[V. 4]

Answer sheet

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 472 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time:** $1\frac{1}{2}$ **hrs.**

Name: **ID:** **Section:** **Signature:**

Note: Please fill in the above columns. Calculators are not allowed.

Question 1: [Marks: 10]

Which of the given choices are correct? **write** the correct choice number in the following table:

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
B	D	D	C	C	C	C	C	B	C

(i) If $B = \{u,v,w\}$ is a basis of $\mathbb{R}^3$ and $ru + sv + tw = 0$ for some scalars r, s, t, then:
a) $r = u, s = v, t = w$ b) $r = s = t = 0$ c) $u = v = w$ d) B is linearly dependent

(ii) If a subset S of the vector space $\mathbb{R}^3$ spans $\mathbb{R}^3$, what must be true?
a) All vectors in S are non-zero.
b) S contains exactly three vectors.
c) S is linearly independent.
d) S contains at least three linearly independent vectors

(iii) If A is a matrix of size 5×7, then its:
a) rank is 5 b) rank is 7 c) rows are linearly independent d) columns are linearly dependent

(iv) If $S = \{u,v,w\}$ is an ordered basis of a vector space V, then the coordinate vector $[u]_S$ with respect to the ordered basis S is:

a) $\begin{bmatrix}0\\0\\1\end{bmatrix}$ b) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ c) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ d) $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(v) If $S = \{u,v,w,x\}$ is a subset of the vector space $\mathbb{R}^3$, then S must be:
a) a basis of $\mathbb{R}^3$ b) a subspace of $\mathbb{R}^3$ c) linearly dependent d) linearly independent

(vi) Which of the following sets is a subspace of the vector space $\mathbb{R}^3$?
a) $\{(x,y,z): x > 0\}$ b) $\{(x,y,z): x + y + z = 1\}$ c) $\{(x,y,z): x + y + z = 0\}$ d) $\{(x,y,z): x^2 + y^2 + z^2 = 1\}$

(vii) If a subset $S \neq \{0\}$ of a vector space V is linearly dependent, then:
a) The set S must span the vector space V.
b) No vector in S can be written as a linear combination of other vectors in S.
c) At least one vector in S is a linear combination of other vectors in S.
d) One vector in S must be 0.

(viii) Any transition matrix P must satisfy that:
a) P is diagonal b) $\det(P) = 0$ c) P is invertible d) P is symmetric

(ix) If A is a 3×6 matrix with linearly independent rows, then:
a) $\text{rank}(A) = 6$ b) $\text{rank}(A) = 3$ c) $\text{rank}(A) = 4$ d) $\text{rank}(A) = 5$

(x) If a matrix A is of size 4×6 such that $\dim(\text{col}(A)) = 3$, then $\dim(N(A))$ is equal to:
a) ≤ 2 b) 1 c) 3 d) 0

Question 2: [Marks: 2 + 2 + 3]

a) Give an example of a linearly dependent subset of the vector space $\mathbb{R}^3$ which spans $\mathbb{R}^3$.

b) Let $\{u_1, u_2, u_3\}$ be a basis of a vector space V. Determine whether the set $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is a basis of V or not.

c) Let $A = \{v_1 = (1,1,1), v_2 = (2,1,1), v_3 = (1,0,0)\}$. Find a basis B of span(A), and then find a basis C of $\mathbb{R}^3$ such that $B \subseteq C$.

Question 3: [Marks: (2+3) + 3]

a) Consider the two ordered bases $B = \{u_1 = (0,1,1), u_2 = (1,1,1), u_3 = (1,0,1)\}$ and $C = \{v_1 = (0,1,-1), v_2 = (1,1,0), v_3 = (-1,0,1)\}$ of the vector space $\mathbb{R}^3$, and the vector $v = (1,2,1) \in \mathbb{R}^3$.

(i) Find the coordinate vector $[v]_B$ of the vector v with respect to the ordered basis B.

(ii) Construct the transition matrix $P_{B \to C}$ from the basis B to the basis C, and then use it to find the coordinate vector $[v]_C$ of the vector v with respect to the ordered basis C.

b) Find the rank and nullity of the matrix $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$. $\sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Reduced Echelon form 2

We get Rank = 4 and nullity = 0 1

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Q2 2 a) $\{(1,1,1), (1,0,0); (0,1,0); (0,0,1)\}$ subset of $\mathbb{R}^3$ which spans $\mathbb{R}^3$

b) $\alpha(2u_1) + \beta(u_1 - 2u_2) + \gamma(2u_1 + u_3) = 0$

So $(2\alpha + \beta + 2\gamma)u_1 - 2\beta u_2 + \gamma u_3 = 0$ then $\begin{cases} 2\alpha + \beta + 2\gamma = 0 \\ -2\beta = 0 \\ \gamma = 0 \end{cases}$ because $\{u_1, u_2, u_3\}$ is a basis

2

$\Rightarrow \alpha = \beta = \gamma = 0$

Hence $\{2u_1; u_1 - 2u_2; 2u_1 + u_3\}$ is linearly independent and $\dim V = 3$ we deduce $\{2u_1; u_1 - 2u_2 + 2u_1 + u_3\}$ is a basis of V.

c) Clearly $v_1 = v_2 - v_3$

and $v_2 \neq \alpha v_3$ ($\{v_2, v_3\}$ is linearly independent)

2

Hence $\{v_2, v_3\}$ is a basis of Span(A)

1

$$\text{As } \begin{array}{c} \begin{array}{ccc} w & v_2 & v_3 \end{array} \\ \begin{vmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} \end{array} = (0+0+1) - (0+0+0) = 1 \neq 0$$

then $C = \{w, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ that contains B.

Q3 a) (i) $[v]_B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = a u_1 + b u_2 + c u_3 = a\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b+c \\ a+b \\ a+b+c \end{pmatrix}$

2

$\begin{cases} b+c = 1 \\ a+b = 2 \\ a+b+c = 1 \Rightarrow a = 0 \end{cases}$ So $b = 2$ and $c = -1$ So $[v]_B = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

(ii)

2 $${}_CP_B = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} 0 & -1/2 & -1 \\ 1 & 3/2 & 1 \\ 1 & 1/2 & 0 \end{pmatrix} \end{array} \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array}$$

- $u_1 = \alpha v_1 + \beta v_2 + \gamma v_3 = \alpha\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\begin{cases} \beta - \gamma = 0 \\ \alpha + \beta = 1 \\ -\alpha + \gamma = 1 \end{cases} \Rightarrow \beta = 1; \gamma = 1; \alpha = 0$

- $u_2 = \alpha v_1 + \beta v_2 + \gamma v_3 \Rightarrow \begin{cases} \beta - \gamma = 1 \\ \alpha + \beta = 1 \\ -\alpha + \gamma = 1 \end{cases} \Rightarrow \beta = \frac{3}{2}; \gamma = \frac{1}{2}, \alpha = \frac{-1}{2}$

- $u_3 = \alpha v_1 + \beta v_2 + \gamma v_3 \Rightarrow \begin{cases} \beta - \gamma = +1 \\ \alpha + \beta = 0 \\ -\alpha + \gamma = 1 \end{cases} \Rightarrow \beta = 1, \gamma = 0; \alpha = -1$

1 $[v]_C = {}_CP_B [v]_B = \begin{pmatrix} 0 & -1/2 & -1 \\ 1 & 3/2 & 1 \\ 1 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$