5.4

(S)

① ⓐ Laurant series for $\frac{1}{z+z^2}$ in

ⓐ $0<|z|<1$, ((since $|z|<1 \Rightarrow \sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$))

المطلوب متسلسلة قوى لـ $(z-0)$

$\therefore \frac{1}{z+z^2} = \frac{1}{z(z+1)} = $ ~~$\frac{1}{z(z-(-1))}$~~

$= \frac{1}{z} \frac{1}{1-(-z)}$ (($\because |z|<1 \Rightarrow |-z|<1$ as $|z|=|-z|$ "

$= \frac{1}{z} \cdot \sum_{j=0}^{\infty} (-z)^j$

$= \sum_{j=0}^{\infty} (-1)^j z^{j-1}$

ⓑ $|z|>1 \Rightarrow |\frac{1}{z}|<1$ (($\because \frac{1}{1-\frac{1}{z}} = \sum_{j=0}^{\infty} (\frac{1}{z})^j$))

المطلوب متسلسلة قوى لـ z

أي المطلوب $\sum \square z^{\square}$

$\therefore \frac{1}{z+z^2} = \frac{1}{z^2} \cdot \frac{1}{1+\frac{1}{z}} = \frac{1}{z^2} \cdot \frac{1}{1-(-\frac{1}{z})}$ $(\because |\frac{1}{z}| = |\frac{1}{-z}| < 1)$

$= \frac{1}{z^2} \cdot \sum_{j=0}^{\infty} (-\frac{1}{z})^j$

$= \sum_{j=0}^{\infty} \frac{(-1)^j}{z^{j+2}} = \sum_{j=0}^{\infty} (-1)^j z^{-j-2}$

$= \sum_{j=2}^{\infty} (-1)^j z^{-j}$

ⓒ $0<|z+1|<1$ (($\Rightarrow \sum (z+1)^j = \frac{1}{1-(z+1)} = \frac{1}{1-z-1} = \frac{-1}{z}$))

المطلوب متسلسلة قوى لـ $z+1$

$\therefore \frac{1}{z^2+z} = \frac{1}{(z+1)z} = \frac{1}{z+1} \cdot (\frac{1}{z})$

$= \frac{1}{z+1} \cdot \left(\frac{-1}{1-(z+1)}\right)$

$= \frac{-1}{z+1} \cdot \sum_{j=0}^{\infty} (z+1)^j = \sum_{j=0}^{\infty} -(z+1)^{j-1} = -\sum_{j=-1}^{\infty} (z+1)^j$

(d) $|z+1| > 1 \Rightarrow \left|\frac{1}{z+1}\right| < 1 \Rightarrow \sum \left(\frac{1}{z+1}\right)^j = \frac{1}{1-\frac{1}{z+1}}$ (6)

$$\therefore \frac{1}{z(z+1)} = \frac{1}{z+1}\left[\frac{1}{z+1-1}\right]$$

$$= \frac{1}{(z+1)^2}\left[\frac{1}{1-1/_{z+1}}\right]$$

$$= \frac{1}{(z+1)^2} \cdot \sum_{j=0}^{\infty} \left(1/_{z+1}\right)^j$$

$$= \sum_{j=0}^{\infty} (z+1)^{-j-2}$$

$$= \sum_{j=0}^{\infty} (z+j)^{-j}$$

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3) $$\frac{z}{(z+1)(z-2)} = \frac{(1/3)}{z+1} + \frac{(2/3)}{z-2}$$

(a) $|z| < 1 \Rightarrow \frac{1}{1-z} = \sum z^j$

نبحث عن صيغة $\sum a_j z^j$

$$\therefore \frac{z}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{(2/3)}{z-2}$$

$$= \frac{1}{3}\left[\frac{1}{1-(-z)}\right] + \frac{2}{3}\left[\frac{1}{-2(1-z/2)}\right] \quad \left(\because |z|<1 \Rightarrow |z/2| < \tfrac{1}{2} < 1\right.$$

$$= \frac{1}{3} \cdot \sum_{j=0}^{\infty} (-z)^j + \frac{1}{3} \sum_{j=0}^{\infty} (z/2)^j$$

$$= \frac{1}{3} \sum_{j=0}^{\infty} (-1)^j z^j - \frac{1}{3} \sum z^j/2^j$$

$$= \frac{1}{3} \cdot \sum_{j=0}^{\infty} \left((-1)^j - 2^{-j}\right) z^j$$

(b) $1 < |z| < 2$

$$\therefore \frac{z}{(z+1)(z-2)} = \frac{(1/3)}{(z+1)} + \frac{(2/3)}{(z-2)} = \frac{1}{3}\left[\frac{1}{-z\text{[illegible]}}\right] + \frac{(2/3)}{-2(1+z/2)}$$

$(1-(-\frac{1}{z}))$

$$= \frac{1}{3z} \cdot \sum_{j=0}^{\infty} \left(-\frac{1}{z}\right)^j + \frac{1}{3} \sum_{j=0}^{\infty} (+z/2)^j$$

as $\left|\frac{1}{z}\right| < 1$ as $|2/z| < 1$

(3)

(c) $|z|>2 \Rightarrow |\frac{1}{z}|<\frac{1}{2} \wedge |2/z|<1$

$$\therefore \frac{z}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{(2/3)}{z-2}$$

$$= \frac{1}{3z}\left[\frac{1}{1-(-\frac{1}{z})}\right] + \left(\frac{2}{3}\right)\left(\frac{1}{z}\right)\left[\frac{1}{1-2/z}\right]$$

$$= \frac{1}{3z} \cdot \sum (-\tfrac{1}{z})^j + \frac{1}{3z}\sum (2/z)^j$$

$$= \frac{1}{3}\sum (-1)^j z^{-j-1} - \frac{1}{3}\sum 2^j (z)^{-j-1}$$

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[4] Laurant Series for $\sin(2z)/z^3$, $|z|>0$

$$\therefore \sin(2z) = \sum_{j=0}^{\infty} \frac{(-1)^j (2z)^{2j+1}}{(2j+1)!}$$

$$\Rightarrow \frac{\sin(2z)}{z^3} = \sum_{j=0}^{\infty} \frac{(-1)^j (2)^{2j+1} \cdot z^{2j-2}}{(2j+1)!}$$

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[5] $\frac{z+1}{z(z-4)^3}$ in $0<|z-4|<4$

$$\text{since } \frac{z+1}{z(z-4)^3} = \frac{A}{z} + \frac{B}{(z-4)} + \frac{C}{(z-4)^2} + \frac{D}{(z-4)^3}$$

$$= \frac{(-\frac{1}{64})}{z} + \frac{(\frac{1}{64})}{z-4} + \frac{(-\frac{1}{16})}{(z-4)^2} + \frac{5/4}{(z-4)^3}$$

$$= \left(-\frac{1}{64}\right)\left[\frac{1}{z-4+4}\right] + \left(\frac{1}{64}\right)(z-4)^{-1} + \left(-\frac{1}{16}\right)(z-4)^{-2} + \left(\frac{5}{4}\right)(z-4)^{-3}$$

$$= \left(-\frac{1}{64}\right)\left(\frac{1}{4}\right)\left[\frac{1}{1-\frac{(-(z-4))}{4}}\right] + \downarrow \quad \downarrow \quad \downarrow$$

$$= \frac{-1}{(64)(4)} \cdot \sum \frac{(-1)^j (z-4)^j}{4^j} + \downarrow \quad \downarrow \quad \downarrow$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^{j+1}(z-4)^j}{4^{j+4}}$$

(B)

$f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_j (z-z_0)^{-j}$

$|f| \le M$ in $r < |z-z_0| < R$

$\because a_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{j+1}} dz$ and $f^{(j)}(z_0) = \frac{j!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{j+1}} dz$

$\therefore |f^{(n)}(z)| \le \frac{n!\, M}{R^n}$

now:

\# $|z-z_0| < R$:

$|a_j| = \frac{|f^{(j)}(z_0)|}{j!} \le \frac{j!\, M}{j!\, R^n} = \frac{M}{R^j}$

وهذا هو المطلوب

\# $|z-z_0| > r$

$|a_j| = \frac{1}{2\pi} \left| \int \frac{f(z)}{(z-z_0)^{-j+1}} dz \right|$ → Theorem 5 page. 121

$\le \frac{1}{2\pi} \frac{M}{r^{-j+1}} \ell(C)$ → length of C, $C: |z-z_0| = r$, $\ell(C) = 2\pi r$

$= \frac{1}{2\pi} \frac{M}{r^{-j+1}} \cdot 2\pi r$

$= M r^j$