

(Draft) **KING SAUD UNIVERSITY**  
**COLLEGE OF SCIENCES**  
**DEPARTMENT OF MATHEMATICS**

**MATH-244 (Linear Algebra); Final Exam; Semester 1 (1443 H)**

**Max. Marks: 40**

**Max. Time: 3 hours**

**Note: Attempt all the five questions!**

**Question 1 [4+2+2 marks]:**

- a) Find adjoint matrix and matrix of cofactors of  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ -2 & 3 & 6 \end{bmatrix}$  and also find  $A^{-1}$ .
- b) Evaluate  $\det(\det(A)B^2A^{-1})$ , where  $A$  and  $B$  are square matrices of order 3 with  $\det(A)=3$  and  $\det(B) = 2$ .
- c) Let  $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 6 & 3 \\ 0 & 2 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ -1 & 0 & 8 \end{bmatrix}$ . Explain why the matrices  $A$  and  $B$  are not row equivalent to each other?

**Question 2 [5+3 marks]:**

- a) Find the values of  $\alpha$  and  $\beta$  such that the following linear system:

$$\begin{aligned} x - 2y + 3z &= 4 \\ 2x - 3y + \alpha z &= 5 \\ 3x - 4y + 5z &= \beta \end{aligned}$$

has:

- i) No solution;
  - ii) Infinitely many solutions.
- b) Let  $s_1 = 3 - 2x$ ,  $s_2 = 2 + x$ ,  $s_3 = 1 + x - x^2$ ,  $s_4 = x + x^2 - x^3$ . Find the values of  $a, b, c$  and  $d$  such that  $1 - 6x - 3x^2 - 4x^3 = as_1 + bs_2 + cs_3 + ds_4$ .

**Question 3 [4+4 marks]:**

- a) Let  $\{u_1 = (1,1,1,1), u_2 = (0,1,2,1), u_3 = (1,0,-2,3), u_4 = (1,1,2,-2)\}$  generate the subspace  $F$  of the Euclidean space  $\mathbb{R}^4$ . Then:
- i) Find  $\dim(F)$
  - ii) Show that  $(1,1,0,1) \notin F$ .
- b) Let  $B = \{v_1 = (1,1,2), v_2 = (3,2,1), v_3 = (2,1,5)\}$  and  $C = \{u_1, u_2, u_3\}$  be two bases for  $\mathbb{R}^3$  such that

$${}_B P_C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

is the transition matrix from  $C$  to  $B$ . Find the vectors  $u_1, u_2$  and  $u_3$ .

**Question 4** [(1+2)+(1+1)+3 marks]:

- a) Let  $w_1 = (0, 0, 1)$ ,  $w_2 = (0, 1, 1)$ ,  $w_3 = (1, 1, 1)$  be vectors in the Euclidean space  $\mathbb{R}^3$ . Then:
- Find the angle between  $w_1$  and  $w_3$ .
  - By applying the Gram-Schmidt process on  $\{w_1, w_2, w_3\}$  to find an orthonormal basis of the Euclidean space  $\mathbb{R}^3$ .
- b) Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation given by  $T(x, y) = (x + 4y, 2x + 3y)$ . Find:
- $\text{Ker}(T)$
  - $\dim \text{Im}(T)$
- c) Let the linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be defined by:

$$T(x, y) = (x + 2y, x - y, 3x + y).$$

Find matrix of the transformation  $[T]_B^C$ , where  $B$  and  $C$  are the standard bases of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , respectively.

**Question 5** [4 + 4 marks]:

- a) Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ . Find eigenvalue/s of the matrix  $A$  and determine one basis of the corresponding eigenspace/s. Then, give reason for the non-diagonalizability of  $A$ .

- b) Show that the matrix  $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$  diagonalizes the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \text{ and then use this fact to compute } A^{-1}.$$

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# **SOLUTION KEY: 1443/Semester-1/Mpsh-244/Final Exam**

Solution of Question 1:

$$a) \text{adj}(A) = C^T = \begin{bmatrix} 30 & -12 & 4 \\ -16 & 6 & -2 \\ 18 & -7 & 2 \end{bmatrix} \quad (2 \text{ marks})$$

$$\text{and } |A| = -2.$$

$$\text{Hence, } A^{-1} = |A|^{-1} \text{adj}(A) = \begin{bmatrix} -15 & 6 & -2 \\ 8 & -3 & 1 \\ -9 & 7/2 & -1 \end{bmatrix} \quad (1 \text{ mark})$$

$$b) \det(\det(A)B^3A^4) = (\det(A))^3(\det(B))^4(\det(A))^4 = (\det(A))^7(\det(B))^4 = 36 \quad (2 \text{ marks})$$

$$c) |A| = -2 \text{ and } |B| = 0 \Rightarrow A \text{ is invertible but } B \text{ is non-invertible} \Rightarrow A \text{ and } B \text{ are not row equivalent.} \quad (2 \text{ marks})$$

Solution of Question 2:

$$a) [A:B] = \left[ \begin{array}{ccc|ccc} 1 & -2 & 3 & 4 & 0 & 0 \\ 0 & 1 & a-6 & -3 & 0 & 0 \\ 0 & 0 & 8-2a & \beta-6 & 0 & 0 \end{array} \right] \quad (2 \text{ marks})$$

Hence, the linear system has:

- i) no solution if  $a = 4$  and  $\beta = 6$ ;
- ii) infinitely many solutions if  $a = 4$  and  $\beta \neq 6$ .

$$b) a = 2, b = -3, c = 1, d = 4.$$

Solution of Question 3:

$$a) i) [u_1 \ u_2 \ u_3 \ u_4] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (R.E.F.)} \quad (1 \text{ mark})$$

$$\Rightarrow F = \text{span}\{u_1, u_2, u_3\} \Rightarrow \dim(F) = 3. \quad (1 \text{ mark})$$

$$ii) \{u_1, u_2, u_3, (1, 1, 0, -1)\} \text{ is linearly independent and } F = \text{span}\{u_1, u_2, u_3\}. \text{ So, } (1, 1, 0, -1) \notin F \quad (2 \text{ marks})$$

$$b) \text{Let } \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = [u_1]_B \ [u_2]_B \ [u_3]_B. \quad (1 \text{ mark})$$

$$\text{Hence, } u_1 = 1v_1 + 1v_2 - 1v_3 = (2, 2, -2). \text{ Similarly, } u_2 = (1, 1, 8) \text{ and } u_3 = (3, 2, 7). \quad (1 \text{ mark})$$

Solution of Question 4:

$$a) i) \theta = \cos^{-1} \frac{1}{\sqrt{3}} \approx 0.955 \text{ rad} \quad (1 \text{ mark})$$

$$ii) u_1 = w_1 = (0, 0, 1), u_2 = w_2 - \frac{\langle w_2, u_1 \rangle}{\|u_1\|^2} u_1 = (0, 1, 0)$$

$$\text{and } u_3 = w_3 - \frac{\langle w_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle w_3, u_2 \rangle}{\|u_2\|^2} u_2 = (1, 0, 0). \quad (3 \text{ marks})$$

$$\text{So, } \{e_1 = u_1 = (0, 0, 1), e_2 = u_2 = (0, 1, 0), e_3 = u_3 = (1, 0, 0)\} \text{ is an orthonormal basis of } \mathbb{R}^3.$$

$$b) i) \text{Ker}(T) = \{(0, 0)\} \quad (1 \text{ mark})$$

$$ii) \text{From Part (i), } T \text{ is one-one and so } \text{Im}(T) = \mathbb{R}^2. \text{ Hence, } \dim \text{Im}(T) = 2. \quad (1 \text{ mark})$$

$$c) [T]_B = [T(1, 0)]_B \ [T(0, 1)]_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}. \quad (2 \text{ marks})$$

Solution of Question 5:

$$a) \text{Eigenvalues} = 1, 1, 1 \quad (1 \text{ mark})$$

$$E_1 = \text{span}\{(0, 0, 1)\}$$

So, the algebraic multiplicity of the eigenvalue 1 is 3 which is different from its geometric multiplicity 1.

Hence, the given matrix  $A$  is not diagonalizable. (2 marks)

$$b) P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \text{ so that } D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \text{ Thus, } A^{-1} = PD^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix} \quad (1+1+2=4 \text{ marks})$$

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