MATH 111 - Integral Calculus
Second Semester - 1447 H
Solution of the First Exam
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Question (1): [$2 + 1 = 3$ marks]

1. Use Riemann Sum to evaluate the definite integral $\displaystyle\int_0^1 x^3\,dx$.

   Solution : $[a, b] = [0, 1]$, $f(x) = x^3$.

   $$\Delta_x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \; .$$

   $$x_k = a + k\;\Delta_x = 0 + k\left(\frac{1}{n}\right) = \frac{k}{n} \; .$$

   $$f\left(x_k\right) = \left(\frac{k}{n}\right)^3 = \frac{k^3}{n^3} \; .$$

   $$R_n = \sum_{k=1}^{n} f\left(x_k\right)\;\Delta_x = \sum_{k=1}^{n}\left(\frac{k^3}{n^3}\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\frac{k^3}{n^4} = \frac{1}{n^4}\sum_{k=1}^{n}k^3$$

   $$= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \frac{1}{4}\left[\frac{n^2(n+1)^2}{n^4}\right]$$

   $$= \frac{1}{4}\frac{(n+1)^2}{n^2} = \frac{1}{4}\left(\frac{n+1}{n}\right)^2 \; .$$

   $$\int_0^1 x^3\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{1}{4}\left(\frac{n+1}{n}\right)^2\right] = \frac{1}{4}(1)^2 = \frac{1}{4} \; .$$

2. Find $F'(x)$, if $F(x) = \displaystyle\int_{\sqrt{3x^2+4}}^{\sec(x^3)} \sqrt{4+t^2}dt$.

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\sqrt{3x^2+4}}^{\sec(x^3)} \sqrt{4+t^2}dt$$

   $$= \sqrt{4+[\sec(x^3)]^2}\,\big(\sec(x^3)\tan(x^3)(3x^2)\big) - \sqrt{4+(\sqrt{3x^2+4})^2}\,\left(\frac{1}{2\sqrt{3x^2+4}}\,(6x)\right)$$

   $$= 3x^2\sec(x^3)\tan(x^3)\sqrt{4+\sec^2(x^3)} - \frac{3x\sqrt{3x^2+8}}{\sqrt{3x^2+4}} \; .$$

Question (2): Find $\dfrac{dy}{dx}$ of the following :[$1+1=2$ marks]

1. $y(x) = \left[\sec^{-1}(\ln x) + e^{x^4}\right]^5$.

   Solution :

   $$y'(x) = 5\left[\sec^{-1}(\ln x) + e^{x^4}\right]^4 \left(\frac{1}{\ln x\sqrt{(\ln x)^2-1}}\ \frac{1}{x} + e^{x^4}(4x^3)\right)$$

   $$= 5\left[\sec^{-1}(\ln x) + e^{x^4}\right]^4 \left(\frac{1}{x\ln x\sqrt{(\ln x)^2-1}} + 4x^3\ e^{x^4}\right) .$$

2. $y(x) = (\tan x)^x$

   Solution :

   $$y(x) = (\tan x)^x \implies \ln|y(x)| = \ln|(\tan x)^x| = x\ \ln|\tan x|$$

   Differentiate both sides.

   $$\frac{y'(x)}{y(x)} = (1)\ln|\tan x| + x\left(\frac{\sec^2 x}{\tan x}\right)$$

   $$y'(x) = y(x)\ \left[\ln|\tan x| + \frac{x\sec^2 x}{\tan x}\right] = (\tan x)^x\left[\ln|\tan x| + \frac{x\sec^2 x}{\tan x}\right] .$$

Question (3): [$1+1+1+1+1+1+2+2=10$ marks]
Evaluate the following integrals :

1. $\displaystyle\int (x^3+1)\sec(x^4+4x)\ dx$.

   Solution :

   $$\int (x^3+1)\sec(x^4+4x)\ dx = \frac{1}{4}\int \sec(x^4+4x)\left[4(x^3+1)\right]\ dx$$

   $$= \frac{1}{4}\int \sec(x^4+4x)(4x^3+4)\ dx = \frac{1}{4}\ln\left|\sec(x^4+4x) + \tan(x^4+4x)\right| + c.$$

2. $\displaystyle\int \frac{x^5+x}{\sqrt{x^6+3x^2+8}}\ dx$.

   Solution : .

   $$\int \frac{x^5+x}{\sqrt{x^6+3x^2+8}}\ dx = \int \left(x^6+3x^2+8\right)^{-\frac{1}{2}}\left(x^5+x\right)\ dx$$

   $$= \frac{1}{6}\int \left(x^6+3x^2+8\right)^{-\frac{1}{2}}\left[6(x^5+x)\right]\ dx = \frac{1}{6}\int \left(x^6+3x^2+8\right)^{-\frac{1}{2}}\left(6x^5+6x\right)\ dx$$

$$= \frac{1}{6} \frac{\left(x^6+3x^2+8\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{3}\left(x^6+3x^2+8\right)^{\frac{1}{2}} + c$$

3. $\displaystyle\int \frac{1}{\mathrm{csch} x(1+\cosh x)}\ dx$.

Solution :

$$\int \frac{1}{\mathrm{csch} x(1+\cosh x)}\ dx = \int \frac{\sinh x}{1+\cosh x}\ dx = \ln|1+\cosh x| + c\ .$$

4. $\displaystyle\int \frac{5^{\tanh x}}{\cosh^2 x}\ dx$.

Solution :

$$\int \frac{5^{\tanh x}}{\cosh^2 x}\ dx = \int 5^{\tanh x}\mathrm{sech}^2 x\ dx = \frac{5^{\tanh x}}{\ln 5} + c\ .$$

5. $\displaystyle\int \frac{2^x}{\sqrt{9-4^x}}\ dx$

Solution :

$$\int \frac{2^x}{\sqrt{9-4^x}}\ dx = \int \frac{2^x}{\sqrt{9-(2^2)^x}}\ dx = \int \frac{2^x}{\sqrt{9-(2^x)^2}}\ dx$$

$$= \frac{1}{\ln 2}\int \frac{2^x\ \ln 2}{\sqrt{(3)^2-(2^x)^2}}\ dx = \frac{1}{\ln 2}\sin^{-1}\left(\frac{2^x}{3}\right) + c$$

6. $\displaystyle\int \frac{(e^x-1)\cos(x+e^{-x})}{e^x}\ dx$

Solution :

$$\int \frac{(e^x-1)\cos(x+e^{-x})}{e^x}\ dx = \int \cos\left(x+e^{-x}\right)\frac{(e^x-1)}{e^x}\ dx$$

$$= \int \cos\left(x+e^{-x}\right)\left(\frac{e^x}{e^x}-\frac{1}{e^x}\right)\ dx = \int \cos\left(x+e^{-x}\right)\left(1-e^{-x}\right)\ dx$$

$$= \int \cos\left(x+e^{-x}\right)\left(1+e^{-x}(-1)\right)\ dx = \sin\left(x+e^{-x}\right) + c\ .$$

7. $\displaystyle\int_0^1 \frac{x}{x+1}\ dx$.

Solution :

$$\int_0^1 \frac{x}{x+1}\ dx = \int_0^1 \frac{(x+1)-1}{x+1}\ dx = \int_0^1 \left(\frac{x+1}{x+1}-\frac{1}{x+1}\right)\ dx$$

$$= \int_0^1 \left(1 - \frac{1}{x+1}\right) \; dx = [x - \ln|x+1|]_0^1$$

$$= (1 - \ln 2) - (0 - \ln(1)) = 1 - \ln 2 \ .$$

8. $\displaystyle\int \frac{1}{x^2 + 8x + 25} \; dx \ .$

Solution : By completing the square.

$x^2 + 8x + 25 = (x^2 + 8x + 16) + 9 = (x+4)^2 + (3)^2$

$$= \int \frac{1}{x^2 + 8x + 25} \; dx = \int \frac{1}{(x+4)^2 + (3)^2} \; dx$$

$$= \frac{1}{3} \tan^{-1}\left(\frac{x+4}{3}\right) + c \ .$$