College of Sciences
Mathematics
Department

MATH 111 - Second Exam
First Semester - 1446H
Duration : 90 minutes.

جامعة الملك سعود
King Saud University

Calculator is not allowed

Question (1) : Find y' of the following

1. $y = csch\left(\frac{2}{x}\right) + \tanh^{-1}\left(e^{5x}\right)$ [2 marks]

2. $y = \sinh^{-1}\left(\sqrt{x}\right) + \coth(4x^2)$ [2 marks]

Question (2) : Evaluate the following integrals

1. $\displaystyle\int \frac{sech^2(x^{-2})}{x^3}\,dx$ [2 marks]

2. $\displaystyle\int \frac{3}{\sqrt{x^2+6x-16}}\,dx$ [3 marks]*

3. $\displaystyle\int e^x \cosh 2x\,dx$ [2 marks]

4. $\displaystyle\int (3x-2)\sinh x\,dx$ [2 marks]

5. $\displaystyle\int x^{-2}\ln x\,dx$ [2 marks]

6. $\displaystyle\int \sin^4 x\cos^5 x\,dx$ [2 marks]

7. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$ [3 marks]*

8. $\displaystyle\int \frac{5x^2+x+8}{x^3+4x}\,dx$ [3 marks]*

9. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,dx$ [2 marks]

MATH 111 - Integral Calculus
First Semester - 1446 H
Solution of the Second Exam
Dr Tariq A. Alfadhel

Question (1): [4 marks]

Find $\dfrac{dy}{dx}$ of the following :

1. $y = csch\left(\dfrac{2}{x}\right) + \tanh^{-1}\left(e^{5x}\right)$. [2]

   Solution :

   $$\frac{dy}{dx} = -csch\left(\frac{2}{x}\right)\coth\left(\frac{2}{x}\right)\ \left(\frac{-2}{x^2}\right) + \frac{1}{1-\left(e^{5x}\right)^2}\ \left(e^{5x}\ (5)\right)$$

   $$= \frac{2}{x^2}\ csch\left(\frac{2}{x}\right)\coth\left(\frac{2}{x}\right) + \frac{5e^{5x}}{1-e^{10x}}\ .$$

2. $y = \sinh^{-1}\left(\sqrt{x}\ \right) + \coth(4x^2)$. [2]

   Solution :

   $$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\sqrt{x}\right)^2}}\ \left(\frac{1}{2\sqrt{x}}\right) + \left(-csch^2(4x^2)\ (8x)\right)$$

   $$= \frac{1}{2\sqrt{x}\sqrt{1+x}} - 8x\ csch^2(4x^2)\ .$$

Question (2): [21 marks]

Evaluate the following integrals :

1. $\displaystyle\int \frac{sech^2(x^{-2})}{x^3}\ dx$. [2]

   Solution :

   $$\int \frac{sech^2(x^{-2})}{x^3}\ dx = \int sech^2(x^{-2})\ x^{-3}\ dx$$

   $$= \frac{1}{-2}\int sech^2(x^{-2})\ \left(-2x^{-3}\right)\ dx = -\frac{1}{2}\tanh(x^{-2}) + c$$

2. $\displaystyle\int \frac{3}{\sqrt{x^2+6x-16}}\ dx$. [3]

   Solution : By completing the square.

   $$x^2+6x-16 = (x^2+6x+9)-16-9 = (x+3)^2-(5)^2\ .$$

$$\int \frac{3}{\sqrt{x^2+6x-16}}\,dx = 3\int \frac{1}{\sqrt{(x+3)^2-(5)^2}}\,dx$$

$$= 3\cosh^{-1}\left(\frac{x+3}{5}\right) + c\,.$$

3. $\int e^x \cosh 2x\, dx$. [2]

Solution :

$$\int e^x \cosh 2x\, dx = \int e^x \left(\frac{e^{2x}+e^{-2x}}{2}\right) dx = \int \left(\frac{e^{3x}+e^{-x}}{2}\right) dx$$

$$\int \left(\frac{e^{3x}}{2}+\frac{e^{-x}}{2}\right) dx = \frac{1}{2}\,\frac{1}{3}\int e^{3x}(3)\,dx + \frac{1}{2}\,\frac{1}{-1}\int e^{-x}(-1)\,dx$$

$$= \frac{1}{6}\,e^{3x} - \frac{1}{2}\,e^{-x} + c = \frac{e^{3x}}{6} - \frac{e^{-x}}{2} + c\,.$$

4. $\int (3x-2)\,\sinh x\, dx$. [2]

Solution : Using integration by parts.

$$u = 3x-2 \qquad dv = \sinh x\, dx$$
$$du = 3\,dx \qquad v = \cosh x$$

$$\int (3x-2)\,\sinh x\,dx = (3x-2)\,\cosh x - \int 3\cosh x\,dx$$

$$= (3x-2)\,\cosh x - 3\int \cosh x\,dx = (3x-2)\,\cosh x - 3\sinh x + c\,.$$

5. $\int x^{-2}\,\ln x\, dx$. [2]

Solution : Using integration by parts.

$$u = \ln x \qquad dv = x^{-2}\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$$

$$\int x^{-2}\,\ln x\,dx = \frac{-1}{x}\ln x - \int \frac{-1}{x}\,\frac{1}{x}\,dx$$

$$= \frac{-\ln x}{x} + \int x^{-2}\,dx = \frac{-\ln x}{x} + \frac{x^{-1}}{-1} + c = \frac{-\ln x}{x} - \frac{1}{x} + c\,.$$

6. $\int \sin^4 x \ \cos^5 x \ dx$. [2]

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x \ dx$.

$$\int \sin^4 x \ \cos^5 x \ dx = \int \sin^4 x \ \cos^4 x \ \cos x \ dx = \int \sin^4 x \ \left(\cos^2 x\right)^2 \cos x \ dx$$

$$= \int \sin^4 x \ \left(1 - \sin^2 x\right)^2 \cos x \ dx = \int u^4 \left(1 - u^2\right)^2 \ du = \int u^4 \left(1 - 2u^2 + u^4\right) \ du$$

$$= \int \left(u^4 - 2u^6 + u^8\right) \ du = \frac{u^5}{5} - 2\ \frac{u^7}{7} + \frac{u^9}{9} + c$$

$$= \frac{\sin^5 x}{5} - 2\ \frac{\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$$

7. $\int \frac{\sqrt{x^2 - 9}}{x} \ dx$. [3]

Solution : Using trigonometric substitutions.

Put $x = 3\sec\theta \implies \sec\theta = \frac{x}{3} \implies \cos\theta = \frac{3}{x}$.

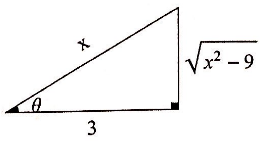
$dx = 3\sec\theta \ \tan\theta \ d\theta$.

$$\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\left(\sec^2\theta - 1\right)} = \sqrt{9\tan^2\theta} = 3\tan\theta$$

$$= \int \frac{\sqrt{x^2 - 9}}{x} \ dx = \int \frac{(3\tan\theta) \ (3\sec\theta \ \tan\theta)}{3\sec\theta} \ d\theta = 3 \int \tan^2\theta \ d\theta$$

$$= 3 \int \left(\sec^2\theta - 1\right) \ d\theta = 3\left(\tan\theta - \theta\right) + c = 3\tan\theta - 3\theta + c$$

$\cos\theta = \frac{3}{x}$.



From the triangle :

$\tan\theta = \frac{\sqrt{9 - x^2}}{3}$

$\theta = \sec^{-1}\left(\frac{x}{3}\right)$

$$\int \frac{\sqrt{x^2 - 9}}{x} \ dx = 3\left(\frac{\sqrt{9 - x^2}}{3}\right) - 3\sec^{-1}\left(\frac{x}{3}\right) + c$$

$$= \sqrt{9 - x^2} - 3\sec^{-1}\left(\frac{x}{3}\right) + c \ .$$

8. $\int \frac{5x^2+x+8}{x^3+4x}\, dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{5x^2+x+8}{x^3+4x} = \frac{5x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{5x^2+x+8}{x(x^2+4)} = \frac{A(x^2+4)}{x(x^2+4)} + \frac{x\,(Bx+C)}{x(x^2+4)}$$

$$5x^2+x+8 = A(x^2+4) + x(Bx+C)$$

$$5x^2+x+8 = Ax^2+4A+Bx^2+Cx = (A+B)x^2+Cx+4A$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A+B=5 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ 4A=8 & \longrightarrow & (3) \end{array}$$

From equation (3) : $4A = 8 \implies A = \frac{8}{4} = 2$.

From equation (1) : $2 + B = 0 \implies B = 3$.

$$\int \frac{5x^2+x+8}{x^3+4x}\, dx = \int \left(\frac{2}{x} + \frac{3x+1}{x^2+4}\right) dx$$

$$= \int \frac{2}{x}\, dx + \int \frac{3x}{x^2+4}\, dx + \int \frac{1}{x^2+4}\, dx$$

$$= 2\int \frac{1}{x}\, dx + \frac{3}{2}\int \frac{2x}{x^2+4}\, dx + \int \frac{1}{x^2+(2)^2}\, dx$$

$$= 2\ln|x| + \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c\,.$$

9. $\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}}$. [2]

Solution : $\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}} = \int \frac{1}{x^{\frac{1}{4}}+x^{\frac{1}{2}}}\, dx$

Using the substitution $x = u^4$, then $u = x^{\frac{1}{4}}$.

$dx = 4u^3\, du$.

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = \int \frac{4u^3}{(u^4)^{\frac{1}{4}}+(u^4)^{\frac{1}{2}}}\, du = \int \frac{4u^3}{u+u^2}\, du$$

$$= \int \frac{4u^3}{u(1+u)}\, du = 4\int \frac{u^2}{u+1}\, du$$

Using long division of polynomials :

$$4\int \frac{u^2}{u+1}\,du = 4\int \left(u-1+\frac{1}{u+1}\right)\,du$$

$$= 4\left(\frac{u^2}{2}-u+\ln|u+1|\right)+c = 2u^2-4u+4\ln|u-1|+c$$

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = 2\left(x^{\frac{1}{4}}\right)^2 - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}}+1\right| + c$$

$$= 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}}+1\right| + c\,.$$