King Saud University
Department of Mathematics

Final Exam in Math 151, S1-1446H.

(This is a two-page long exam)

Calculators are not allowed

9 **Q1.** (a) Without using truth tables, show that $[(p \to \neg q) \wedge p] \to \neg q$ is a tautology. (3)

(b) Use induction to prove that $3 \mid (16^n + 2)$ for all $n \geq 0$. (4)

(c) Let a and b be integers such that $4 \mid (a^2 + 5b^2)$. Use a proof by contradiction to show that a is even or b is even. (2)

11 **Q2.** (a) Let $E = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$ be a relation on $A = \{1, 2, 3, 4, 5\}$.

(i) Represent E with a digraph. (1)

(ii) Show that E is an equivalence relation. (3)

(iii) Find all distinct equivalence classes of E. (1)

(b) Let P be the relation on the set $\mathbb{Z}^+$ of positive integers defined by aPb if and only if there exists a an integer $n \geq 0$ such that $a = 2^n b$.

(i) Show that P is a partial ordering. (3)

(ii) Is P a total ordering? (Justify your answer.) (1)

(iii) Draw the Hasse diagram of P on the subset $\{1, 2, 3, 4, 5, 6, 7, 8\}$ of $\mathbb{Z}^+$. (2)

7 **Q3.** (a) Does a bipartite graph have to always be connected? (Justify your answer.) (1)

(b) Let G be a simple graph with 15 edges such that its complement $\overline{G}$ has 13 edges. Find the number of vertices of G. (Justify your answer.) (2)

(c) Let J be the (undirected) graph represented with the following adjacency matrix.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(i) Determine whether J is bipartite. (Justify your answer.) (1)

(ii) Draw the complement $\overline{J}$ of J. (1)

(iii) Determine if J is isomorphic to $\overline{J}$. (Justify your answer.) (2)

6

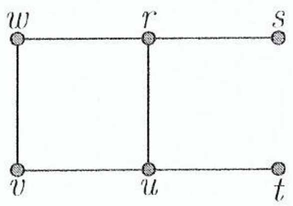
Q4. (a) Let T be a tree with degree-sequence $3, 2, 2, 2, 2, 2, x, y, z$. Find all possible solutions for the triple (x, y, z). (Justify your solutions.) (2)

(b) For the graph H below, find a spanning tree with root r,

(i) using *depth-first* search; (1)

(ii) using *breadth-first* search. (1)

H



(c) Using alphabetical order, form a binary search tree for the words: *Car, Train, Boat, Aeroplane, Bus, Helicopter, Bicycle.* (2)

7

Q5. (a) For the Boolean function $f(x, y, z) = x + \overline{y}z$, find

(i) the complete sum-of-products expansion (CSP); (2)

(ii) the complete product-of-sums expansion (CPS). (2)

(b) Let $g(x, y, z) = xyz + x\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of g. (1)

(ii) Simplify g (i.e., write it in MSP form). (2)

Final Math 151
461.

Q1) 9

a)

$[(p \to \neg q) \wedge p] \to \neg q$

$\equiv [(\neg p \vee \neg q) \wedge p] \to \neg q$ 3

$\equiv \neg[(\neg p \vee \neg q) \wedge p] \vee \neg q$.

$\equiv \neg(\neg p \vee \neg q) \vee \neg p \vee \neg q \equiv T$.

b)

$P(n): 3 \mid (16^n + 2)$.

B.S: $P(0): 16^0 + 2 = 1 + 2 = 3$

$3 \mid 3 \Rightarrow 3 \mid (16^0 + 2)$

$\Rightarrow P(0)$ is true.

I.S: Let $k \geq 0$, we assume that $P(k)$ is true:

$P(k): 3 \mid (16^k + 2) \Rightarrow 16^k + 2 = 3q;\ q \in \mathbb{Z} \Rightarrow 16^k = 3q - 2$.

and we prove that $P(k+1)$ is true: $3 \mid (16^{k+1} + 2)$?

4 $16^{k+1} + 2 = 16(16^k) + 2 = 16(3q - 2) + 2$.

$= 48q - 32 + 2$

$= 48q - 30 = 3(\underbrace{16q - 10}_{\in \mathbb{Z}})$

$\Rightarrow 3 \mid (16^{k+1} + 2)$

$\Rightarrow P(k+1)$ is true

$\Rightarrow P(n)$ is true $\forall n \geq 0$.

c) Let $a, b \in \mathbb{Z}$. We assume that $4 \mid (a^2 + 5b^2)$.

By contradiction, we assume that a is odd and b is odd

2 $\Rightarrow a = 2k + 1;\ k \in \mathbb{Z}$ and $b = (2t + 1),\ t \in \mathbb{Z}$.

$\Rightarrow a^2 = 4k^2 + 4k + 1$ and $b^2 = 4t^2 + 4t + 1 \Rightarrow 5b^2 = 20t^2 + 20t + 5$.

$\Rightarrow a^2 + 5b^2 = 4k^2 + 4k + 20t^2 + 20t + 6$

$= 4(k^2 + k + 5t^2 + 5t + 1) + 2$

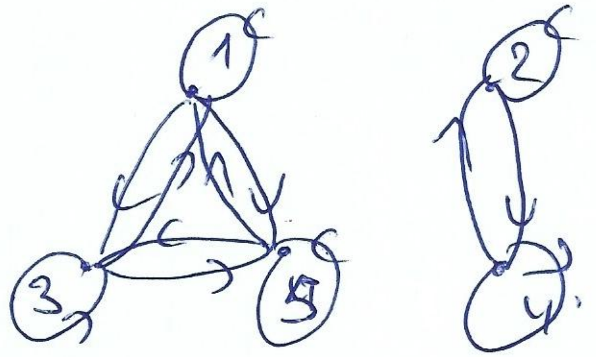
$\Rightarrow 4 \nmid (a^2 + 5b^2) \Rightarrow$ contradiction

$\Rightarrow a$ is even or b is even.

1)

Q2) 11)

a) i)



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ii)

- $(1,1), (2,2), (3,3), (4,4), (5,5) \in E \Rightarrow E$ is reflexive.
- $(1,3) \in E, (3,1) \in E, (1,5) \in E, (5,1) \in E$; $(3,5) \in E, (5,3) \in E$; $(4,2) \in E, (2,4) \in E$ $\Rightarrow E$ is symmetric.
- $(1,3),(3,5) \rightarrow (1,5) \in E$.
  $(1,5),(5,3) \rightarrow (1,3) \in E$.
  $(1,3)(3,1) \rightarrow (1,1) \in E$.
  $(1,5)(5,1) \rightarrow (1,1) \in E$
  $(2,4)(4,2) \Rightarrow (2,2) \in E$.
  $(3,1)$ $(1,3) \Rightarrow (3,3) \in E$.
   $(1,5) \Rightarrow (3,5) \in E$
  $(3,5)$ $(5,1) \Rightarrow (3,1) \in E$.
   $(5,3) \Rightarrow (3,3) \in E$

3

$(4,2),(2,4) \in E \Rightarrow (4,4) \in E$.

$(5,1) \in E$ $(1,5) \in E \Rightarrow (5,5) \in E$
 $(1,3) \in E \Rightarrow (5,3) \in E$.

$(5,3) \in E$ $(3,5) \in E \Rightarrow (5,5) \in E$
 $(3,1) \in E \Rightarrow (5,1) \in E$

$\Rightarrow$ E is transitive

$\Rightarrow$ E is equivalence relation on A.

iii) the distinct classes of E are:
$\{1,3,5\}$; $\{2,4\}$

1

2)

b)

i) • $a \in \mathbb{Z}^+$; $\quad a = 2^0 \cdot a \Rightarrow \exists\, n = 0 \in \mathbb{N}$ s.t $a = 2^n a$

$\Rightarrow aPa$

$\Rightarrow P$ is reflexive.

• $a, b \in \mathbb{Z}^+$; aPb and $bPa \Rightarrow a = 2^n b$ and $b = 2^m a$, $\quad m, n \in \mathbb{N}$

$\Rightarrow a = 2^n \cdot 2^m a = 2^{n+m} a$.

$\Rightarrow n + m = 0 \Rightarrow m = n = 0$

$\Rightarrow a = b$.

$\Rightarrow P$ is antisymmetric.

• $a, b, c \in \mathbb{Z}^+$; aPb and $bPc \Rightarrow a = 2^n b$ and $b = 2^m c$, $\quad n, m \in \mathbb{N}$

3

$\Rightarrow a = 2^n 2^m c$

$= 2^{n+m} c \quad n+m \in \mathbb{N}$

$\Rightarrow aPc$

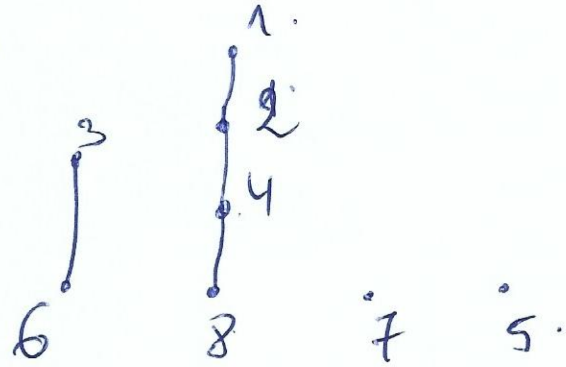
$\Rightarrow P$ is transitive

$\Rightarrow P$ is a partial ordering relation on $\mathbb{Z}^+$.

ii) $2 \not P 3$ and $3 \not P 2$

$\Rightarrow P$ is not total order 1

iii)



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3)

7

P3) a) No, for example.

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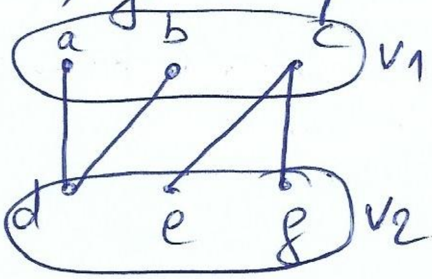


bipartite
not connected.

b)

$$|E(G)| = 15;\quad |E(\bar{G})| = 13.$$

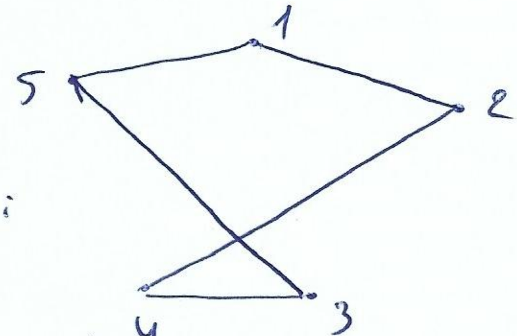
$$|V(G)| = n \Rightarrow |E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}.$$

$$\Rightarrow \frac{n(n-1)}{2} = 28 \Rightarrow n(n-1) = 28 \times 2 = 56$$

$$\Rightarrow \boxed{n = 8.}$$ 2

c)

J:



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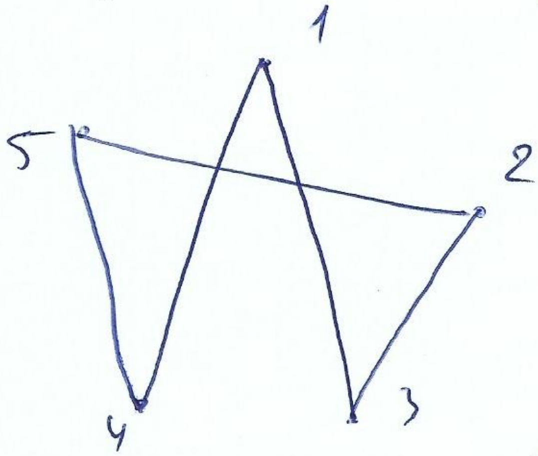
i) J is not bipartite because $J = C_5$. an odd cycle.

1

ii) $\bar{J}$:



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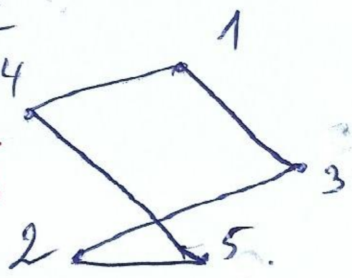
iii) yes J and $\bar{J}$ are isomorphic

	x	1	2	3	4	5
$\bar{J}$	$f(x)$	1	3	5	2	4

2

$\bar{J}$



6 Q4) a) $n = 8 \Rightarrow |E| = 7 = n - 1$.

$$\Rightarrow 3 + 2 + 2 + 2 + 2 + x + y + z = 2 \times 7 = 14$$

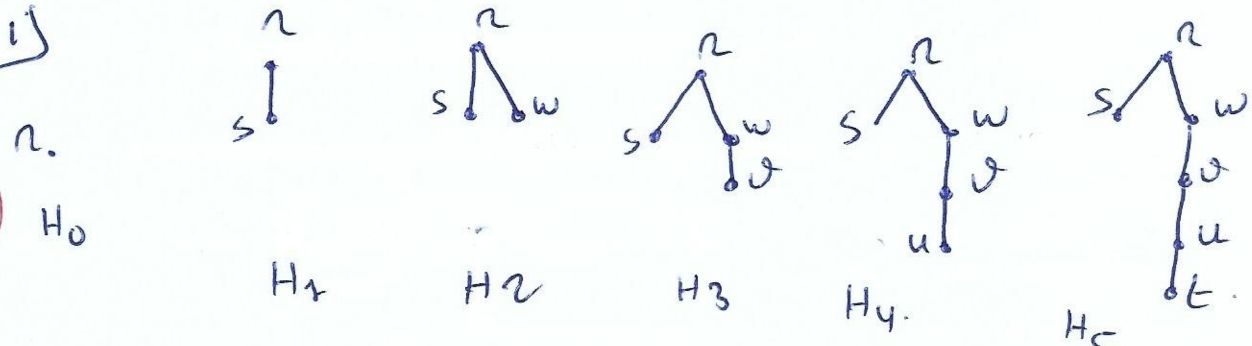
$$\Rightarrow 11 + x + y + z = 14 \Rightarrow x + y + z = 3$$

2 $\Rightarrow (x, y, z) = (1, 1, 1)$.

b)

i)

1



ii)

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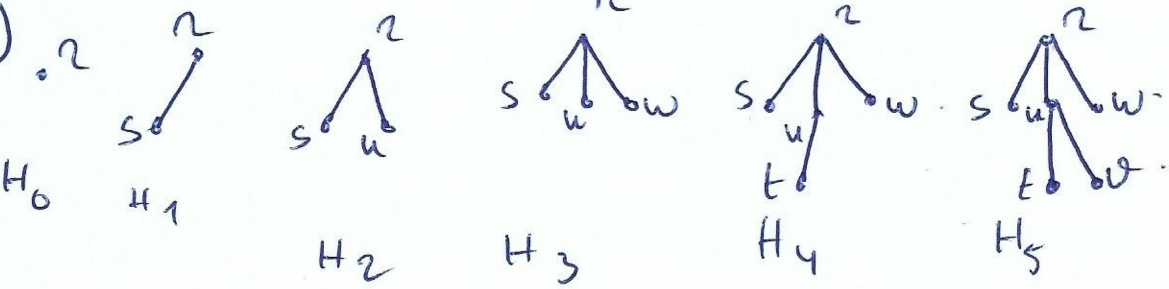

c)

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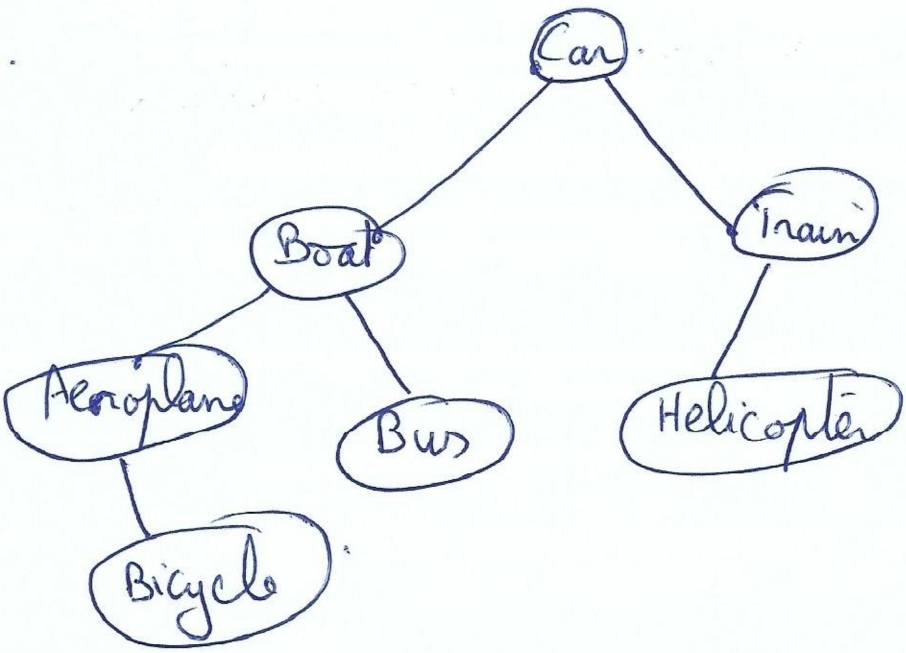

7 Q5

a)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$				1

i) $CSP(f) = xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}\bar{y}z$

2

ii) $CSP(\bar{f}) = \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$ 2

$\Rightarrow CPS(f) = (x+\bar{y}+\bar{z})(x+\bar{y}+z)(x+y+z)$

b) i)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1		1	
$\bar{x}$		1	1	1

1

ii)

$MSP(f) = xyz + \bar{x}\bar{z} + \bar{x}\bar{y} + \bar{y}\bar{z}$