Solution of the second midterm examination Math 106 November 2024

Question 1 : (3+3+3)

a) $(1+4x)^{\cot x} = e^{\cos x \frac{\ln(1+4x)}{\sin x}}$. Then $\lim_{x \longrightarrow 0^+} (1+4x)^{\cot x} = e^4$. **(1)+(2)**

b)

$$\begin{aligned} \int x(\ln x)^2 dx &= \frac{x^2}{2}(\ln x)^2 - \int x \ln x dx \quad (\mathbf{1.5}) \\ &= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + c. \quad (\mathbf{1.5}) \end{aligned}$$

c)

$$\begin{aligned} \int \sec^7 x \tan^5 x dx &\overset{u=\sec x}{=} \int u^6(u^2-1)^2 du \quad (\mathbf{1.5}) \\ &= \frac{(\sec x)^{11}}{11} - 2\frac{(\sec x)^9}{9} + \frac{(\sec x)^7}{7} + c. \quad (\mathbf{1.5}) \end{aligned}$$

Question 2 : (3+3+2)

a) Let $x = \sin\theta$,

$$\begin{aligned} \int x^3(1-x^2)^{\frac{5}{2}} dx &= \int \sin^3\theta \cos^5\theta \cos\theta d\theta \quad (\mathbf{1}) \\ &\overset{u=\cos\theta}{=} -\int u^6(1-u^2)du = -\frac{1}{7}\cos^7\theta + \frac{1}{9}\cos^9\theta + c \\ &= -\frac{1}{7}(1-x^2)^{\frac{7}{2}} + \frac{1}{9}(1-x^2)^{\frac{9}{2}} + c. \quad (\mathbf{2}) \end{aligned}$$

b) $\frac{1}{(x^2-2x+2)(x-1)} = \frac{1}{x-1} + \frac{1-x}{(x-1)^2+1}$. Then **(1)**

$$\int \frac{dx}{(x^2-2x+2)(x-1)} = \ln|x-1| - \frac{1}{2}\ln(x^2-2x+2) + c. \quad \mathbf{(2)}$$

c) Let $t = \tan(\dfrac{x}{2})$,

$$\begin{aligned}
\int \frac{dx}{\sin x - \cos x} &= \int \frac{2dt}{2t + t^2 - 1} \quad (\mathbf{2}) \\
&= \int \frac{2dt}{(t+1)^2 - 2} = \sqrt{2}\ln\left|\frac{1-\sqrt{2}\tan(\frac{x}{2})}{1+\sqrt{2}\tan(\frac{x}{2})}\right| + c. \quad (\mathbf{1}) \\
&= -\sqrt{2}\tanh^{-1}(\frac{\tan(\frac{x}{2})}{\sqrt{2}}) + c.
\end{aligned}$$

Question 3 : (2+3+3)

a) $\displaystyle \int \frac{dx}{\sqrt{x^2 - 4x}} = \int \frac{dx}{\sqrt{(x-2)^2 - 4}} = \cosh^{-1}(\frac{x-1}{2}) + c.$ **(1+1)**

b)

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^{\frac{4}{7}} - x^{\frac{9}{7}}}} &\overset{x=t^7}{=} \int \frac{7t^6}{\sqrt{t^4 - t^9}}dt \quad (\mathbf{1.5}) \\
&= 7\int \frac{t^4}{\sqrt{1-t^5}}dt = -\frac{14}{5}(1-t^5)^{\frac{1}{2}} + c \\
&= -\frac{14}{5}(1 - x^{\frac{5}{7}})^{\frac{1}{2}} + c \quad (\mathbf{1.5})
\end{aligned}$$

c) $\displaystyle \int_1^{+\infty} \frac{dx}{x(1+(\ln x)^2)} \overset{u=\ln x}{=} \tan^{-1} u]_0^{+\infty} = \frac{\pi}{2}$. The integral converges with value $\frac{\pi}{2}$. **(2+1)**