, we present here only for the choice of $(\theta_1, \theta_2) = (2,5)$ for the sake of brevity. To examine the sensitivity of the Bayesian estimates with respect to the hyperparameters (a_1, b_1, a_2, b_2), we used four different choices of the hyperparameters (a_1, b_1, a_2, b_2): $(2,1,5,1), (1.9,1,4.8,1), (2,1.1,5,1.2)$ and $(0,0,0,0)$ (with the last one corresponding to Jeffreys' non-informative prior). For these cases, we computed the ML and the Bayesian estimates of θ_1 and θ_2 under the SE, LINEX and GE loss functions. We repeated this process 10,000 times and computed the average values of all the estimates and the estimated risk (ER) for each estimate by using the root mean square error. The average values of all the estimates

Table 2. The average values of the ML and Bayesian estimates of θ_2 for different choices of m, n, r, a_1, b_1, a_2 and b_2.

(m,n)	r	a_1	b_1	a_2	b_2	$\hat{\theta}_{2M}$	$\hat{\theta}_{2BS}$	$\hat{\theta}_{2BL}$ $v=0.1$	$\hat{\theta}_{2BL}$ $v=0.5$	$\hat{\theta}_{2BL}$ $v=1$	$\hat{\theta}_{2BE}$ $c=-0.5$	$\hat{\theta}_{2BE}$ $c=0.1$	$\hat{\theta}_{2BE}$ $c=0.5$
(10,10)	8	2	1	5	1	5.7139	5.1453	5.0176	4.5813	4.1572	5.0227	4.8742	4.7744
		1.9	1	4.8	1	–	5.0460	4.9207	4.4928	4.0769	4.9234	4.7749	4.6750
		2	1.1	5	1.2	–	4.6669	4.5620	4.1985	3.8378	4.5556	4.4209	4.3303
		0	0	0	0	–	5.7139	5.3873	4.4776	3.7792	5.4553	5.1396	4.9253
	12	2	1	5	1	5.5802	5.1920	5.0851	4.7106	4.3329	5.0909	4.9686	4.8865
		1.9	1	4.8	1	–	5.1103	5.0050	4.6364	4.2647	5.0091	4.8868	4.8047
		2	1.1	5	1.2	–	4.7872	4.6965	4.3755	4.0468	4.6939	4.5811	4.5054
		0	0	0	0	–	5.5802	5.3665	4.6997	4.1204	5.4028	5.1875	5.0422
	16	2	1	5	1	5.5489	5.2120	5.1162	4.7762	4.4267	5.1222	5.0137	4.9409
		1.9	1	4.8	1	–	5.1395	5.0450	4.7097	4.3651	5.0497	4.9412	4.8685
		2	1.1	5	1.2	–	4.8472	4.7646	4.4688	4.1608	4.7637	4.6628	4.5951
		0	0	0	0	–	5.5489	5.3740	4.8067	4.2885	5.4034	5.2272	5.1085
(15,15)	12	2	1	5	1	5.4765	5.1563	5.0538	4.6933	4.3275	5.0585	4.9404	4.8611
		1.9	1	4.8	1	–	5.0773	4.9764	4.6214	4.2611	4.9796	4.8614	4.7821
		2	1.1	5	1.2	–	4.7675	4.6802	4.3696	4.0499	4.6772	4.5679	4.4946
		0	0	0	0	–	5.4765	5.2808	4.6606	4.1113	5.3097	5.1073	4.9708
	18	2	1	5	1	5.3450	5.1536	5.0722	4.7784	4.4689	5.0760	4.9825	4.9198
		1.9	1	4.8	1	–	5.0911	5.0106	4.7204	4.4146	5.0135	4.9199	4.8572
		2	1.1	5	1.2	–	4.8415	4.7698	4.5094	4.2322	4.7686	4.6807	4.6218
		0	0	0	0	–	5.3450	5.2160	4.7757	4.3469	5.2303	5.0915	4.9982
	24	2	1	5	1	5.3247	5.1645	5.0933	4.8336	4.5544	5.0971	5.0159	4.9615
		1.9	1	4.8	1	–	5.1102	5.0399	4.7828	4.5065	5.0429	4.9617	4.9073
		2	1.1	5	1.2	–	5.5489	5.3740	4.8067	4.2884	5.4034	5.2272	5.1085
		0	0	0	0	–	5.3247	5.2197	4.8511	4.4781	5.2309	5.1176	5.0416
(20,20)	16	2	1	5	1	5.3606	5.1563	5.0713	4.7659	4.4460	5.0752	4.9774	4.9118
		1.9	1	4.8	1	–	5.0909	5.0070	4.7055	4.3896	5.0098	4.9120	4.8464
		2	1.1	5	1.2	–	4.8310	4.7565	4.4871	4.2020	4.7550	4.6633	4.6019
		0	0	0	0	–	5.3606	5.2223	4.7554	4.3071	5.2379	5.0894	4.9895
	24	2	1	5	1	5.2603	5.1404	5.0746	4.8328	4.5703	5.0776	5.0018	4.9511
		1.9	1	4.8	1	–	5.0898	5.0247	4.7853	4.5253	5.0270	4.9513	4.9006
		2	1.1	5	1.2	–	4.8863	4.8269	4.6079	4.3684	4.8265	4.7545	4.7063
		0	0	0	0	–	5.2603	5.1675	4.8369	4.4953	5.1754	5.0728	5.0040
	32	2	1	5	1	5.2482	5.1490	5.0923	4.8814	4.6484	5.0950	5.0300	4.9865
		1.9	1	4.8	1	–	5.1056	5.0494	4.8403	4.6092	5.0516	4.9866	4.9431
		2	1.1	5	1.2	–	4.9288	4.8769	4.6834	4.4684	4.8772	4.8149	4.7733
		0	0	0	0	–	5.2482	5.1725	4.8972	4.6038	5.1787	5.0949	5.0387

of θ_1 and θ_2 so computed are summarized in Tables 1 and 2, respectively, and the corresponding ER values are summarized in Tables 3 and 4, respectively.

From these results, it is clear that the bias of the Bayesian estimates based on the LINEX and GE loss functions are smaller than those of the ML estimates and the Bayesian estimates based on the SE loss function. Moreover, we observe that the ML estimates and the Bayesian estimates have a moderate bias when the essential sample size r is small even when the sample sizes m and n are not small. However, the bias of all estimates become negligible when r increases, as is evident from Tables 1 and 2.

From Tables 3 and 4, we observe that the ERs of the Bayesian estimates based on the LINEX and GE loss functions are smaller than those of the ML estimates and the Bayesian estimates based on the SE loss function. We also observe that the ERs of all the estimates decrease with increasing r even when the sample sizes m and n are small. Moreover, the ERs of the Bayesian estimates based on the LINEX loss function decrease in v, while the ERs of the Bayesian estimates based on the GE loss function for positive values of c are less than the ERs for negative values of c.

Table 3. The values of the ERs of the ML and Bayesian estimates of θ_1 for different choices of m, n, r, a_1, b_1, a_2 and b_2.

(m,n)	r	a_1	b_1	a_2	b_2	ER_M	ER_{BS}	ER_{BL} $v=0.1$	ER_{BL} $v=0.5$	ER_{BL} $v=1$	ER_{BE} $c=-0.5$	ER_{BE} $c=0.1$	ER_{BE} $c=0.5$
(10,10)	8	2	1	5	1	1.6473	0.7206	0.6900	0.6141	0.5940	0.6922	0.6817	0.6906
		1.9	1	4.8	1	–	0.7072	0.6799	0.6161	0.6070	0.6857	0.6844	0.6994
		2	1.1	5	1.2	–	0.6635	0.6421	0.5966	0.6012	0.6514	0.6605	0.6814
		0	0	0	0	–	1.6473	1.4901	1.1318	0.9657	1.5341	1.4336	1.3957
	12	2	1	5	1	1.1975	0.6905	0.6663	0.5980	0.5631	0.6649	0.6470	0.6438
		1.9	1	4.8	1	–	0.6790	0.6567	0.5956	0.5684	0.6572	0.6444	0.6448
		2	1.1	5	1.2	–	0.6445	0.6257	0.5766	0.5599	0.6283	0.6222	0.6268
		0	0	0	0	–	1.1975	1.1254	0.9274	0.8074	1.1339	1.0739	1.0460
	16	2	1	5	1	0.9675	0.6457	0.6257	0.5632	0.5195	0.6213	0.5990	0.5890
		1.9	1	4.8	1	–	0.6352	0.6162	0.5582	0.5196	0.6129	0.5935	0.5856
		2	1.1	5	1.2	–	0.6070	0.5903	0.5400	0.5093	0.5881	0.5729	0.5679
		0	0	0	0	–	0.9675	0.9241	0.7908	0.6917	0.9224	0.8761	0.8507
(15,15)	12	2	1	5	1	1.2639	0.7166	0.6927	0.6266	0.5953	0.6941	0.6812	0.6822
		1.9	1	4.8	1	–	0.7063	0.6844	0.6259	0.6023	0.6880	0.6807	0.6854
		2	1.1	5	1.2	–	0.6713	0.6531	0.6067	0.5935	0.6587	0.6581	0.6670
		0	0	0	0	–	1.2639	1.1887	0.9869	0.8691	1.2031	1.1496	1.1283
	18	2	1	5	1	0.9158	0.6413	0.6242	0.5718	0.5371	0.6222	0.6067	0.6013
		1.9	1	4.8	1	–	0.6330	0.6169	0.5687	0.5388	0.6160	0.6034	0.6001
		2	1.1	5	1.2	–	0.6084	0.5943	0.5530	0.5301	0.5947	0.5859	0.5851
		0	0	0	0	–	0.9158	0.8814	0.7758	0.6995	0.8807	0.8475	0.8316
	24	2	1	5	1	0.7412	0.5744	0.5613	0.5182	0.4834	0.5577	0.5416	0.5336
		1.9	1	4.8	1	–	0.5673	0.5548	0.5141	0.4823	0.5518	0.5373	0.5305
		2	1.1	5	1.2	–	0.5490	0.5011	0.5011	0.4737	0.5352	0.5231	0.5178
		0	0	0	0	–	0.7412	0.7199	0.6499	0.5910	0.7161	0.6905	0.6765
(20,20)	16	2	1	5	1	1.0140	0.6693	0.6516	0.6005	0.5731	0.6530	0.6435	0.6438
		1.9	1	4.8	1	–	0.6619	0.6455	0.5999	0.5781	0.6486	0.6429	0.6458
		2	1.1	5	1.2	–	0.6355	0.6216	0.5843	0.5702	0.6262	0.6252	0.6310
		0	0	0	0	–	1.0140	0.9724	0.8522	0.7753	0.9783	0.9486	0.9381
	24	2	1	5	1	0.7553	0.5820	0.5699	0.5318	0.5048	0.5686	0.5576	0.5536
		1.9	1	4.8	1	–	0.5761	0.5647	0.5295	0.5058	0.5642	0.5552	0.5527
		2	1.1	5	1.2	–	0.5586	0.5485	0.5178	0.4987	0.5489	0.5424	0.5416
		0	0	0	0	–	0.7553	0.7350	0.6702	0.6205	0.7338	0.7140	0.7048
	32	2	1	5	1	0.6139	0.5116	0.5027	0.4726	0.4471	0.5001	0.4889	0.4833
		1.9	1	4.8	1	–	0.5067	0.4982	0.4697	0.4462	0.4960	0.4859	0.4811
		2	1.1	5	1.2	–	0.4942	0.4863	0.4604	0.4397	0.4846	0.4760	0.4722
		0	0	0	0	–	0.6139	0.6012	0.5582	0.5204	0.5982	0.5823	0.5737

4.2. *Illustrative example*

In order to illustrate the usefulness of the prediction procedures developed in the preceding sections, we consider two samples of size $m = n = 10$ each from Nelson's data (groups 3 and 5 in Table 4.1, [27, p.462]) which correspond to breakdown in minutes of an insulating fluid subjected to high voltage stress. These failure times, denoted here as groups X and Y, are presented in Table 5. Table 6 presents the jointly type-II censored data that have been obtained from the two samples in Table 5 with $r = 15$. We then computed the ML and Bayesian estimates of θ_1 and θ_2 (with the choice of $(a_1, b_1, a_2, b_2) = (1, 1.75, 1, 3)$ as hyperparameters) based on the data in Table 6, and these results are presented in Table 7.

From Table 6, it is clear that $m_r = 9 < m$ and $n_r = 6 < n$. Hence, we cannot say precisely whether the future failures come from the first group X or the second group Y, and then we have to use the predictive density and survival functions given in Equations (30) and (34), respectively, for predicting the future failures. Furthermore, based on the data in Table 6, we computed the

Table 4. The values of the ERs of the ML and Bayesian estimates of θ_2 for different choices of m, n, r, a_1, b_1, a_2 and b_2.

(m,n)	r	a_1	b_1	a_2	b_2	ER_M	ER_{BS}	ER_{BL} $v=0.1$	ER_{BL} $v=0.5$	ER_{BL} $v=1$	ER_{BE} $c=-0.5$	ER_{BE} $c=0.1$	ER_{BE} $c=0.5$
(10,10)	8	2	1	5	1	2.8673	1.1281	1.0715	1.0111	1.1537	1.0993	1.0828	1.0835
		1.9	1	4.8	1	–	1.1038	1.0594	1.0401	1.2071	1.0860	1.0835	1.0936
		2	1.1	5	1.2	–	1.0037	1.0111	1.1292	1.3518	1.0315	1.0802	1.1212
		0	0	0	0	–	2.8673	2.4955	1.8438	1.8181	2.7182	2.5668	2.4861
	12	2	1	5	1	2.2934	1.1482	1.0943	0.9981	1.0652	1.1186	1.0948	1.0866
		1.9	1	4.8	1	–	1.1236	1.0778	1.0116	1.1022	1.1012	1.0866	1.0847
		2	1.1	5	1.2	–	1.0032	0.9955	1.0467	1.2056	1.0133	1.0371	1.0597
		0	0	0	0	–	2.2934	2.0879	1.6398	1.5617	2.2001	2.1039	2.0509
	16	2	1	5	1	2.1148	1.1409	1.0884	0.9780	1.0088	1.1107	1.0836	1.0713
		1.9	1	4.8	1	–	1.1160	1.0699	0.9843	1.0371	1.0914	1.0714	1.0642
		2	1.1	5	1.2	–	0.9885	0.9744	0.9939	1.1183	0.9906	1.0028	1.0167
		0	0	0	0	–	2.1148	1.9445	1.5402	1.4273	2.0349	1.9501	1.9011
(15,15)	12	2	1	5	1	2.1541	1.1315	1.0835	1.0023	1.0726	1.1066	1.0879	1.0827
		1.9	1	4.8	1	–	1.1107	1.0703	1.0171	1.1089	1.0927	1.0827	1.0834
		2	1.1	5	1.2	–	1.0054	1.0003	1.0543	1.2090	1.0176	1.0430	1.0662
		0	0	0	0	–	2.1541	1.9799	1.6036	1.5532	2.0753	1.9963	1.9546
	18	2	1	5	1	1.7104	1.0774	1.0389	0.9591	0.9874	1.0567	1.0391	1.0321
		1.9	1	4.8	1	–	1.0603	1.0267	0.9659	1.0115	1.0440	1.0321	1.0290
		2	1.1	5	1.2	–	0.9668	0.9569	0.9749	1.0781	0.9708	0.9830	0.9956
		0	0	0	0	–	1.7104	1.6140	1.3805	1.3294	1.6630	1.6156	1.5903
	24	2	1	5	1	1.5595	1.0448	1.0091	0.9255	0.9301	1.0245	1.0058	0.9969
		1.9	1	4.8	1	–	1.0282	0.9963	0.9279	0.9476	1.0113	0.9969	0.9911
		2	1.1	5	1.2	–	0.9361	0.9232	0.9213	0.9956	0.9353	0.9403	0.9472
		0	0	0	0	–	1.5595	1.4815	1.2784	1.2115	1.5193	1.4777	1.4544
(20,20)	16	2	1	5	1	1.7649	1.0858	1.0461	0.9672	1.0033	1.0647	1.0474	1.0410
		1.9	1	4.8	1	–	1.0683	1.0341	0.9756	1.0297	1.0521	1.0410	1.0387
		2	1.1	5	1.2	–	0.9740	0.9652	0.9898	1.1023	0.9796	0.9943	1.0089
		0	0	0	0	–	1.7649	1.6608	1.4147	1.3689	1.6647	1.6647	1.6386
	24	2	1	5	1	1.4340	1.0145	0.9846	0.9159	0.9238	0.9983	0.9838	0.9773
		1.9	1	4.8	1	–	1.0011	0.9746	0.9192	0.9404	0.9879	0.9773	0.9735
		2	1.1	5	1.2	–	0.9245	0.9142	0.9157	0.9846	0.9253	0.9315	0.9388
		0	0	0	0	–	1.4340	1.3745	1.2213	1.1790	1.4033	1.3727	1.3565
	32	2	1	5	1	1.3070	0.9723	0.9452	0.8766	0.8680	0.9567	0.9418	0.9344
		1.9	1	4.8	1	–	0.9595	0.9350	0.8769	0.8794	0.9463	0.9344	0.9289
		2	1.1	5	1.2	–	0.8867	0.8750	0.8631	0.9083	0.8844	0.8857	0.8891
		0	0	0	0	–	1.3070	1.2588	1.1268	1.0754	1.2808	1.2538	1.2388

Table 5. The failure time data for groups X and Y.

Group	Data									
X	1.99,	0.64,	2.15,	1.08,	2.57,	0.93,	4.75,	0.82,	2.06,	0.49
Y	8.11,	3.17,	5.55,	0.80,	0.20,	1.13,	6.63,	1.08,	2.44,	0.78

point predictors and their standard errors, as well as the lower and upper 95% prediction bounds for W_s, by using the survival and HPD methods (for $s = 16, 17, 18, 19$ and 20), and these results are all presented in Table 8.

From Table 8, we observe the HPD method to be more precise than the method based on the survival function. Also, the width of the prediction intervals (based on both methods) and the corresponding standard errors increase when s increases, as one would expect.

Table 6. The jointly type-II censored data, with $r = 15$, from groups X and Y in Table 5.

w	0.20	0.49	0.64	0.78	0.80	0.82	0.93	1.08	1.08	1.13	1.99	2.06	2.15	2.44	2.57
z	0	1	1	0	0	1	1	1	0	0	1	1	1	0	1

Table 7. The ML and Bayesian estimates of θ_1 and θ_2.

		$\hat{\theta}_1$	$\hat{\theta}_2$
ML		0.5882	0.3591
SE		0.5865	0.3551
LINEX	$v = 0.1$	0.5848	0.3542
LINEX	$v = 0.5$	0.5781	0.3507
LINEX	$v = 1$	0.5699	0.3464
GE	$c = -0.5$	0.5720	0.3427
GE	$c = 0.1$	0.5545	0.3276
GE	$c = 0.5$	0.5427	0.3174

Table 8. The point predictors and their standard errors, and the lower and upper 95% prediction bounds for W_s for $s = 16, \dots, 20$.

			Survival method		HPD method	
s	$\hat{W}_s$	$SE(\hat{W}_s)$	L_{W_s}	U_{W_s}	L_{W_s}	U_{W_s}
16	3.4180	1.0386	2.5879	6.1710	2.5700	5.3156
17	4.4780	1.7926	2.7507	9.1447	2.5844	7.7718
18	5.8913	2.7492	3.0767	12.9946	2.7142	10.9769
19	8.0113	4.2214	3.6031	18.8894	2.9869	15.8503
20	12.2513	7.5048	4.5475	31.6345	3.4614	26.1665

5. Conclusions and discussion

In this paper, the Bayesian estimation based on the SE, LINEX and GE loss functions for the unknown parameters of two exponential distributions has been discussed based on a jointly type-II censored sample. Both Bayesian point and interval predictions of the future failures have been developed based on the observed joint type-II censored data. The ML estimates, developed by Balakrishnan and Rasouli,[8] and the Bayesian estimates have then been compared through a Monte Carlo simulation study and a numerical example has also been presented to illustrate all the inferential results established here. The computational results show that the Bayesian estimation based on the SE, LINEX and GE loss functions is more precise than the ML estimation. Also, the ERs of all the estimates decrease with increasing r even when the sample sizes m and n are small and the ERs of the Bayesian estimates based on the LINEX loss function decrease in v, while the ERs of the Bayesian estimates based on the GE loss function for positive values of c are less than the ERs for negative values of c. Moreover, the HPD prediction intervals seem to be more precise than the equi-tailed prediction intervals.

In the case when $m_r = m$, we are certain that the future failures come from the second group Y, and similarly in the case when $n_r = n$, we are certain that the future failures come from the first group X. But, in the case when $m_r < m$ and $n_r < n$, we do not know precisely whether the future failures come from the first group X or the second group Y. It may be of interest to develop a method for this prediction problem. We are currently working on this problem and hope to report these findings in a future paper.

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