Thus, the alternating series converges.

(b) We can show that $a_k \geq a_{k+1}$ for every k; however,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n-3} = \frac{1}{2} \neq 0,$$

and hence the series diverges, by the nth-term test (8.17)(i).

The alternating series test (8.30) may be used if condition (i) holds for $k > m$ for some positive integer m, because this corresponds to deleting the first m terms of the series.

If a series converges, then the nth partial sum S_n can be used to approximate the sum S of the series. In many cases, it is difficult to determine the accuracy of the approximation. However, for an *alternating series*, the next theorem provides a simple way of estimating the error that is involved.

Theorem 8.31

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series that satisfies conditions (i) and (ii) of the alternating series test. If S is the sum of the series and S_n is a partial sum, then

$$|S - S_n| \leq a_{n+1};$$

that is, the error involved in approximating S by S_n is less than or equal to a_{n+1}.

PROOF The series obtained by deleting the first n terms of $\sum(-1)^{n-1}a_n$, namely,

$$(-1)^n a_{n+1} + (-1)^{n+1}a_{n+2} + (-1)^{n+2}a_{n+3} + \cdots,$$

also satisfies the conditions of (8.30) and therefore has a sum R_n. Thus,

$$S - S_n = R_n = (-1)^n(a_{n+1} - a_{n+2} + a_{n+3} - \cdots)$$

and

$$|R_n| = a_{n+1} - a_{n+2} + a_{n+3} - \cdots.$$

Employing the same argument used in the proof of the alternating series test, we see that $|R_n| \leq a_{n+1}$. Consequently,

$$E = |S - S_n| = |R_n| \leq a_{n+1},$$

which is what we wished to prove. ■

In the next example, we use Theorem (8.31) to approximate the sum of an alternating series. In order to discuss the accuracy of an approximation, we must first agree on what is meant by one-decimal-place accuracy, two-decimal-place accuracy, and so on. Let us adopt the following convention. If E is the error in an approximation, then the approximation will be considered accurate to k decimal places if $|E| < 0.5 \times 10^{-k}$. For example, we have

1-decimal-place accuracy if $|E| < 0.5 \times 10^{-1} = 0.05$
2-decimal-place accuracy if $|E| < 0.5 \times 10^{-2} = 0.005$
3-decimal-place accuracy if $|E| < 0.5 \times 10^{-3} = 0.0005$.

EXAMPLE ▪ 2 Prove that the series

$$1 - \frac{1}{3!} + \frac{1}{5!} - \cdots + (-1)^{n-1}\frac{1}{(2n-1)!} + \cdots$$

is convergent, and approximate its sum S to five decimal places.

SOLUTION The nth term $a_n = 1/(2n-1)!$ has limit 0 as $n \to \infty$, and $a_k > a_{k+1}$ for every positive integer k. Hence the series converges, by the alternating series test. If we use S_n to approximate S, then, by Theorem (8.31), the error involved is less than or equal to $a_{n+1} = 1/(2n+1)!$. Calculating several values of a_{n+1}, we find that for $n = 4$,

$$a_5 = \frac{1}{9!} \approx 0.0000028 < 0.000005.$$

Hence, the partial sum S_4 approximates S to five decimal places. Since

$$S_4 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$$
$$= 1 - \tfrac{1}{6} + \tfrac{1}{120} - \tfrac{1}{5040} \approx 0.841468,$$

we have $S \approx 0.84147$.

It will follow from (8.48)(a) that the sum of the series is $\sin 1$, and hence $\sin 1 \approx 0.84147$.

The following concept is useful in investigating a series that contains both positive and negative terms but is not alternating. It allows us to use tests for positive-term series to establish convergence for other types of series (see Theorem 8.34).

Definition 8.32

A series $\sum a_n$ is **absolutely convergent** if the series

$$\sum |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots$$

is convergent.

Note that if $\sum a_n$ is a positive-term series, then $|a_n| = a_n$, and in this case, absolute convergence is the same as convergence.

EXAMPLE ▪ 3 Prove that the following alternating series is absolutely convergent:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + (-1)^n\frac{1}{n^2} + \cdots$$