Answer Sheet (Dr Bu

King Saud University
College of Sciences
Department of Mathematics
Semester 452 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: *Attempt all the five questions. Scientific calculators are not allowed.
* Please fill in the above columns and give your answer to Question 1 on this page of the paper.

Question 1 [Marks: 10×1]: Choose the correct answer:

$\text{nullity}(B^T) + \text{rank}\, B^T = 5$
$\text{nullity}(B^T) + \overset{4}{\text{rank}\, B} = 5$
$? + 4 = 5$
So $\text{nullity}(B^T) = 1$

(i) If B is a 5×7 matrix and $nullity(B) = 3$, then $nullity(B^T)$ equals:

a) 2 b) 5 c) 3 d) 1

(ii) If C is a 3×3 matrix with $det(C) = 2$, then $rank(C)$ equals:

a) 1 b) 3 c) 2 d) 0

(iii) Which of the following matrices cannot be a transition matrix:

a) $\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 5 \\ 2 & 6 & 0 \end{bmatrix}$ $|A| = 0$

(iv) If u and v are orthogonal unit vectors in an inner product space, then $\|v - u\|$ equals:

$\|v-u\|^2 = \langle v-u | v-u \rangle = \|v\|^2 + \|u\|^2 = 1 + 1 = 2$

a) 1 b) $\sqrt{2}$ c) 0 d) $\sqrt{3}$

(v) If $u = (1,-1,0)$, $v = (3,-1,2)$ and $w = (-2,0,1)$ in Euclidean space $\mathbb{R}^3$, then $\frac{\langle u,v \rangle}{\langle w,w \rangle} v$ equals:

$\langle u|v \rangle = 4$
$\langle w|w \rangle = 5$

a) $\frac{4}{5}(3,-1,2)$ b) $\frac{4}{5}$ c) $\frac{5}{4}(-2,0,1)$ d) $\frac{4}{5}(1,-1,0)$

(vi) If v and w are orthogonal nonzero vectors in any Euclidean space $\mathbb{R}^n$, then the angle θ between v and w equals:

a) 45° b) 90° c) 180° d) 0°

(vii) The set $\{\underbrace{(x, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}})}_{u}, \underbrace{(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)}_{v}\}$ of vectors in the Euclidean space $\mathbb{R}^3$ is (ortho)normal if x equals:

$x^2 + \frac{1}{6} + \frac{4}{6} = 1$
So $x = \pm \frac{1}{\sqrt{6}}$
$\langle u|v \rangle = 0$
$\Rightarrow x = \frac{1}{\sqrt{6}}$

a) 0 b) $\frac{1}{\sqrt{6}}$ c) $\frac{-1}{\sqrt{6}}$ d) $\frac{1}{2}$

(viii) Consider the vector space P_n of real polynomials of degree $\leq n$. If the linear transformation $T: P_2 \to P_1$ is given by $T(a_0 + a_1x + a_2x^2) = a_0 + a_1 + a_2x$, then $ker(T)$ equals:

$T(a_0 + a_1x + a_2x^2) = 0$
$\begin{cases} a_0 + a_1 = 0 \\ a_2 = 0 \end{cases}$

a) $\{0\}$ b) $span\{-1 + x\}$ c) $span\{-1, x\}$ d) P_2

(ix) Consider the basis $\{(1,2),(3,0)\}$ for the space $\mathbb{R}^2$. If the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(1,2) = (1,5)$ and $T(3,0) = (-4,6)$, then $T(7,8)$ equals:

As $(7,8) = 4(1,2) + 1(3,0)$

a) (13,26) b) (−15,35) c) (−3,11) d) (0,26)

So $T(7,8) = 4T(1,2) + 1T(3,0)$
$T(7,8) = 4(1,5) + (-4,6) = (0,26)$

(x) If $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$, then the eigenvalues of A^4 are:

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 3 & -1-\lambda \end{vmatrix} = 0$

a) 2, 16 b) -1, 8 c) 1, 16 d) 4, 16

$= (2-\lambda)(-1-\lambda) = 0$
So $\lambda = 2$ or $\lambda = -1$
A is diagonalizable
$A = PDP^{-1}$
So $A^4 = PD^4P^{-1}$

Question 2 [Marks: 2+2+3]**:**

(a) Given: the polynomial $p(x) = x^3 - 3x^2 - 4x + 13$ and the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Compute $p(A)$.

(b) Show that $\begin{vmatrix} 1 & 2 & 2 \\ x+1 & 2x+1 & 2x+2 \\ x+1 & x+1 & 2x+1 \end{vmatrix}$ is constant, for all $x \in \mathbb{R}$.

(c) Let A be $m \times n$ matrix and let B and C be linearly independent vectors in $\mathbb{R}^m$. Suppose X_1 is a solution of $AX_1 = B$ and X_2 is a solution of $AX_2 = C$. Then, show that the vectors X_1 and X_2 are linearly independent in $\mathbb{R}^n$. Explain why $rank(A) \geq 2$.

Question 3 [Marks: 2+3+3]**:**

Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{bmatrix}$. Then:

(a) Compute the *RREF* of A.

(b) Find a basis of the column space *col* (A) and a basis of the null space $N(A)$.

(c) Find $rank(A)$ and $nullity(A)$.

Question 4: [Marks: 3+2+3]

(a) Consider the vector space $\boldsymbol{P_2}$ of real polynomials of degree ≤ 2 equipped with the inner product:

$$< p(x), q(x) > = p(-1)q(-1) + p(o)q(0) + p(1)q(1) \text{ for all } p(x), q(x) \in \boldsymbol{P_2}$$

Explain why $\{1, x, x^2\}$ is not orthogonal. Apply the Gram-Schmidt process to transform the basis $\{1, x, x^2\}$ of $\boldsymbol{P_2}$ to an orthogonal basis.

(b) Let $\boldsymbol{S} = \{u, v, w\}$ be any orthonormal subset of the above inner product space $\boldsymbol{P_2}$. Show that $\boldsymbol{S}$ is a basis for $\boldsymbol{P_2}$.

(c) Consider the basis $\{v_1 = (2,2,1), v_2 = (2,1,0), v_3 = (1,0,0)\}$ for the vector space $\mathbb{R}^3$. Let the linear transformation T: $\mathbb{R}^3 \to \mathbb{R}^2$ be given by: $T(v_1) = (3,-1)$, $T(v_2) = (6,2)$ and $T(v_3) = (4,3)$. Then find:

(i) *ker* (T) (ii) *rank* (T).

Question 5: [Marks: 3+4]

(a) Let A, B and C be square matrices of size n, where C is invertible satisfying $B = C^{-1}AC$. If λ is an eigenvalue of A and X is its corresponding eigenvector, then find a nonzero vector $Y \in \mathbb{R}^n$ such that $BY = \lambda Y$.

(b) Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ is diagonalizable and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

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Q2 (a) $P(A) = A^3 - 3A^2 - 4A + 13I = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & -8 \end{pmatrix} - 3\begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix} - 4\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} + 13\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ So A has inverse $A^{-1} = -\frac{1}{12}(A^2 - 3A - 4I)$

because $-\frac{1}{12} A(A^2 - 3A - 4I) = I$

(b) $\begin{vmatrix} 1 & 2 & 2 \\ x+1 & 2x+1 & 2x+2 \\ x+1 & x+1 & 2x+1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ x+1 & 2x+1 & 2x+2 \\ 0 & -x & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -x & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -x & -1 \end{vmatrix} = 1$

(c). If $\alpha X_1 + \beta X_2 = 0$ then $\alpha AX_1 + \beta AX_2 = A0 = 0$

so $\alpha B + \beta C = 0$ but $\{B, C\}$ are independent

Hence $\alpha = \beta = 0$ ($\{X_1, X_2\}$ is linearly independent)

- As B & C are in the image space of A and linearly independent
  Then $\text{rank}(A) \geq 2$.

Q3

(a) $\begin{pmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(b) $Col(A) = \text{span}\langle (1,-1,0,1) ; (-1,1,1,0) \rangle$

$N(A) = \{ X \in \mathbb{R}^4 / AX = 0 \}$

Let $X(x,y,z,t) \in N(A)$ then $\begin{cases} x - y - t = 0 \\ y + z + 3t = 0 \end{cases} \Rightarrow \begin{cases} x = y + t \\ z = -y - 3t \end{cases}$

$(y+t ; y ; -y-3t ; t) = y(1,1,-1,0) + t(1,0,-3,1)$

$N(A) = \text{span} \langle (1,1,-1,0) ; (1,0,-3,1) \rangle$

(c) $\text{Rank } A = \dim Col(A) = 2$; $\text{nullity}(A) = \dim N(A) = 2$

Q4 (a) $\langle 1, x \rangle = 0$; $\langle 1, x^2 \rangle = 1 + 1 = 2$ so $\{\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}\}$ is not orthogonal.

- $u_1 = \boxed{1}$; $w_1 = \frac{1}{\sqrt{\langle u_1 | u_1 \rangle}} u_1 = \frac{1}{\sqrt{3}}$
- $u_2 = v_2 - \frac{\langle v_2 | u_1 \rangle}{\langle u_1 | u_1 \rangle} u_1 = x - \frac{\langle x | 1 \rangle}{\langle 1 | 1 \rangle} 1 = \boxed{x}$; $w_2 = \frac{1}{\sqrt{\langle x | x \rangle}} x = \frac{x}{\sqrt{2}}$
- $u_3 = v_3 - \frac{\langle v_3 | u_2 \rangle}{\|u_2\|^2} u_2 - \frac{\langle v_3 | u_1 \rangle}{\|u_1\|^2} u_1$

$u_3 = x^2 - \frac{\langle x^2 | x \rangle}{\langle x | x \rangle} \frac{x}{\cancel{\ }} - \frac{\langle x^2 | 1 \rangle}{3} 1$

$u_3 = x^2 - 0 \frac{x}{2} - \frac{2}{3} = \boxed{x^2 - 2/3}$; $w_3 = \frac{3}{\sqrt{6}}(x^2 - 2/3)$

$\left\{ \frac{1}{\sqrt{3}} ; \frac{x}{\sqrt{2}} ; \frac{1}{\sqrt{6}}(3x^2 - 2) \right\}$ orthonormal basis.

(b) If $\alpha u + \beta v + \lambda w = 0$ then $\langle \alpha u + \beta v + \lambda w | u \rangle = \langle 0 | u \rangle = 0$

$\alpha \|u\|^2 = 0 \Rightarrow \alpha = 0$

Also $\langle \beta v + \lambda w | v \rangle = 0 \Rightarrow \beta = 0$

$\langle \lambda w | w \rangle = 0 \Rightarrow \lambda = 0$

we deduce that S is a basis of P_2.

(c) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

Matrix of transformation $A_T = \begin{array}{c} \begin{array}{ccc} T(v_1) & T(v_2) & T(v_3) \end{array} \\ \begin{pmatrix} 3 & 6 & 4 \\ -1 & 2 & 3 \end{pmatrix} \end{array}$

(i) $\operatorname{Ker} T = \{ X(x,y,z) \in \mathbb{R}^3 \,/\, T(X) = A_T X = 0 \}$

Let $X(x,y,z) \in \operatorname{Ker} T$ then $\begin{cases} 3x+6y+4z = 0 \quad (1) \\ -x+2y+3z = 0 \quad (2) \times 3 \end{cases}$

$(1) + 3\times(2) \Rightarrow 12y + 13z = 0$ $\qquad -3x+6y+9z=0$

$y = -\frac{13}{12} z$; $x = +2y+3z = -\frac{26}{12}z + 3z = \frac{10}{12} z$

$\left(\frac{10}{12}z ; -\frac{13}{12}z , z\right) = \frac{z}{12}(10,-13,12)$

$\operatorname{Ker} T = \operatorname{span}\langle (10,-13,12) \rangle$ = span by $10v_1 - 13v_2 + 12v_3$

(ii) $\operatorname{rank}(T) = 2$

Q5 (a) As $AX = \lambda X$ then $\underbrace{C^{-1}A}X = \lambda C^{-1}X$. As $B = C^{-1}AC$ so $C^{-1}A = BC^{-1}$

$B\underbrace{C^{-1}X} = \lambda C^{-1}X$ $\qquad$ take $Y = C^{-1}X$

$BY = \lambda Y$

(b) $q_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 2 & 3 \\ 2 & (2-\lambda) & 2 \\ 3 & 2 & (1-\lambda) \end{vmatrix} = \left[\left((1-\lambda)^2(2-\lambda)+12+12\right) - \left(9(2-\lambda)+4(1-\lambda)+4(1-\lambda)\right)\right]$

Characteristic equation

$= (1-\lambda)^2(2-\lambda) + 24 - 9(2-\lambda) - 8(1-\lambda)$

$= (\lambda^2 - 2\lambda + 1)(2-\lambda) + \boxed{24} - 18 + 9\lambda - 8 + 8\lambda$

$= 2\lambda^2 - 4\lambda + \cancel{2} - \lambda^3 + 2\lambda^2 - \lambda + 17\lambda - \cancel{2}$

$= -\lambda^3 + 4\lambda^2 + 12\lambda = -\lambda(\lambda^2 - 4\lambda - 12)$

$= -\lambda(\lambda+2)(\lambda-6)$

A has a different eigenvalues $\{0,-2,6\}$ then A is diagonalizable.

- $E_0 = \{X \in \mathbb{R}^3 / AX = 0\} = \operatorname{span}(1;-2;1)$

$\begin{array}{c} -2 \;\; -4 \;\; -6 \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix} \\ -3 \;\; -6 \;\; -9 \end{array} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{cases} x = -2y - 3z = z \\ y = -2z \end{cases}$

$(z, -2z, z) = z(1,-2,1)$

- $E_{-2} = \{X \in \mathbb{R}^3 / AX = -2X\} = \{X \in \mathbb{R}^3 / (A+2I)X = 0\} = \operatorname{span}(-1,0,1)$

$\begin{pmatrix} 3 & 2 & 3 \\ 2 & 4 & 2 \\ \cancel{3} & \cancel{2} & \cancel{3} \end{pmatrix} \to \begin{array}{c} -3 \;\; -6 \;\; -3 \\ \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x + z = 0 \Leftrightarrow x = -z$; $y = 0$

$(x,y,z) = (-z,0,z) = z(-1,0,1)$

- $E_6 = \{X \in \mathbb{R}^3 / (A - 6I)X = 0\} = \{X \in \mathbb{R}^3 / AX = 6X\} = \operatorname{span}\langle (1,1,1) \rangle$

$\begin{pmatrix} -5 & 2 & 3 \\ 2 & -4 & 2 \\ 3 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ -5 & 2 & 3 \\ 3 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & -8 & 8 \\ 0 & 8 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{cases} x = 2y - z = z \\ y = z \end{cases}$ $\quad (x = z ; y = z ; z = z)$

$D = P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$P = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$