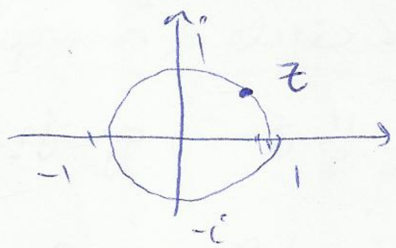
① ⓐ $\sum_{j=0}^{\infty} z^j$ convergence at no points on its circle of convergence $|z|=1$ « diverges on the circle of convergence »

Ans: since $|z^j| = |z|^j$

$= 1^j$

$= 1 \geq 1$



$\Rightarrow$ the geometric series is diverges

ⓑ the series $\sum_{j=1}^{\infty} \frac{z^j}{j^2}$ convergence on the circle of convergence $|z|=1$

Ans:

Since $\left|\frac{z^j}{j^2}\right| = \frac{|z|^j}{j^2} = \frac{1}{j^2}$

and $\sum \frac{1}{j^2}$ is p-series and $p = 2 > 1$

$\Rightarrow \sum \frac{1}{j^2}$ convergence

$\Rightarrow \sum z^j/j^2$ converges

« في هذا السؤال يوضح لنا أنه لا يمكن الحكم على تقارب متسلسلة القوى إذا كانت على دائرة التقارب. »

③ ⓐ $\sum_{j=0}^{\infty} j^3 z^j \quad (z_0 = 0)$

$\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty} \left|\frac{(j+1)^3}{j^3}\right| = 1 = L$

$\Rightarrow$ the radius of convergence is $R = 1/L = 1/1 = 1$

the circle of convergence is $\boxed{|z - 0| = 1}$

(2)

(b) $\sum_{k=0}^{\infty} \underbrace{2^k}_{a_k}(z-1)^k$ «clear that $z_0 = 1$»

$L = \lim_{k\to\infty} \left|\frac{2^{k+1}}{2^k}\right| = \lim_{k\to\infty} |2| = 2$

$\Rightarrow R = 1/L = 1/2$

$\therefore$ the circle of convergence is $|z-1| = \frac{1}{2}$

(c) $\sum_{j=0}^{\infty} \underbrace{j!}_{a_j} z^j$, $a_j = j!$

$L = \lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty} \left|\frac{(j+1)!}{j!}\right| = \lim_{j\to\infty} (j+1) = \infty$

$\Rightarrow R = 1/L = 1/\infty = 0$

$\therefore$ the circle of convergence is $|z-0| = 0$

$\Rightarrow |z| = 0 \Rightarrow z = 0$

the series converge to only at $z=0$

(d) $\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k}(z-i)^k$, $a_k = \frac{(-1)^k}{3^k}$

$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}}{3^{k+1}} \cdot \frac{3^k}{(-1)^k}\right| = 1/3$

$\Rightarrow R = 1/L = 3$

$\therefore |z-i| < 3$

(e) $\sum_{k=1}^{\infty} \frac{(3-i)^k}{k^2}(z+2)^k$, $a_k = \frac{(3-i)^k}{k^2}$

$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(3-i)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(3-i)^k}\right| = \lim_{k\to\infty} \left|\frac{(3-i)\,k^2}{(k+1)^2}\right|$

$= \lim_{k\to\infty} |3-i| = \sqrt{9+1} = \sqrt{10}$

$\Rightarrow R = 1/L = 1/\sqrt{10}$, $|z+2| = \sqrt{10}$

(f) $\sum_{j=0}^{\infty} \frac{z^{2j}}{4^j}$, let $y = z^2$ then we have $\sum_{j=0}^{\infty} \frac{(y)^j}{4^j}$

$L = \lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty} \left|\frac{1}{4^{j+1}} \cdot 4^j\right| = 1/4 \Rightarrow R = 1/(1/4) = 4$

$\therefore |y-0| = 4 \Rightarrow |z^2| = 4 \Rightarrow |z| = 2$

5 (a) $f(z) = \sum_{k=0}^{\infty} (k^3/3^k)\, z^k$

$f^{(6)}(0)$

$f' = \sum_{k=0}^{\infty} (k^3/3^k)\, k z^{k-1} = \sum (k^4)/3^k \, z^{k-1}$

$f'' = \sum_{k=0}^{\infty} \frac{k^4 (k-1)}{3^k} z^{k-2}$

$\vdots$

$f^{(4)} = \sum_{k=0}^{\infty} \frac{k^4 (k-1)(k-2)(k-3)}{3^k} z^{k-4}$

$\vdots$

$f^{(6)} = \sum_{k=0}^{\infty} \frac{k^4 (k-1)(k-2)(k-3)(k-4)(k-5)}{3^k} z^{k-6}$

$\cancel{f^{(6)}(0)} = 0+0+0+0+0+0+ \underset{\text{at } k=6}{\uparrow} + \boxed{\cdot}\, z + \boxed{\ }\, z^2 + \dots$

$$f^{(6)}(0) = \frac{6^4 (5)(4)(3)(2)(1)}{3^6}$$

(b) $\int_{|z|=1} e^z f(z)\, dz = \int \sum_{k=0}^{\infty} (k^3/3^k)\, e^z z^k \, dz$

$= \sum_{k=0}^{\infty} (k^3/3^k) \int_{|z|=1} e^z z^k \, dz$

analytic on $|z|=1$

$= 0$

6 $f(z) = \begin{cases} \sin z/z & z \neq 0 \\ 1 & z = 0 \end{cases}$

$\sin z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}$

$\Rightarrow \sin z/z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!} = 1 - \frac{z}{3!} + \frac{z^4}{5!} - - - - -$

$\because f(0) = 1 \Rightarrow f(z) = 1 - z/3! + \frac{z^4}{5!} - - - - \quad \forall z$

(b) because it is a power series converges inside circle of convergence to analytic function.

(c) $f'(z) = \frac{-2z}{3!} + \frac{4z^3}{5!} - \frac{6z^5}{7!} + \cdots$

$\rightarrow f''(z) = \frac{-2}{3!} + \frac{4(3)z^2}{5!} - \frac{6(5)z^4}{7!} + \cdots$

$\rightarrow f^{(3)}(z) = \frac{24z}{5!} - \frac{6(5)(4)z^3}{7!} + \cdots$

$\Rightarrow f^{(3)}(0) = 0$

$f^{(4)} = 24/5! - \frac{6(5)(4)z^2}{7!} + \cdots$

$\rightarrow f^{(4)}(0) = 24/5! = 1/5$

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9) Let $P_n(z)$ be a sequance of polynomial converges uniformally to $g(z)$

we know $\int_{|z|=1} P_n(z)\,dz = 0 \quad \forall n$ as P_n analytic on loop

Now, by Theorom 8

$$\int P_n(z)\,dz \longrightarrow \int g(z)\,dz$$

but $\int P_n(z)\,dz = 0 \longrightarrow 0$

$\therefore \int g(z)\,dz = 0$ (سو حلانية نهائية)

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10) let R_1 be the radius of convergence of $\sum_{k=0}^{\infty} a_k z^k$

and R_2 " " " " $\sum_{k=1}^{\infty} a_k(k) z^{k-1}$

هذا يعني ان نصف قطر التقارب للمتسلسلة ومشتقتها متساوية

Now, $R_1 = 1/L_1$, where $L_1 = \lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$

and $R_2 = 1/L_2$, " ", $L_2 = \lim \left|\frac{a_{k+1}(k+1)}{a_k\, k}\right| = \lim\left|\frac{a_{k+1}}{a_k}\right| = L_2$

$\Rightarrow R_2 = R_1$