KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
M204.TIME 3H, FULL MARKS 40, FINAL EXAM T1-2022/23

Question 1. [4,4,4] **a)** Show whether the functions $f_1(x) = 2\cos^2 x$, $f_2(x) = 9\cos 2x$, $f_3(x) = 3\sin^2 x$ are linearly dependent or linearly independent on $\mathbb{R}$.

b) Find the orthogonal trajectories for the family of curves $y = \ln(x - C)$.

c) Solve the differential equation

$$2xy^{-1}y' + (\sec^2 x + \tan x)y^2 = 1 + x, \quad x > 0, \ y \neq 0.$$

Question 2. [4,5] **a)** Find the general solution of the differential equation

$$y'' - 2y' + y = xe^x \ln x, \quad x > 0.$$

b) Solve the initial value problem

$$\begin{cases} (x+2)y'' - y' + \frac{y}{x+2} = 0, & x + 2 > 0 \\ y(-1) = 1, \ y'(-1) = 0. \end{cases}$$

Question 3. [4,5] **a)** Use undetermined coefficients method to find the general solution of the differential equation

$$y'' - 2y' - 3y = 2e^x - 10\sin x.$$

b) Use power series method to find the first four terms of the solution for the initial value problem

$$(1 - x^2)y'' - (2x + 1)y' - y = 0, \quad y(0) = 2, \ y'(0) = 0.$$

Question 4. [5,5] **a)** Let f be a function defined on $[0, \pi]$ defined by

$$f(x) = x(\pi - x)$$

Find the Fourier sine series for f.

Deduce the value of the numerical series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \qquad \text{Hint} : \sin \frac{(2n+1)\pi}{2} = (-1)^n$$

b) Consider the function: $f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Sketch the graph of f, find the Fourier integral representation, and deduce the value of the integral $\displaystyle\int_0^{\infty} \frac{\sin^2(\lambda)}{\lambda^2} d\lambda$.

Complete solution of final

exam M. 204, First semester 1444

Question (1)

(a) $f_1(x) = 2\cos^2 x$, $f_2(x) = 9\cos(2x)$, $f_3(x) = 3\sin^2 x$

As $\cos(2x) = \cos^2 x - \sin^2 x$, then (2)

$$f_2(x) = 9\cos(2x) = 9\cos^2 x - 9\sin^2 x$$

$$= \frac{9}{2}(2)\cos^2 x - 3(3\sin^2 x)$$

$$f_2(x) = \frac{9}{2} f_1(x) - 3 f_3(x)$$ (2)

or $\boxed{\frac{9}{2} f_1(x) - f_2(x) - 3 f_3(x) = 0}$ for all $x \in \mathbb{R}$

Then, these functions are linearly dependent on $\mathbb{R}$

(c) $2xy^{-1}y' + (\sec^2 x + \tan x) y^2 = 1 + x$, $y \neq 0$, $x > 0$

$$y' - \frac{1+x}{2x} y = -\frac{\sec^2 x + \tan x}{2x} y^3$$ is B. equation

$$y' y^{-3} - \frac{1+x}{2x} y^{-2} = -\frac{\sec^2 x + \tan x}{2x}$$ (1)

We substitute $u = y^{-2}$, then $u' = -2y^{-3}y'$ or $-\frac{u'}{2} = y^{-3}y'$

$$-\frac{u'}{2} - \frac{1+x}{2x} u = -\frac{\sec^2 x + \tan x}{2x}$$

$$u' + \left(1 + \frac{1}{x}\right) u = \frac{\sec^2 x + \tan x}{x}$$ is Linear D.E

$$\mu(x) = e^{\int (1 + \frac{1}{x}) dx} = e^{x + \ln x} = e^x x$$ (1)

$$u x e^x = \int e^x (\sec^2 x + \tan x)\, dx$$

$u x e^x = e^x \tan x + C$ or $\boxed{y^{-2} x e^x = e^x \tan x + C}$ (2)

is the solution of the D.E

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(b) $y = \ln(x-c)$, where c is an arbitrary constant

$y' = \frac{1}{x-c}$, $x - c = e^{y}$ (1)

$y' = \frac{1}{e^{y}} = e^{-y}$

Now we have to solve the D.E $y' = \frac{-1}{f(x,y)} = -e^{+y}$ (1)

or $y' = -e^{+y} \Rightarrow \frac{dy}{e^{+y}} = -dx \Rightarrow \int e^{-y} dy = -\int dx$

$-e^{-y} = -x + c_1$ or $e^{-y} = (x - c_1)$ (2)

Then $\boxed{e^{y}(x-c_1) = 1}$ is the family of curve which is orthogonal to $y = \ln(x-c)$.

Question (2)

(a) $y'' - 2y' + y = xe^{x}\ln x$, $x > 0$

1) $y'' - 2y' + y = 0$, $m^2 - 2m + 1 = (m-1)^2 = 0$) $m = 1, 1$

$y_c = c_1 e^{x} + c_2 x e^{x}$, $y_1 = e^{x}$, $y_2 = xe^{x}$, $f(x) = xe^{x}\ln x$

2) $y_p = y_1 u_1 + y_2 u_2$, $W = \begin{vmatrix} e^{x} & xe^{x} \\ e^{x} & e^{x} + xe^{x} \end{vmatrix} = e^{2x}$ (1)

$$u_1' = \frac{\begin{vmatrix} 0 & xe^{x} \\ xe^{x}\ln x & xe^{x} + e^{x} \end{vmatrix}}{W} = \frac{-x^2 e^{2x}\ln x}{e^{2x}} = -x^2 \ln x$$

$$u_1 = -\int x^2 \ln x \, dx = -\left\{ \ln x \cdot \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \frac{1}{x} dx \right\}$$

$$\boxed{u_1 = -\frac{x^3}{3}\ln x + \frac{1}{9}x^3}$$

(1)

$$u_2' = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & xe^{x}\ln x \end{vmatrix}}{W} = \frac{e^{2x} x \ln x}{e^{2x}} = x \ln x$$

$$u_2 = \int x \ln x = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x} dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

Then $y_p = e^{x}\left(-\frac{x^3}{3}\ln x + \frac{x^3}{9}\right) + xe^{x}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)$ (1)

$$\boxed{y_p = \frac{1}{6}x^3 e^x \ln x - \frac{5}{36}x^3 e^x}$$

Then the general solution of the D.E. is

$$y = y_c + y_p = \boxed{c_1 e^x + c_2 x e^x + \frac{1}{6}x^3 e^x \ln x - \frac{5}{36}x^3 e^x}$$ 1

(b)
$$\begin{cases} (x+2)y'' - y' + \dfrac{y}{x+2} = 0, & x+2>0 \\ y(-1)=1, \quad y'(-1)=0 \end{cases}$$

$(x+2)^2 y'' - (x+2)y' + y = 0$, substitute $y = (x+2)^m$

then, $m(m-1) - m + 1 = 0$ or $m^2 - 2m + 1 = (m-1)^2 = 0$ 1

$m = 1, 1$

$y = c_1(x+2) + c_2(x+2)\ln(x+2)$ 1

$y' = c_1 + c_2 \ln(x+2) + c_2$

So $y(-1) = c_1 = 1$, $y'(-1) = c_1 + c_2 = 0 \Rightarrow c_2 = -1$

Then the solution of the IVP is 2

$y = (x+2) - (x+2)\ln(x+2)$ or $\boxed{y = (x+2)[1 - \ln(x+2)]}$

Question (3)

(a) $y'' - 2y' - 3y = 2e^x - 10\sin x$

1) $y'' - 2y' - 3y = 0$, $y = e^{mx}$,

$m^2 - 2m - 3 = (m-3)(m+1) = 0$, $m = 3$, $m = -1$

$y_c = c_1 e^{-x} + c_2 e^{3x}$ 1

2) $y_p = Ae^x + B\sin x + C\cos x$

$y_p' = Ae^x + B\cos x - C\sin x$, $y_p'' = Ae^x - B\sin x - C\cos x$

$$y_p'' - 2y_p' - 3y_p = Ae^x - B\sin x - C\cos x - 2Ae^x - 2B\cos x$$
$$+ 2C\sin x - 3Ae^x - 3B\sin x - 3C\cos x$$
$$= -4Ae^x + (-4B + 2C)\sin x$$
$$+ (-4C - 2B)\cos x$$
$$= 2e^x - 10\sin x$$

(1)

Then $-4A = 2$, $A = \frac{-1}{2}$

$-4B + 2C = -10$, $\quad -4C - 2B = 0 \Rightarrow B = 2$, $C = -1$

Hence $\boxed{y_p = -\frac{1}{2}e^x + 2\sin x - \cos x}$

and the general solution of the D.E is

$$\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} - \frac{1}{2}e^x + 2\sin x - \cos x}$$

(2)

(b) $(1-x^2)y'' - (2x+1)y' - y = 0$, $\quad y(0) = 2$, $\quad y'(0) = 0$

Solution: $\frac{a_1}{a_2}(x) = -(2x+1)\frac{1}{1-x^2} = -(2x+1)\sum_0^\infty x^{2n}$, $\quad |x|<1$

$$\frac{a_0}{a_2} = \frac{-1}{1-x^2} = -\sum_0^\infty x^{2n}, \quad |x|<1$$

Then the solution of the D.E is the form $y = \sum_0^\infty a_n x^n$, $|x|<1$

$y(0) = 2 \Rightarrow a_0 = 2$, $\quad y'(0) = 0 \Rightarrow a_1 = 0$

$$(1-x^2)\sum_2^\infty n(n-1)a_n x^{n-2} - (2x+1)\sum_1^\infty n a_n x^{n-1} - \sum_0^\infty a_n x^n = 0$$

$$\sum_2^\infty n(n-1)a_n x^{n-2} - \sum_2^\infty n(n-1)a_n x^n - \sum_1^\infty 2n a_n x^n - \sum_1^\infty n a_n x^{n-1} - \sum_0^\infty a_n x^n = 0$$

$n-2 = k$, $n = k+2$ | $n = k$ | $n = k$ | $n-1 = k$, $n = k+1$ | $n = k$

$$\sum_0^\infty (k+1)(k+2)a_{k+2}x^k - \sum_2^\infty k(k-1)a_k x^k - \sum_1^\infty 2k a_k x^k - \sum_0^\infty (k+1)a_{k+1}x^k$$
$$- \sum_0^\infty a_k x^k = 0$$

$$(2a_2 - a_1 - a_0) + (6a_3 - 2a_1 - 2a_2 - a_1)x$$

$$+ \sum_{2}^{\infty} \left[(k+1)(k+2)a_{k+2} - k(k-1)a_k - 2k a_k - (k+1)a_{k+1} - a_k \right] x^k = 0$$

$2a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = 1$

$6a_3 - 3a_1 - 2a_2 = 0 \Rightarrow 6a_3 = 2, \quad a_3 = \frac{1}{3}$

$$a_{k+2} = \frac{(k+1)a_{k+1} + (k^2+k+1)a_k}{(k+1)(k+2)}, \quad k \geq 2$$ 2

For $k=2$, $a_4 = \dfrac{3a_3 + 7a_2}{3\cdot 4} = \dfrac{8}{12} = \dfrac{2}{3} = a_4$

$k=3$ $\quad a_5 = \dfrac{4a_4 + 13a_3}{4\cdot 5} = \dfrac{\frac{8}{3} + \frac{13}{3}}{20} = \dfrac{7}{20}$

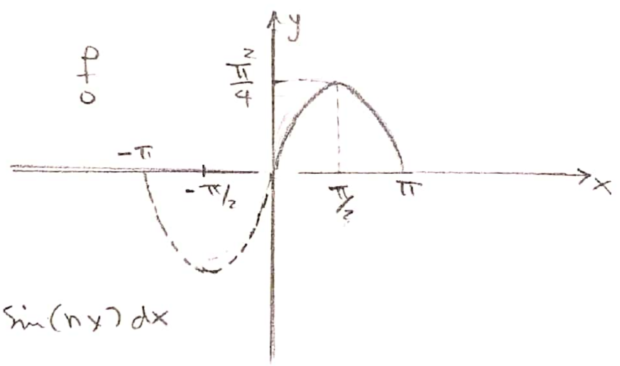
So the solution of the IVP is 1

$$\boxed{y = 2 + x^2 + \frac{1}{3}x^3 + \frac{2}{3}x^4 + \frac{7}{20}x^5 + \cdots} \qquad |x| < 1$$

Question (4)

(a) $f(x) = x(\pi - x), \qquad 0 \leq x \leq \pi$

Solution



1

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin(nx)\, dx$$

$$\frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi} (\pi - 2x)\frac{\cos(nx)}{n}\, dx$$

$$= \frac{2}{\pi}\left[(\pi - 2x)\frac{\sin nx}{n^2}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} (-2)\frac{\sin nx}{n^2}\, dx$$

$$b_n = +\frac{4}{\pi}\int_0^{\pi} \frac{\sin nx}{n^2}\,dx = \frac{-4}{\pi n^3}\left[\cos nx\right]_0^{\pi} = \boxed{\frac{4}{\pi n^3}\left[1-(-1)^n\right]}$$ 2

$$\boxed{f(x) = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left[1-(-1)^n\right]\sin nx,} \qquad 0 \le x \le \pi$$

$$f(x) = (\pi x - x^2) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^3}\left[1-(-1)^{2n-1}\right]\sin((n-1)x)$$

$$= \frac{8}{\pi}\sum_{1}^{\infty} \frac{1}{(2n-1)^3}\sin((n-1)x) \qquad 0 \le x \le \pi$$

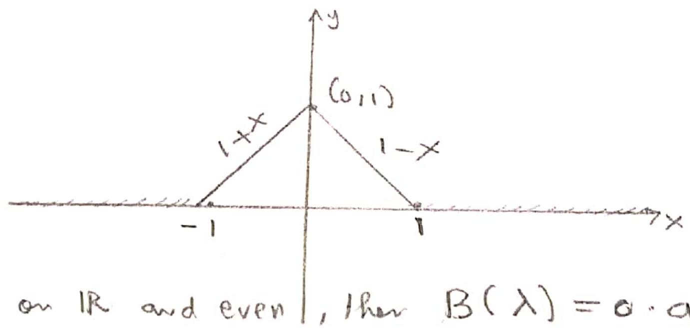
If $x = \pi/2$, then

$$f(\pi/2) = \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{1}^{\infty}\frac{1}{(2n-1)^3}\sin\left(\frac{n-1}{2}\pi\right)$$ 2

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}$$

Hence $\boxed{\frac{\pi^3}{32} = \sum_{1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}}$ or $\boxed{\frac{\pi^3}{32} = \sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3}}$

(b)



1

f is continuous on $\mathbb{R}$ and even, then $B(\lambda) = 0$ and

$$A(\lambda) = \int_{-\infty}^{\infty} f(x)\cos(\lambda x) = 2\int_0^1 (1-x)\cos(\lambda x)\,dx$$

$$= 2\left[(1-x)\left(\cancel{\frac{\sin\lambda x}{\lambda}}\right)\right]_0^1 - 2\int_0^1 \frac{\sin\lambda x}{\lambda}(-1)\,dx$$

$$= \frac{2}{\lambda}\int_0^1 \sin\lambda x\,dx = -\frac{2}{\lambda^2}\left[\cos\lambda x\right]_0^1$$ 2

$$\boxed{A(\lambda) = \frac{2}{\lambda^2}\left[1-\cos\lambda\right]}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\,d\lambda$$

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\cos(\lambda x)\,d\lambda.}$$

For $x=0$ we have:

$$f(0) = 1 = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\,d\lambda.$$

But $1-\cos\lambda = 2\sin^2(\lambda/2)$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\cancel{2}\sin^2(\lambda/2)}{\cancel{4}\left(\frac{\lambda}{2}\right)^2}\,d(\lambda/2)(2)\text{, and we put } \frac{\lambda}{2} = \beta$$

2

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin^2(\beta)}{\beta^2}\,d\beta$$

Hence $\boxed{\frac{\pi}{2} = \int_0^\infty \frac{\sin^2(\beta)}{\beta^2}\,d\beta = \int_0^\infty \frac{\sin^2(\lambda)}{\lambda^2}\,d\lambda}$