King Saud University
Department of Mathematics

First Midterm Exam

Course Title: Math 111 (Integral Calculus)

Date: Wednesday 14 October 2015; 3–4$\frac{1}{2}$ pm

Instructions:

- Students are NOT allowed to use CALCULATORS.

Name .... ID ....

Question	Grade
Q1	
Q2	
Q3	
Q4	
Total	

Question 1	(a)	(b)	(c)	(d)	(e)
Answer	(ii)	(ii)	(ii)	(i)	(iii)

Question 1

Choose the correct answer (write it down on the table above):

(a) $F(x) = (x+1)^2$ is an antiderivative of

- (i) $x^2 + 2x + 1$
- (ii) $2(x+1)$
- (iii) $\frac{1}{3}(x+1)^3 + c$
- (iv) None.

Solution: (ii) $F'(x) = f(x) \implies f(x) = 2(x+1)$.

(b) $\sum_{i=3}^{7} i(i-1)$ equals

- (i) 35
- (ii) 110
- (iii) 40
- (iv) None.

Solution: (ii)

$$\sum_{i=3}^{7} i(i-1) = \sum_{i=1}^{7}(i^2-i) - \sum_{i=1}^{2}(i^2-i) = [\frac{7(8)(15)}{6} - \frac{7(8)}{2}] - [\frac{2(3)(5)}{6} - \frac{2(3)}{2}] = 112 - 2 = 110.$$

(c) If $f(x) = x$ and $g(x) = \sqrt{x}$, then

- (i) $\int_1^2 \sqrt{x}dx \geq \int_1^2 xdx$
- (ii) $\int_1^2 \sqrt{x}dx \leq \int_1^2 xdx$
- (iii) $\int_1^2 \frac{1}{x}dx \geq \int_1^2 \sqrt{x}dx$
- (iv) None.

Solution: (ii) Since $\sqrt{x} \leq x$ on the interval $[1,2]$ then $\int_1^2 \sqrt{x}dx \leq \int_1^2 xdx$.

(d) $\int_0^{2\pi} \sin x dx$ equals

(i) 0

(ii) 1

(iii) 2

(iv) None.

Solution: **(i)** $\int_0^{2\pi} \sin x dx = -\cos x\Big|_0^{2\pi} = -(1-1) = 0$

(e) If $\int_1^6 g(x)dx = 3$ and $\int_1^9 g(x)dx = 10$, then $\int_6^9 g(x)dx$ equals

(i) 3

(ii) -7

(iii) 7

(iv) None

Solution: **(iii)** $\int_1^6 g(x)dx + \int_6^9 g(x)dx = \int_1^9 g(x)dx \implies \int_6^9 g(x)dx = \int_1^9 g(x)dx - \int_1^6 g(x)dx = 10 - 3 = 7$

Question 2

(a) If $f(x)$ is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, prove that $F'(x) = f(x)$ on $[a, b]$.

Solution: We do a regular partition From the definition of the integral

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\Big[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\Big] \\ &= \lim_{h\to 0} \frac{1}{h}\Big[\int_a^{x+h} f(t)dt + \int_x^a f(t)dt\Big] \\ &= \lim_{h\to 0} \frac{1}{h}\Big[\int_x^{x+h} f(t)dt\Big] \end{aligned}$$

Since $h = x + h - x$ then by the Integral mean value theorem we have

$$F'(x) = \lim_{h\to 0} f(c)$$

for number $c \in (x, x + h)$. Since c between x and $x + h$ we have $c \to x$ as $h \to 0$ and since f is continuous

$$F'(x) = f(x).$$

(b) If $F(x) = (\sin x)^2 \int_0^x \sqrt{t}dt$

(i) Find $F'(x)$.
Solution: Let $g(x) = (\sin x)^2$ and $h(x) = \int_0^x \sqrt{t}dt$, then

$$\begin{aligned} F(x) &= g(x)h(x) \qquad \text{"Product of two functions"} \\ &= g'(x)h(x) + g(x)h'(x) \end{aligned}$$

Since $g'(x) = -2\cos x \sin x$ and by The Fundamental Theorem of Calculus $h'(x) = \sqrt{x}$, Then

$$F'(x) = -2\cos x \sin x \int_0^x \sqrt{t}dt + (\sin x)^2\sqrt{x}.$$

(ii) Find $F'(0)$.
Solution: $F(0) = -2\cos(0)\sin(0) \int_0^0 \sqrt{t}dt + (\sin(0))^2\sqrt{0}. = 0 + 0 = 0$

Question 3

(a) Find $\int_0^3 (x^2+1)dx$ using limit of Riemann sum and right endpoints. **Solution:**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$a = 0,\ b = 3,\ f(x) = (x^2+1).$$

1. $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$
2. $c_i = x_i = x_{i-1} + \Delta x$

   $$x_0 = a = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{3}{n}+\frac{3}{n} = 2\frac{3}{n},\ x_3 = \frac{3}{n}+2\frac{3}{n} = 3\frac{3}{n}, \dots,\ x_n = n\frac{3}{n} = 3 = b$$

   this implies that $c_i = x_i = i(\frac{3}{n})$.
3.

$$\begin{aligned}
\int_0^3 (x^2+1)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{3i}{n}) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \big((\frac{3i}{n})^2 + 1\big) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \big(\frac{9i^2}{n^2}\big) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n}\Big[\sum_{i=1}^{n} \frac{9}{n^2} i^2 + \sum_{i=1}^{n} 1\Big] \\
&= \lim_{n\to\infty} \frac{3}{n}\Big[\frac{9}{n^2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\Big] \\
&= \lim_{n\to\infty} \frac{3}{n}\Big[\frac{9}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)(2n+1)}{6} + n\Big] \\
&= \lim_{n\to\infty} \frac{3}{n}\Big[\frac{3}{n}\frac{(2n^2+3n+1)}{2} + n\Big] \\
&= \lim_{n\to\infty} \frac{3}{n}\Big[\frac{3(2n^2+3n+1)+2n^2}{2n}\Big] \\
&= \frac{3}{2}\lim_{n\to\infty} \Big[\frac{(6n^2+9n+3+2n^2)}{n^2}\Big] \\
&= \frac{3}{2}\lim_{n\to\infty} \Big[\frac{8n^2+9n+3}{n^2}\Big] \\
&= \frac{3}{2}(8) = 12.
\end{aligned}$$

(b) Find the value of c that satisfies the Integral Mean Value Theorem for $f(x) = 4x^3+1$ on $[0, 1]$.

Solution:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \tag{1}$$

$a = 0,\ b = 1$ and

$$\int_0^1 f(x)dx = \int_0^1 (4x^3+1)dx = (4.\frac{x^4}{4}+x)\Big|_0^1 = \cancel{4}\frac{x^4}{\cancel{4}}\Big|_0^1 + (1-0) = (1^4-0)+1 = 2$$

Substitute in (1)

$$\begin{aligned} f(c) = 4c^3+1 = 2 &\implies 4c^3 = 1 \\ &\implies c^3 = \frac{1}{4} \\ &\implies c = \sqrt[3]{\frac{1}{4}} \in (0,1) \end{aligned}$$

Question 4

Evaluate the following integrals:

(i) $\int \sqrt{x}\sqrt[3]{x}dx = \int (x)^{\frac{1}{2}}(x)^{\frac{1}{3}}dx = \int x^{\frac{1}{2}+\frac{1}{3}}dx = \int x^{\frac{5}{6}}dx = \frac{x^{\frac{5}{6}+1}}{\frac{5}{6}+1} + c$

(ii) $\int \frac{1}{x\ln x}dx$

Let $u = \ln x$ then $\frac{du}{dx} = \frac{1}{x} \implies du = \frac{1}{x}dx$

Substitute in the integral

$$\int \frac{1}{u}du = \ln|u| + c = \ln|\ln x| + c$$

(iii) $\int_{-1}^{1} f(x)dx$, where $f(x) = \begin{cases} 2x & x > 0 \\ 1 & x \le 0 \end{cases}$

$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} dx + \int_{0}^{1} 2xdx = x\Big|_{-1}^{0} + x^2\Big|_{0}^{1} = (0-(-1)) + (1-0) = 1+1 = 2.$

(iv) $\int \frac{\cos x - \sin x}{\sin x + \cos x}dx.$

Let $u = \sin x + \cos x$ then $\frac{du}{dx} = \cos x - \sin x \implies \int \frac{\cos x - \sin x}{\sin x + \cos x}dx = \int \frac{1}{u}\cdot\frac{du}{dx}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|\sin x + \cos x| + c$