جامعة الملك سعود
King Saud University

Name: Answer Sheet (Dr Borhen)
ID:............................
Num of attendance:.....

Time: 3h **Final Exam - Math 218- Semester II- 1443 H** Marks: 40

Question	1	2	3	4	5	6	7	8	9	10
Answer	D	A	D	A	B	A	B	C	A	C

I) Choose the correct answer (write it in the table above): [1 × 10 Marks]

1) The distance between the points $A(2, 1)$ and $B(-1, -3)$ is equal to

$\vec{AB} = \binom{-3}{-4}$ then $AB = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$

(a) $\sqrt{13}$	(b) $\sqrt{8}$	(c) 3	(d) 5

2) The equation of the line that is perpendicular to the line $4x + 6y + 5 = 0$ and passes through the origin is $y = -2/3\, x - 5/6$ so $m = -2/3$

As the line passes through origin $y = 3/2\, x$ $mm' = -1 \Rightarrow m' = 3/2$

(a) $y = \frac{3}{2}x$	(b) $y = \frac{2}{3}x$	(c) $y = -x$	(d) $y = x$

3) The remainder when $P(x) = 8x^4 + 6x^2 - 3x + 1$ is divided by $2x^2 - x + 2$ is

$$\begin{array}{l|l} 8x^4+6x^2-3x+1 & 2x^2-x+2 \\ -8x^4+4x^3-8x^2 & 4x^2+2x \\ \hline 4x^3-2x^2-3x+1 & \\ -4x^3+2x^2-4x & \\ \hline -7x+1 & \end{array}$$

(a) $x - 5$	(b) $7x - 1$	(c) $-x + 5$	(d) $-7x + 1$

4) If $\lim_{x \to 0} \dfrac{\sin(3x)}{3x^2 + 2x} = \dfrac{0}{0} = \lim_{x\to 0} \dfrac{3\cos(3x)}{6x+2} = \dfrac{3}{2}$

(a) $\frac{3}{2}$	(b) 0	(c) $\frac{1}{2}$	(d) 1

5) If $2 - \cos x \leq f(x) \leq 1 + sin^2 x$, then the value of $\lim_{x \to 0} f(x) =$

(a) 0	(b) 1	(c) 2	(d) 3

$2 - \cos x \leq f(x) \leq 1 + \sin^2 x$

$\lim_{x\to 0} (2 - \cos x) \leq \lim_{x\to 0} f(x) \leq \lim_{x\to 0} (1 + \sin^2 x)$

$1 \leq \lim_{x\to 0} f(x) \leq 1$

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6) The horizontal asymptote to the graph of $f(x) = \frac{1-x^2}{1+x^2}$ is

$\lim_{x\to\infty} \frac{1-x^2}{1+x^2} = -1$

(a) $y = -1$	(b) $x = -1$	(c) $y = 1$	(d) $x = 1$

7) The graph of $f(x) = \frac{1-x}{1-x^2}$ has vertical asymptotes

$\lim \frac{1-x}{1-x^2} = \lim_{x\to -1} \frac{1}{1+x} = \infty$

(a) $y = -1$	(b) $x = -1$	(c) $y = 1$	(d) $x = 1$

8) The inflection point for the function $f(x) = x^3 - 3x + 27$ is

$f'(x) = 3x^2 - 3$
$f''(x) = 6x$

x		0	
f''	$-$		$+$
		27	

(a) $(27, 0)$	(b) $(1, 25)$	(c) $(0, 27)$	(d) $(-1, 29)$

9) The equation of tangent line for the curve $y = x\sqrt{1+x^2}$ at the point $P(1, \sqrt{2})$ is

put $f(x) = x\sqrt{1+x^2}$ so $f'(x) = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} \Rightarrow f'(1) = 3/\sqrt{2}$

Tp $y - \sqrt{2} = 3/\sqrt{2}\,(x-1) \Leftrightarrow \sqrt{2}y - 2 = 3x - 3 \Leftrightarrow -2\sqrt{2}y + 6x = 2$

(a) $6x - 2\sqrt{2}y = 2$	(b) $6x + 2\sqrt{2}y = 2$	(c) $2\sqrt{2}x - 6y = 2$	(d) $2\sqrt{2}x + 6y = 2$

10) The function $f(x) = x^2 - 2x$ has a local extremum at

(a) $x = -2$	(b) $x = -1$	(c) $x = 1$	(d) $x = 2$

$f'(x) = 2x - 2$
$f'(x) = 0 \Leftrightarrow x = 1$ (critical pt)

x	$-\infty$		1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	-1	$\nearrow$	$+\infty$

II) [8 Marks]

A) Find the absolute maximum and minimum values of $f(x) = -2x^2 + 4x - 5$ for $x \in [0, 2]$

$f(x) = -2x^2 + 4x - 5$

$f'(x) = -4x + 4$; $f'(x) = 0 \Leftrightarrow x = 1$

. f has a maximum at $x = 1$

x	0		1		2
$f'(x)$		+	0	−	
$f(x)$	−5	↗	−3	↘	−5

3

. $f(0) = f(2) = -5$

for $0 \leq x \leq 2$, $-5 \leq f(x) \leq -3$

B) Solve in $\mathbb{R}$ the equation $3^{x+2} = 7$.

1

$\ln(3^{x+2}) = \ln 7$

$(x+2)\ln 3 = \ln 7$

$x + 2 = \frac{\ln 7}{\ln 3}$ so $x = \frac{\ln 7}{\ln 3} - 2 = \log_3(7) - 2$

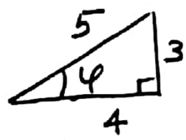
C) Evaluate $\sin(\theta + \phi)$ where $sin\theta = \frac{12}{13}$ with θ in the second quadrant and $\tan\phi = \frac{3}{4}$ with ϕ in the third quadrant.

$\sin(\theta + \varphi) = \sin\theta \cos\varphi + \cos\theta \sin\varphi$

2

As θ in II then $\cos\theta < 0$

$\cos\theta = -\sqrt{1 - \sin^2\theta} = -\sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{-5}{13}$.



As $\tan\varphi = \frac{3}{4}$

so $\cos\varphi = -4/5$

$\sin\varphi = -3/5$

because $\varphi \in$ III

We deduce

$\sin(\theta + \varphi) = \frac{12}{13} \times \left(-\frac{4}{5}\right) + \left(\frac{-5}{13}\right) \times \left(\frac{-3}{5}\right) = -\frac{33}{65}$

D) Solve the equation: $5 \sin\theta \cos\theta + 4 \cos\theta = 0$.

$\cos\theta\,(5\sin\theta + 4) = 0$

2

so $\cos\theta = 0$ or $5\sin\theta + 4 = 0$

$\theta = \frac{\pi}{2} + k\pi$ with $k \in \mathbb{Z}$ or $\theta = \sin^{-1}(-4/5)$

III) Evaluate the following limits, provided they exist. [5 Marks]

i) $\lim_{x\to 1} \frac{2x+3}{x-2}$ $= \frac{(2\times 1)+3}{1-2} = \frac{5}{-1} = -5$

1

ii) $\lim_{x\to\infty} \frac{1+x+3x^2}{1-2x^2}$ $= \frac{\infty}{\infty}$ IF

$\overset{\text{Hospital's Rule}}{=} \lim_{x\to\infty} \frac{1+6x}{-4x} = \frac{-3}{2}$

1

iii) $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$ $= \frac{0}{0}$

$= \lim_{x\to 3} \frac{(x-3)(x+1)}{(x-3)} = \lim_{x\to 3}(x+1) = 4$

1

iv) $\lim_{x\to 0} \frac{\sin 3x}{5x}$ $= \frac{0}{0}$ IF

$\overset{\text{Hospital's Rule}}{=} \lim_{x\to 0} \frac{3\cos(3x)}{5} = \frac{3}{5}$

1

v) $\lim_{x\to\infty} \frac{\ln x}{x}$ $= \frac{\infty}{\infty}$ IF

$\overset{\text{Hospital's Rule}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$

1

IV) Differentiate the following functions: [5 marks]

i) $f(x) = x^3 - 2x^2 + 2\sqrt{x}$

1 $f'(x) = 3x^2 - 4x + \frac{1}{\sqrt{x}}$

ii) $f(x) = (\sin x)e^x$

$f'(x) = (\cos x)\, e^x + (\sin x)\, e^x$

1 $f'(x) = (\cos x + \sin x)\, e^x$

iii) $f(x) = 3^{\cos x}$

1 $f'(x) = (\ln 3)(-\sin x) \cdot 3^{\cos x}$

iv) $f(x) = [\ln(1 + 4x^2)]^7$

2 $f'(x) = 7\left[\ln(1+4x^2)\right]^6 \frac{d}{dx}\left[\ln(1+4x^2)\right]$

$f'(x) = 7 \left(\ln(1+4x^2)\right)^6 \frac{8x}{1+4x^2}$

V) [4 Marks]

A) Use Gauss-Jordan elimination or Gaussian elimination to find the solution of the system of equations

$$\begin{cases} x + 4y = -2 \\ 3x + 5y = 1 \end{cases}$$

The augmented matrix is $\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & 5 & 1 \end{array}\right)$

It is equivalent to $\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & -7 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \end{array}\right)$

2

$\to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$ So $x = 2$ and $y = -1$

B) Use Cramer's method to find z (**only** z) for the system

$$\begin{cases} 2x - y + z = 5 \\ -x + y + 2z = 2 \\ x + 2y - z = -3 \end{cases}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = (-2-2-2) - (1+8-1) = -14 \neq 0$$

So the system has a unique solution. By Cramer's Rule,

$$z = \frac{1}{-14} \begin{vmatrix} 2 & -1 & 5 \\ -1 & 1 & 2 \\ 1 & 2 & -3 \end{vmatrix} = -\frac{1}{14}\left((-6-2-10) - (5+8-3)\right)$$

2

$$z = \frac{28}{14} = 2$$

⚠ $x = -\frac{1}{14} \begin{vmatrix} 5 & -1 & 1 \\ 2 & 1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = -\frac{1}{14}\left[(-5+6+4) - (-3+20+2)\right] = 1$

$y = -\frac{1}{14} \begin{vmatrix} 2 & 5 & 1 \\ -1 & 2 & 2 \\ 1 & -3 & -1 \end{vmatrix} = -\frac{1}{14}\left[(-4+10+3) - (2-12+5)\right] = -1$

VI) Let $f(x) = \dfrac{x^2 - 1}{x}$.

[8 Marks]

i) Find the domain of f.

ii) Find the x-intercepts and the y-intercepts (if any).

iii) Find the horizontal and vertical asymptotes (if any).

iv) Find the intervals on which f is increasing and the intervals on which f is decreasing.

v) Find the local minimum and the local maximum values (if any).

vi) Find the intervals on which f is concave upward and the intervals on which f is concave downward.

vii) Find the inflexion points (if any).

viii) Sketch the graph of f.

0,5 (i) The domain of f is $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

1 (ii) x-intercepts : $f(x) = 0 \Leftrightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1 \in D_f$.

y-intercept : It doesnot exist.

(iii) $\lim\limits_{x \to -\infty} f(x) = -\infty$; $\lim\limits_{x \to +\infty} f(x) = +\infty$ $\left(\frac{x^2-1}{x} = x - \frac{1}{x}, \text{if } x \neq 0 \right)$

$\lim\limits_{x \to \infty} \frac{f(x)}{x} = 1$ No horizontal asymptotes

1,5 $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \left(x - \frac{1}{x}\right) = +\infty$ so $\boxed{x = 0}$ is a Vertical asymptote

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \left(x - \frac{1}{x}\right) = -\infty$

(iv) let $x \neq 0$; $f(x) = \frac{x^2 - 1}{x} = x - \frac{1}{x}$.

So $f'(x) = 1 + \frac{1}{x^2} > 0$

1,5

x	$-\infty$	-1		0		1	$+\infty$
$f(x)$	$-\infty$ ↗	0	↗ $+\infty$	‖	$-\infty$ ↗	0	↗ $+\infty$

• f is increasing function on $(-\infty, 0) \cup (0, +\infty)$.

⚠ f is odd because $f(-x) = -f(x)$ for $x \in D_f$ so the graph is symmetric about the origin.

1,5 (v) No local minimum and no local maximum for f

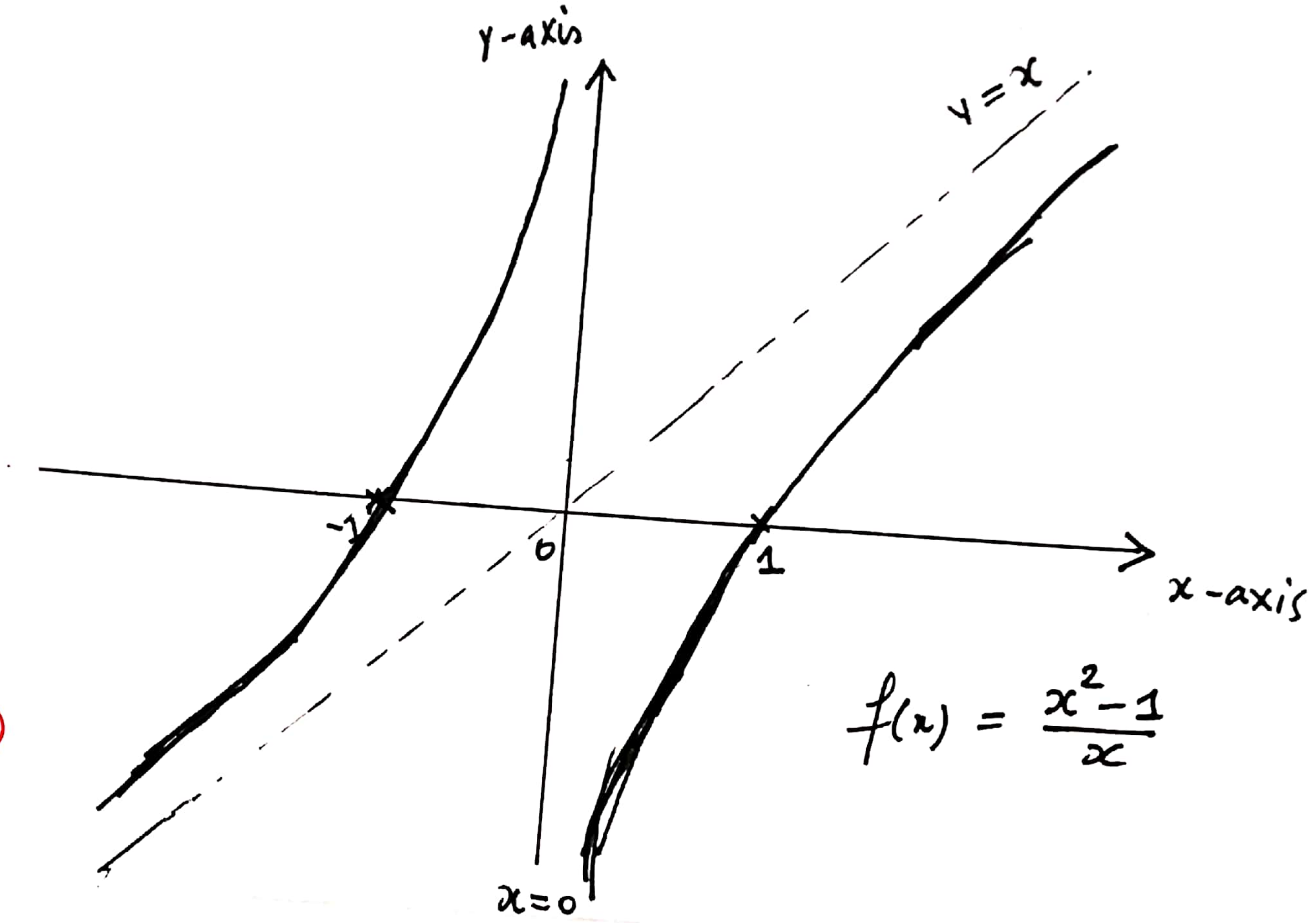
(vi) $f''(x) = \frac{d}{dx}\left(1 + \frac{1}{x^2}\right) = \frac{-2}{x^3}$. So,

x	$-\infty$	0	$+\infty$
$f''(x)$	$+$ ∪	‖	$-$ ∩

So the graph of f is Concave downward on $(0, \infty)$ and concave upward on $(-\infty, 0)$

1 (vii) No inflection pts.

1



$$f(x) = \frac{x^2 - 1}{x}$$