King Saud University, College of Sciences
Mathematical Department.
Mid-Term 2/S1/2023 Full Mark:25. Time 1H30mn

Question 1[5,4]. **a)** Find the largest interval about $x_0 = 0$ for which the following initial value problem has a unique solution

$$\begin{cases} y'' + (\tan x)y = \frac{1}{1-x^2} \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

b) Show that the functions: $f(x) = x$, $g(x) = e^x$, $h(x) = \ln(x+2)$ are linearly independent on $(-2, \infty)$.

Question 2[4,4]. **a)** Verify that the function $y = e^x$ is a solution of

$$xy'' - 2(x+1)y' + (x+2)y = 0,$$

then use the reduction of order method to solve the nonhomogeneous equation

$$xy'' - 2(x+1)y' + (x+2)y = e^x, \quad x \neq 0.$$

b) Find the general solution of the differential equation

$$y'' + \frac{5}{x-2}y' + \frac{8}{(x-2)^2}y = 0, \qquad x > 2.$$

Question 3[4,4]. **a)** Solve the differential equations

$$4y'' + y = \sin x + \cos x.$$

b) Find the general solution of a linear homogeneous differential equation with a characteristic equation having the roots: $0, 0, 0, 2+3i, 2-3i, -1+2i, -1-2i$, and then obtain the differential equation.

$$\begin{cases} y'' + (\tan x)\,y = \dfrac{1}{1-x^2} \\ y(0) = 1, \; y'(0) = 0 \end{cases}$$

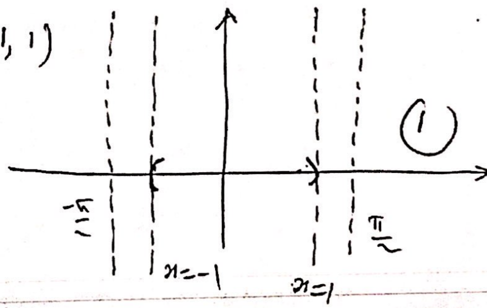
$a_2(x) = 1 \neq 0 \;\; \forall x \in \mathbb{R}$ and continuous on $\mathbb{R}$ (1)

$a_0(x) = \tan x$ is continuous on $\mathbb{R} - \{\frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}\}$ (1)

$g(x) = \dfrac{1}{1-x^2}$ is cont on $\mathbb{R} - \{1, -1\}$

All functions are cont on $(-1, 1)$ which contains $x_0 = 0$. (2)

Hence the largest interval for which the I.V.P admits unique solution is $I = (-1, 1)$



Q1b) $c_1 x + c_2 e^x + c_3 \ln(x+2) = 0 \quad \forall x \in (-2, \infty)$

$$\begin{array}{ll} x=0: & c_2 + c_3 \ln 2 = 0 \longrightarrow (1) \\ x=-1: & -c_1 + c_2 e^{-1} = 0 \longrightarrow (2) \\ x=1: & c_1 + c_2 e + c_3 \ln 3 = 0 \longrightarrow (3) \end{array}$$ (1)

(2) and (3) imply that $c_3 = -\left(\dfrac{e + e^{-1}}{\ln 3}\right) c_2$, then (1) gives (1)

$c_2 \left[1 - \left(\dfrac{e+e^{-1}}{\ln 3}\right) \ln 2\right] = 0 \Rightarrow c_2 = 0$, then from (2) (2)

we have $c_1 = 0$, and from (1) $c_3 = 0$.

Hence $x, e^x, \ln(x+2)$ are LI on $(-2, \infty)$

Q) a)

1) verify that $y_1(x) = e^x$ is a solution of $xy'' - 2(x+1)y' + (x+2)y = 0$ (H)

2) use the reduction order method to solve the equation:

(E) $\boxed{xy'' - 2(x+1)y' + (x+2)y = e^x}$ $x \neq 0$

Solution.

1) $y_1 = e^x$: $xy_1'' - 2(x+1)y_1' + (x+2)y_1$
$= (x - 2(x+1) + x + 2)e^x = 0 \cdot e^x = 0$

then y_1 is a solution of (H) 1

2) Set $y = u \cdot y_1$.

(E) $\Rightarrow x(u''y_1 + uy_1'' + 2u'y_1') - 2(x+1)(u'y_1 + uy_1') + (x+2)uy_1 = e^x$

$\Rightarrow xy_1 u'' + (2xy_1' - 2(x+1)y_1)u' = e^x$ 1

$\Rightarrow xu'' - 2u' = 1$

$\Rightarrow \boxed{u'' - \frac{2}{x}u' = \frac{1}{x}}$

set $W = u'$; then $\boxed{W' - \frac{2}{x}W = \frac{1}{x}}$ (linear equation) 1

- $\mu(x) = e^{-\int \frac{2}{x}dx} = e^{\ln x^{-2}} = \frac{1}{x^2}$
- $\frac{1}{x^2}W(x) = \int \frac{1}{x} \cdot \frac{1}{x^2}dx + \frac{C_2}{2} = \int \frac{dx}{x^3} + C_2 = \frac{-1}{2x^2} + \frac{C_2}{2}$

$\Rightarrow W(x) = -\frac{1}{2} + C_2 x^2$ 1

$\Rightarrow u(x) = -\frac{1}{2}x + C_2\frac{x^3}{3} + C_1$

Hence $\boxed{y_E(x) = C_1 e^x + C_2 \cdot \frac{x^3 e^x}{3} - \frac{xe^x}{2}}$

method 2

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$$

$y'' - \frac{2(x+1)}{x}y' + \frac{x+2}{x}y = 0$

$P(x) = -2\left[1 + \frac{1}{x}\right]$

$y_2(x) = e^x \int \frac{e^{2\int(1+\frac{1}{x})dx}}{e^{2x}}dx$

$= e^x \int x^2 dx = \frac{x^3}{3}e^x$

The general solution for $xy'' - 2(x+1)y' + (x+2)y = 0$ is $y(x) = C_1 e^x + C_2 \frac{x^3}{3}e^x$

For y_p. using method of variation of parameters you get $y_p = -\frac{x}{2}e^x$.

Q2 b)

(Cauchy-Euler DE)

$y'' + \frac{5}{x-2}y' + \frac{8}{(x-2)^2}y = 0$, $x > 2$ (*) 1

$\Rightarrow (x-2)^2 y'' + 5(x-2)y' + 8y = 0$

let $u = x-2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{du}$ 2

$\Rightarrow u^2 y'' + 5uy' + 8y = 0$

$\Rightarrow m^2 + 4m + 8 = 0$

$\Rightarrow m = -2 \pm 2i$

$\Rightarrow y_c = u^{-2}(C_1\cos(2\ln u) + C_2\sin(2\ln u))$

$y_c = (x-2)^{-2}(C_1\cos(2\ln(x-2)) + C_2\sin(2\ln(x-2)))$ 2

- $y(x) = (x-2)^m$
  $y'(x) = m(x-2)^{m-1}$
  $y''(x) = m(m-1)(x-2)^{m-2}$

by substitution in (*)

$(x-2)^m[m(m-1) + 5m + 8] = 0$

$m^2 + 4m + 8 = 0$

$(m+2)^2 = -4$

$m = -2 \pm 2i$

$\alpha = -2$; $\beta = 2$

The general solution is

$$y_c(x) = \frac{1}{(x-2)^2}\left[C_1\cos(2\ln(x-2)) + C_2\sin(2\ln(x-2))\right]$$

Q3 a) $4y''+y=\sin x+\cos x$

$$y_g = y_{gh} + y_p$$

$4y''+y=0$. The Ch Eq $4m^2+1=0 \Rightarrow m=\pm\frac{i}{2}$ (1)

$$\boxed{y_{gh} = C_1\cos\left(\frac{x}{2}\right) + C_2\sin\left(\frac{x}{2}\right)}$$

$y_p = x^s\left[A\cos x + B\sin x\right]e^{0x}$

$\alpha+i\beta = i$ is not a root for the Ch Eq so

then $s=0$. (1)

$$\boxed{y_p = A\cos x + B\sin x}$$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

Then $-4A\cos x - 4B\sin x + A\cos x + B\sin x = \sin x + \cos x$ (2)

$\Rightarrow A=-\frac{1}{3},\quad B=-\frac{1}{3}$

$$\boxed{y_p = -\frac{1}{3}\cos x - \frac{1}{3}\sin x}$$

Hence $y_g = \underbrace{C_1\cos\left(\frac{x}{2}\right) + C_2\sin\left(\frac{x}{2}\right)}_{y_c} \underbrace{- \frac{1}{3}\cos x - \frac{1}{3}\sin x}_{y_p}$

Q3 b). $m_1=0,\ m_2=0,\ m_3=0,\ m_4=2+3i,\ m_5=2-3i$.

$m_6=-1+2i,\ m_7=-1-2i$.

$y_1=1,\ y_2=x,\ y_3=x^2,\ y_4=e^{2x}\cos(3x),\ y_5=e^{2x}\sin(3x)$ (2)

$y_6=e^{-x}\cos(2x),\ y_7=e^{-x}\sin(2x)$

$$y_{gh} = C_1 + C_2x + C_3x^2 + \left[C_4\cos(3x) + C_5\sin(3x)\right]e^{2x} + \left[C_6\cos(2x) + C_7\sin(2x)\right]e^{-x}.$$

Q b) The Ch Eq

$m^3(m-2-3i)(m-2+3i)(m+1-2i)(m+1+2i)=0$

$m^3\left[(m-2)^2+9\right]\left[(m+1)^2+4\right]=0$ (1)

$m^3\left[m^2-4m+13\right]\left[m^2+2m+5\right]=0$

$m^3\left[m^4-2m^3+10m^2+6m+65\right]$

$m^7-2m^6+10m^5+6m^4+65m^3=0$

The DE is

$y^{(7)}-2y^{(6)}+10y^{(5)}+6y^{(4)}+65y^{(3)}=0$ (1)

(!) $Z=\alpha+i\beta \quad \alpha\in\mathbb{R}$

$\bar{Z}=\alpha-i\beta \quad \beta\in\mathbb{R}$

$(m-Z)(m-\bar{Z}) = m^2-2\operatorname{Re}Z\, m + |Z|^2 = m^2-2\alpha m+(\alpha^2+\beta^2)$