MATH 111 - Integral Calculus
Second Semester - 1447 H
Solution of the Second Exam
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Question (1): [$1 + 1 = 2$ marks]
Find the derivative of the following functions :

1. $f(x) = \sinh^{-1}\left(\ln|x+1|\right)$.

   Solution :

   $$f'(x) = \frac{1}{\sqrt{1 + \left(\ln|x+1|\right)^2}} \; \frac{1}{x+1} = \frac{1}{(x+1)\sqrt{1 + \left(\ln|x+1|\right)^2}} .$$

2. $g(x) = \operatorname{csch}^{-1}\left(\sqrt{x+2}\right)$.

   Solution :

   $$g'(x) = \frac{-1}{\sqrt{x+2}\;\sqrt{\left(\sqrt{x+2}\right)^2 + 1}} \; \frac{1}{2\sqrt{x+2}} = \frac{-1}{2(x+2)\;\sqrt{x+3}} .$$

Question (2): [2 marks]
Calculate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right)$.

Solution :

$$\lim_{x\to 0^+}\left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right) \qquad (\infty - \infty)$$

$$\lim_{x\to 0^+}\left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x - \ln(x+1)}{x\;\ln(x+1)} \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule.

$$\lim_{x\to 0^+}\frac{1 - \frac{1}{x+1}}{\ln(x+1) + x\;\frac{1}{x+1}} = \lim_{x\to 0^+}\frac{\left(\frac{(x+1)-1}{x+1}\right)}{\left(\frac{(x+1)\ln(x+1)+x}{x+1}\right)}$$

$$= \lim_{x\to 0^+}\frac{x}{(x+1)\ln(x+1) + x} \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule.

$$\lim_{x\to 0^+}\frac{1}{\ln(x+1) + (x+1)\;\frac{1}{x+1} + 1} = \lim_{x\to 0^+}\frac{1}{\ln(x+1) + 2} = \frac{1}{0+2} = \frac{1}{2} .$$

Therefore, $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right) = \frac{1}{2}$.

Question (3): [$1.5 + 1 + 1 + 2 + 2 + 2 + 1.5$ marks]
Evaluate the following integrals :

1. $\displaystyle\int x \;\text{sech}^{-1}x \; dx$.

   Solution : Using integration by parts.

   $$u = \text{sech}^{-1}x \qquad dv = x\; dx$$
   $$du = \frac{-1}{x\sqrt{1-x^2}}\; dx \qquad v = \frac{x^2}{2}$$

   $$\int x \;\text{sech}^{-1}x \; dx = \frac{x^2}{2}\;\text{sech}^{-1}x - \int \frac{x^2}{2}\;\frac{-1}{x\sqrt{1-x^2}}\; dx$$
   $$= \frac{x^2}{2}\;\text{sech}^{-1}x + \frac{1}{2}\int \frac{x}{\sqrt{1-x^2}}\; dx$$
   $$= \frac{x^2}{2}\;\text{sech}^{-1}x + \frac{1}{2}\;\frac{1}{-2}\int \left(1-x^2\right)^{-\frac{1}{2}}(-2x)\; dx$$
   $$= \frac{x^2}{2}\;\text{sech}^{-1}x - \frac{1}{4}\;\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{x^2}{2}\;\text{sech}^{-1}x - \frac{\sqrt{1-x^2}}{2} + c$$

2. $\displaystyle\int \frac{1}{x\sqrt{\left(\ln|x|\right)^2 - 1}}\; dx$

   Solution :

   $$\int \frac{1}{x\sqrt{\left(\ln|x|\right)^2 - 1}}\; dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{\left(\ln|x|\right)^2 - 1}}\; dx = \cosh^{-1}\left(\ln|x|\right) + c\;.$$

3. $\displaystyle\int \sec^3 x\;\tan^3 x\; dx$

   Solution :

   Using the substitution $u = \sec x$.

   Hence $du = \sec x\;\tan x\; dx$.

   $$\int \sec^3 x\;\tan^3 x\; dx = \int \sec^2 x\;\tan^2 x\;\sec x\;\tan x\; dx$$
   $$= \int \sec^2 x\;\left(\sec^2 x - 1\right)\;\sec x\;\tan x\; dx = \int u^2\left(u^2 - 1\right)\; du$$
   $$= \int \left(u^4 - u^2\right)\; du = \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$$

4. $\displaystyle\int \frac{1}{(x-2)\sqrt{-x^2+4x-2}}\ dx$.

Solution : By completing the square.

$-x^2+4x-2 = -(x^2-4x)-2 = -(x^2-4x+4)-2+4$

$= 2-(x^2-4x+4) = \left(\sqrt{2}\right)^2 - (x-2)^2$.

$$\int \frac{1}{(x-2)\sqrt{-x^2+4x-2}}\ dx = \int \frac{1}{(x-2)\sqrt{\left(\sqrt{2}\right)^2-(x-2)^2}}\ dx$$

$$= -\frac{1}{\sqrt{2}}\ \operatorname{sech}^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+c\ .$$

5. $\displaystyle\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\ dx$.

Solution : Using trigonometric substitutions.

Put $x = 2\sin\theta \implies \sin\theta = \dfrac{x}{2}$.

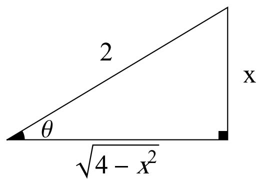
$dx = 2\cos\theta\ d\theta$.

$$\left(4-x^2\right)^{\frac{3}{2}} = \left(4-4\sin^2\theta\right)^{\frac{3}{2}} = \left[4\left(1-\sin^2\theta\right)\right]^{\frac{3}{2}} = \left[(2)^2\ \cos^2\theta\right]^{\frac{3}{2}} = 2^3\ \cos^3\theta.$$

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\ dx = \int \frac{2\cos\theta}{2^3\ \cos^3\theta}\ d\theta = \frac{1}{2^2}\int \frac{1}{\cos^2\theta}\ d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\ d\theta = \frac{1}{4}\tan\theta + c$$

$\sin\theta = \dfrac{x}{2}$.



From the triangle :
$\tan\theta = \dfrac{x}{\sqrt{4-x^2}}$

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\ dx = \frac{1}{4}\ \frac{x}{\sqrt{4-x^2}}+c$$

6. $\displaystyle\int \frac{3x^2+x+4}{x(x^2+4)}\ dx$

Solution : Using the method of partial fractions.

$$\frac{3x^2+x+4}{x(x^2+4)} = \frac{A}{x}+\frac{Bx+C}{x^2+4}$$

$$\frac{3x^2+x+4}{x(x^2+4)} = \frac{A(x^2+4)}{x(x^2+4)}+\frac{(Bx+C)\ x}{x(x^2+4)}$$

$3x^2 + x + 4 = Ax^2 + 4A + Bx^2 + Cx$

$3x^2 + x + 4 = (A + B)x^2 + Cx + 4A$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A + B = 3 & \longrightarrow & (1) \\ C = 1 & \longrightarrow & (2) \\ 4A = 4 & \longrightarrow & (3) \end{array}$$

From equations (3) : $4A = 4 \implies A = \dfrac{4}{4} = 1$.

From equations (1) : $A + B = 3 \implies 1 + B = 3 \implies B = 2$.

$$\int \frac{3x^2 + x + 4}{x(x^2 + 4)}\, dx = \int \left(\frac{1}{x} + \frac{2x + 1}{x^2 + 4} \right) dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{2x + 1}{x^2 + 4}\, dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{2x}{x^2 + 4}\, dx + \int \frac{1}{(x)^2 + (2)^2}\, dx$$

$$= \ln|x| + \ln(x^2 + 4) + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c \ .$$

7. $\displaystyle\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{4}}}\, dx$.

Solution :

Using the substitution $x = u^4$, then $u = x^{\frac{1}{4}}$.

$dx = 4u^3\ du$.

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{4}}}\, dx = \int \frac{4u^3}{(u^4)^{\frac{1}{2}} + (u^4)^{\frac{3}{4}}}\, du = 4 \int \frac{u^3}{u^2 + u^3}\, du$$

$$= 4 \int \frac{u^3}{u^2\ (1 + u)}\, du = 4 \int \frac{u}{u + 1}\, du = 4 \int \frac{(u + 1) - 1}{u + 1}\, du$$

$$= 4 \int \left[\frac{u + 1}{u + 1} - \frac{1}{u + 1} \right] du = 4 \int \left(1 - \frac{1}{u + 1} \right) du$$

$$= 4\left(u - \ln|u + 1|\right) + c = 4u - 4\ln|u + 1| + c = 4x^{\frac{1}{4}} - 4\ln\left|x^{\frac{1}{4}} + 1\right| + c \ .$$