[FV]

King Saud University
College of Sciences
Department of Mathematics
Semester 462 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Calculators are not allowed.***

Question 1 [Marks 10]: Which of the given choices are correct?

$(I-A)(I+A) = I-A^2$

(i) If square of a matrix A is zero matrix, then $I-A$ is equal to:

As $A^2=0$ then $(I-A)=(I+A)^{-1}$

a) 0 b) $(A-I)^{-1}$ c) $(A+I)^{-1}$ d) $A+I$

(ii) If A is a square matrix of order 3 with $det(A)=2$, then $det(det(\frac{1}{det(A)}A^3)A^{-1})$ is equal to: $\left|\frac{1}{|A|}A^3\right| = 1$

a) 1/4 b) 1/2 c) 1/3 d) 1/16 $|A^{-1}| = \frac{1}{|A|}$

(iii) If the general solution of $AX = 0$ is $\underbrace{(-2r, 4r, r)}_{X_H}$, $r \in \mathbb{R}$, and $\underbrace{(1,0,-2)}_{X_P}$ is a solution of $AX = B$, then the general solution of $AX = B$ is:

$X_G = X_P + X_H$

a) $(1-2r, 4r, r-2)$ b) $(-2r, 0, -2r)$ c) $(-2r, 4r, r)$ d) $(-2r-1, 4r, r-2)$

(iv) A subset S of $\mathbb{R}^3$ is a basis of the vector space $\mathbb{R}^3$ if S is equal to:

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 1 & 0 \end{vmatrix} = -6 \neq 0$

a) $\{(1,0,0),(0,2,1),(0,6,0)\}$ b) $\{(1,1,0),(2,1,0),(3,2,0)\}$ c) $\{(1,1,0),(0,0,0),(3,2,1)\}$ d) $\{(1,1,0),(0,0,1),(2,2,1)\}$

(v) If $B = \{u_1 = (2,1), u_2 = (4,3)\}$ and $C = \{v_1 = (0,1), v_2 = (6,0)\}$ are ordered bases of $\mathbb{R}^2$, then the transition matrix $P_{C\to B}$ from C to B is equal to:

$P_C^B = \begin{array}{c} \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{pmatrix} -2 & 9 \\ 1 & -3 \end{pmatrix} \end{array} \begin{matrix} u_1 \\ u_2 \end{matrix}$

a) $\begin{bmatrix} 1 & -1/2 \\ -2 & 3/2 \end{bmatrix}$ b) $\begin{bmatrix} -2 & 9 \\ 1 & -3 \end{bmatrix}$ c) $\begin{bmatrix} -2/3 & 3 \\ 1/3 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

(vi) If B is a square matrix of order 3 with $det(B) = 2$, then $nullity(B)$ is equal to:

a) 2 b) 1 c) 3 d) 0

(vii) If $\langle , \rangle$ is an inner product on $\mathbb{R}^n$ and $u, v \in \mathbb{R}^n$ such that $\|u\|^2 = 5$, $\|v\|^2 = 1$, $\langle u,v\rangle = -2$, then $\langle u+2v, 5u-v\rangle$ is equal to: $\langle u+2v | 5u-v\rangle = 5\|u\|^2 + 9\langle u|v\rangle - 2\|v\|^2 = 25-18-2 = 5$

a) $\sqrt{5}$ b) 5 c) 9 d) 41

(viii) If $S = \{A, I_2\} \subseteq M_{2\times 2}(\mathbb{R})$, where A is a non-symmetric matrix, then S must be:

a) linearly dependent b) a spanning set for $M_{2\times 2}(\mathbb{R})$ c) linearly independent d) orthogonal

(ix) Let T be the transformation from the Euclidean space $\mathbb{R}^2$ to $\mathbb{R}$ given by $T(u) = \|u\|$ for all $u \in \mathbb{R}^2$, where $\|u\|$ is the Euclidean norm of u. Then, for $v, w \in \mathbb{R}^2$ and $k \in \mathbb{R}$, T satisfies:

a) $T(u+v) = T(u) + T(v)$ b) $T(u+v) \leq T(u) + T(v)$ c) $T(0) > 0$ d) $T(ku) = kT(u)$

$\|u+v\| \leq \|u\| + \|v\|$

(x) Zero is an eigenvalue of the matrix $\begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}$ with geometric multiplicity equal to:

a) 1 b) 2 c) 3 d) 4

$\begin{vmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0$ so 0 is an eigenvalue

$E_0 = \{X(x,y,z) \in \mathbb{R}^3 / AX = 0\}$

$x+y+z=0$

$X \in E_0 \Leftrightarrow z = -x-y$

$(x, y, -x-y) = x(1,0,-1) + y(0,1,-1)$

Question 2 [Marks $2+2+3$]:

(*a*) Find the square matrix A of order **3** such that $A^{-1}(A-I)=\begin{bmatrix}1&1&1\\2&1&1\\1&1&2\end{bmatrix}$ and evaluate *det*(*A*).

(*b*) Let $A=\begin{bmatrix}1&0&-1\\1&1&-2\\-2&-1&2\end{bmatrix}$ and $B=\begin{bmatrix}-2&1&1\\-1&1&-2\\1&-1&-2\end{bmatrix}$. Find a matrix X that satisfies $XA=B$.

(c) Solve the following system of linear equations:

$$\begin{aligned} x+\ y+\ z&=1\\ 2x\qquad +2z&=3\\ 3x+\ 5y\ +4z&=2. \end{aligned}$$

Question 3 [Marks $3+3+2$]:

Let $A=\begin{bmatrix}1&0&1\\1&2&3\\2&0&2\end{bmatrix}$ and $B=\begin{bmatrix}1&0&1\\0&2&2\\0&0&0\end{bmatrix}$. Then:

(a) Find a basis and the dimension for each of the vector spaces *row*(A), *col*(A), and *N*(*A*).

(b) Decide with justification whether the following statements are true or false:

(i) *row*(*A*) = *row*(*B*) (ii) *col*(*A*) = *col*(*B*) (iii) *N*(*A*).= *N*(**B**).

(c) Find all square matrices Z of order **3** such that $AZ=O$.

Question 4 [Marks $3+(1+3)$]:

(a) Construct an orthonormal basis C of the Euclidean space $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on the given basis $B=\{v_1=(1,1,0), v_2=(1,0,1), v_3=(0,1,1)\}$, and then find the coordinate vector of $v=(1,2,0)\in\mathbb{R}^3$ relative to the orthonormal basis C.

(b) Let $\mathscr{P}_2$ denote the vector space of real polynomials with degree ≤ 2. Consider the linear transformation $\boldsymbol{T}:\mathbb{R}^3\to\mathscr{P}_2$ defined by: $\boldsymbol{T}(1,0,0)=x^2+1, \boldsymbol{T}(0,1,0)=3x^2+2, \boldsymbol{T}(0,0,1)=-x^2$. Then:

(i) Compute $\boldsymbol{T}(a,b,c)$, for all $(a,b,c)\in\mathbb{R}^3$.

(ii) Find a basis for each of the vector spaces *Im*(**T**) and *ker*(**T**).

Question 5 [Marks $3+2+3$]: Let $A=\begin{bmatrix}2&2&-2\\2&1&-1\\2&2&-2\end{bmatrix}$. Then:

(a) Find the eigenvalues of A.

(b) Find algebraic and geometric multiplicities of all the eigenvalues of A.

(c) Is the matrix A diagonalizable? If yes, find a matrix P that diagonalizes A.

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Q2 (a) $A^{-1}(A-I)=I-A^{-1}=\begin{pmatrix}1&1&1\\2&1&1\\1&1&2\end{pmatrix}$ So $A^{-1}=I-\begin{pmatrix}1&1&1\\2&1&1\\1&1&2\end{pmatrix}=\begin{pmatrix}0&-1&-1\\-2&0&-1\\-1&-1&-1\end{pmatrix}$

$2\ 2\ 2 \quad 0\ 0\ -2$

$\left(\begin{array}{ccc|ccc}0&-1&-1&1&0&0\\-2&0&-1&0&1&0\\-1&-1&-1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&1&0&0&-1\\0&-1&-1&1&0&0\\-2&0&-1&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&1&0&0&-1\\0&-1&-1&1&0&0\\0&2&1&0&1&-2\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&1&0&0&-1\\0&1&1&-1&0&0\\0&0&-1&2&1&-2\end{array}\right)$

$0\ -2\ -2\ 2\ 0\ 0$

$\to\left(\begin{array}{ccc|ccc}1&1&0&2&1&-3\\0&1&0&1&1&-2\\0&0&1&-2&-1&2\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&0&-1\\0&1&0&1&1&-2\\0&0&1&-2&-1&2\end{array}\right)$ So $A=\begin{pmatrix}1&0&-1\\1&1&-2\\-2&-1&2\end{pmatrix}$

$\det A=[2+0+1]-[2+2+0]=-1$

(b) $XA=B$ then $XAA^{-1}=BA^{-1}$ So $X=BA^{-1}=\begin{pmatrix}-2&1&1\\-1&1&-2\\1&-1&-2\end{pmatrix}\begin{pmatrix}0&-1&-1\\-2&0&-1\\-1&-1&-1\end{pmatrix}$

$X=\begin{pmatrix}-3&1&0\\0&3&2\\4&1&2\end{pmatrix}$

$-3\ -3\ -3\ -3$

$-2\ -2\ -2\ -2$

(c) $\left(\begin{array}{ccc|c}1&1&1&1\\2&0&2&3\\3&5&4&2\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&1\\0&-2&0&1\\0&2&1&-1\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&1\\0&1&0&-1/2\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&0&1\\0&1&0&-1/2\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&3/2\\0&1&0&-1/2\\0&0&1&0\end{array}\right)$

The system has unique Solution $x=3/2$; $y=-1/2$; $z=0$

Q3 (a) $A=\begin{array}{c} \begin{matrix}-2 & 0 & -2\end{matrix} \\ \begin{pmatrix}1 & 0 & 1\\ 1 & 2 & 3\\ 2 & 0 & 2\end{pmatrix}\end{array} \rightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 2\\ 0 & 0 & 0\end{pmatrix} \rightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$

- $Row(A) = span\langle (1,0,1) ; (1,2,3)\rangle$
- $Col(A) = span\langle (1,1,2) ; (0,2,0)\rangle$
- $N(A) = \{X(x,y,z) \,/\, AX=0\} = span\langle -1,-1,1\rangle$

Let $X \in N(A)$: $\begin{cases} x+z=0 \\ y+z=0 \end{cases} \Rightarrow \begin{cases} x=-z \\ y=-z \end{cases}$ $(-z,-z,z)$

(b) Yes because B is the reduced echelon form for A.

(c) $AZ=0$; $\begin{pmatrix}1 & 0 & 1\\ 1 & 2 & 3\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = \begin{pmatrix}a+g & b+h & c+i\\ a+2d+3g & b+2e+3h & c+2f+3i\\ 2(a+g) & 2(b+h) & 2(c+i)\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

$$\begin{cases} a+g=0 \\ b+h=0 \\ c+i=0 \\ a+2d+3g=0 \\ b+2e+3h=0 \\ c+2f+3i=0 \end{cases} \qquad \begin{cases} a=-g \\ b=-h \\ c=-i \\ d=-g \\ e=-h \\ f=-i \end{cases} \qquad Z=\begin{pmatrix}-g & -h & -i\\ -g & -h & -i\\ g & h & i\end{pmatrix}$$

$$Z = \alpha\begin{pmatrix}-1 & 0 & 0\\ -1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} + \beta\begin{pmatrix}0 & -1 & 0\\ 0 & -1 & 0\\ 0 & 1 & 0\end{pmatrix} + \lambda\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & -1\\ 0 & 0 & 1\end{pmatrix}$$

Q4 (a) • $\boxed{u_1 = v_1} = \boxed{(1,1,0)} \longrightarrow \boxed{w_1 = \frac{(1,1,0)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1,1,0)}$

$\boxed{u_2 = v_2 - \frac{\langle v_2 | u_1\rangle}{\|u_1\|^2} u_1} = (1,0,1) - \frac{\langle (1,0,1) | (1,1,0)\rangle}{2}(1,1,0)$

• $u_2 = (1,0,1) - \frac{1}{2}(1,1,0) = \boxed{\left(\frac{1}{2}, -\frac{1}{2}, 1\right)} \rightarrow w_2 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\left(\frac{1}{2}; \frac{-1}{2}, 1\right)$

$\boxed{w_2 = \frac{2}{\sqrt{6}}\left(\frac{1}{2}; -\frac{1}{2}, 1\right)}$

• $\boxed{u_3 = v_3 - \frac{\langle v_3 | u_2\rangle}{\|u_2\|^2}u_2 - \frac{\langle v_3 | u_1\rangle}{\|u_1\|^2}u_1}$.

$u_3 = (0,1,1) - \frac{\langle (0,1,1) | (\frac{1}{2}, \frac{-1}{2}, 1)\rangle}{\frac{1}{4}+\frac{1}{4}+1}\left(\frac{1}{2}, \frac{-1}{2}, 1\right) - \frac{\langle (0,1,1) | (1,1,0)\rangle}{2}(1,1,0)$

$u_3 = (0,1,1) - \frac{1}{3}\left(\frac{1}{2}, \frac{-1}{2}, 1\right) - \frac{1}{2}(1,1,0) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \boxed{\frac{2}{3}(-1,1,1)}$

$w_3 = \frac{1}{\sqrt{3}}(-1,1,1)$

* Orthonormal basis $C = \left\{\frac{1}{\sqrt{2}}(1,1,0) ; \frac{2}{\sqrt{6}}\left(\frac{1}{2}; -\frac{1}{2}, 1\right); \frac{1}{\sqrt{3}}(-1,1,1)\right\}$

* $v = (1,2,0) = \langle v | w_1\rangle w_1 + \langle v | w_2\rangle w_2 + \langle v | w_3\rangle w_3$

$[v]_C = \begin{pmatrix}\langle v | w_1\rangle\\ \langle v | w_2\rangle\\ \langle v | w_3\rangle\end{pmatrix} = \begin{pmatrix}3/\sqrt{2}\\ -1/\sqrt{6}\\ 1/\sqrt{3}\end{pmatrix}$

$T: \mathbb{R}^3 \longrightarrow \mathcal{P}_2$

(i) $T(a,b,c) = a\,T(1,0,0) + b\,T(0,1,0) + c\,T(0,0,1)$

$= a(x^2+1) + b(3x^2+2) + c(-x^2)$

$= (a+3b-c)x^2 + (2b+a)$

(ii) $\operatorname{Im} T = \{ T(a,b,c) \,;\, (a,b,c) \in \mathbb{R}^3 \}$

$= \{ (a+3b-c)x^2 + (a+2b) \,;\, \forall (a,b,c) \in \mathbb{R}^3 \}$

$= \operatorname{Span}\{1, x^2\}$

basis for Im T is $\{1, x^2\}$

$\operatorname{Ker} T = \{ (a,b,c) \in \mathbb{R}^3 / T(a,b,c) = 0 \}$

Let $(a,b,c) \in \operatorname{Ker} T$; Then $\begin{cases} a+2b=0 \\ a+3b-c=0 \end{cases} \Leftrightarrow \begin{cases} a=-2b \\ b-c=0 \end{cases}$ so $\begin{cases} a=-2b \\ \text{and } c=b \end{cases}$

$(-2b, b, b) = b\langle(-2,1,1)\rangle$

$\boxed{\operatorname{Ker} T = \operatorname{span}\langle(-2,1,1)\rangle}$

P5

(a) $\det(A - \lambda I) = 0$

$$\begin{vmatrix} (2-\lambda) & 2 & -2 \\ 2 & (1-\lambda) & -1 \\ 2 & 2 & (-2-\lambda) \end{vmatrix} = [(2-\lambda)(1-\lambda)(-2-\lambda) - 4 - 8] - [-4(1-\lambda) - 2(2-\lambda) + 4(-2-\lambda)]$$

$= [(\lambda+2)(\lambda-1)(2-\lambda) - 12] - [\cancel{-4} + \cancel{4\lambda} - 4 + 2\lambda - 8 - \cancel{4\lambda}]$

$= (\lambda+2)(\lambda-1)(2-\lambda) - 12 - 2\lambda + 16$

$= (\lambda+2)(\lambda-1)(2-\lambda) - 2(\lambda-2)$

$= (\lambda+2)(\lambda-1)(2-\lambda) - 2(\lambda-2) = (\lambda-2)[-\lambda^2-\lambda]$

$= (\lambda-2)[(1-\lambda)(\lambda+2) - 2]$

$= -\lambda(\lambda+1)(\lambda-2) = 0$

The eigenvalues of A are $0, -1, 2$.

(b) As the eigenvalues of A are different then the algebraic and geometric multiplicities are equal to 1.

(c) Yes A is diagonalizable because the eigenvalues are different.

$A = PDP^{-1}$ where $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $P = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$

$E_0 = \{X \in \mathbb{R}^3 / AX = 0\} = \langle(0,1,1)\rangle$

$E_{-1} = \{X \in \mathbb{R}^3 / AX = -X\} = \langle(-2,1,-2)\rangle$

$E_2 = \{X \in \mathbb{R}^3 / AX = 2X\} = \langle(1,1,1)\rangle$

$P^{-1} = \frac{1}{3}\begin{pmatrix} -3 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & 2 & -2 \end{pmatrix}$