King Saud University
Faculty of Sciences
Department of Mathematics

Mid-Exam	Math 204	Semester I/ 07/10/2024
	Full Mark 25	Time: 1.5H

Question 1 :

1. [4] Determine and sketch the largest local region of the $xy-$plane for which the initial problem

$$\begin{cases} \sqrt{x^2-4}\,\frac{dy}{dx} & = & 1+e^x\ln y \\ y(-3) & = & 4. \end{cases}.$$

has a unique solution.

2. [4] Find the general solution of the differential equation

$$(xy+x)dx-(x^2y^2+x^2+y^2+1)dy=0.$$

Question 2 :

1. [4] Obtain the general solution of the differential equation

$$\begin{cases} y(x\sqrt{y}+x^3\sqrt{y}-4x)dx & = & (1+x^2)dy \\ y(0)=1. \end{cases}.$$

2. [4] Solve the differential equation

$$\sin(\pi x)\cos(3\pi y)dx+3\cos(\pi x)\sin(3\pi y)dy=0.$$

Question 3 : [4]

Solve the initial value problem

$$\begin{cases} (x+ye^{\frac{y}{x}})dx & = & xe^{\frac{y}{x}}dy \\ y(1)=0. \end{cases}.$$

Question 4 : [5]

A man with a thermometer in his pocket goes out from his room to outside where the temperature is 50^oC. The thermometer reads 25^oC and 30^oC after 1 minute, 2minutes respectively.
What was the room temperature?

Question 1 :

1. $y' = f(x,y) = \dfrac{1+e^x \ln y}{\sqrt{x^2-4}}$. $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{y+e^x}{y\sqrt{x^2-4}}$. The functions f
2 and $\dfrac{\partial f}{\partial y}$ are continuous on the region $\Omega = \{(x,y) \in \mathbb{R}^2 : \; y > 0, |x| >$
2 $2\}$. Then the largest local region of the $xy-$plane for which the initial problem has a unique solution is $D = \{(x,y) \in \mathbb{R}^2 : \; y > 0, x < -2\}$.

2. $(xy+x)dx-(x^2y^2+x^2+y^2+1)dy = 0 \iff x(y+1)dx = (x^2+1)(y^2+1)dy$. $y = -1$ is a solution (singular solution).
1 The equation becomes $\dfrac{x}{x^2+1}dx = \dfrac{y^2+1}{y+1}dy$. After integration we get $\dfrac{1}{2}y^2 - y + 2\ln|y+1| = \dfrac{1}{2}\ln(x^2+1) + c$.
3 The general solution of the differential equation is defined by

$$\frac{1}{2}y^2 - y + 2\ln|y+1| = \frac{1}{2}\ln(x^2+1) + c.$$

Question 2 :

1. The differential equation $y(x\sqrt{y}+x^3\sqrt{y}-4x)dx = (1+x^2)dy$ is equivalent to

1
$$\frac{dy}{dx} + \frac{4x}{x^2+1}y = xy^{\frac{3}{2}},$$

which is a Bernoulli's equation. If $z = y^{-\frac{1}{2}}$,, we get

1
$$\frac{dz}{dx} - \frac{2x}{x^2+1}z = -\frac{1}{2}x,$$

which is a linear differential equation. The general solution is $z = -\frac{1}{4}(x^2+1)\ln(x^2+1) + c$. As $z(0) = 1$, we get $c = 1$ and the general solution of the differential equation is

2
$$y^{-\frac{1}{2}} = -\frac{1}{4}(x^2+1)\ln(x^2+1) + 1.$$

2. The differential equation is equivalent to

2

$$3\tan(3\pi y)dy = -\tan(\pi x)dx.$$

2 The $\ln|\sec(3\pi y)| = \ln|\cos(\pi x)|+c$ and the general solution is $\sec(3\pi y) = C\cos(\pi x)$.

Question 3 :
The differential equation is equivalent to

1

$$\frac{dy}{dx} = \frac{x + ye^{\frac{y}{x}}}{xe^{\frac{y}{x}}}.$$

If $z = \frac{y}{x}$, we get

1

$$e^z\frac{dz}{dx} = \frac{1}{x}.$$

2 The unique solution is defined by $\ln x = e^{\frac{y}{x}} + 1 = 0$.

Question 4 :
1 Let $\theta(t)$ is the temperature at time t. Then $\theta(t) = 50 + ce^{kt}$. Suppose the room temperature is θ_0, then $c = \theta_0 - 50$ and the equation becomes

$$\theta(t) = 50 + (50 - \theta_0)e^{kt}.$$

1 For $t = 1$, we get $25 = 50 - (50 - \theta_0)e^k$.
2 For $t = 2$, we get $30 = 50 - (50 - \theta_0)e^{2k}$.
1 Then $\theta_0 = 50 - \frac{125}{4} \approx 18.75$.