M-203 FINAL EXAMINATION (Semester-I, 1441/1442)
Department of Mathematics, King Saud University

Time: 3 Hours　　　　　　　　　　　　　　　　　　Max. Marks-40

Q1. (a) Determine whether the sequence $\left\{\left(1+\frac{3}{n}\right)^{2n}\right\}$ converges or diverges and if it converges, find its limit. [3]

(b) Find the interval and radius of convergence for the power series [5]

$$\sum_{n=1}^{\infty}(-1)^n \frac{(x-4)^n}{2^{n+1}(n+3)}.$$

(c) Find the power series representation for the function $f(x) = \tan^{-1} x$ and approximate $\tan^{-1}(0.1)$ usng two non-zero terms. [4]

Q2. (a) Evaluate the double integral $\int_0^1 \int_y^1 (2x^2)e^{xy}\,dx\,dy.$ [4]

(b) Use triple integral to find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$. [4]

(c) Find the mass of the solid that lies under $z = \sqrt{4x^2 + 4y^2}$, inside $x^2 + y^2 = x$ and above $z = 0$, where density is given by $\delta = \sqrt{x^2 + y^2}$. [4]

Q3. (a) Show that the line integral $\int_C (y-1)dx + x\,dy$ is independent of path and evaluate the integral $\int_{(1,1)}^{(2,3)} (y-1)dx + x\,dy.$ [4]

(b) Use Green's theorem to find the area bounded by the circle $x^2 + y^2 = 9$. [4]

(c) Use divergence theorem to find the flux of the force $\vec{F} = 2x\,\vec{i} - y\,\vec{j} + z\,\vec{k}$ through the surface of the solid bounded by $z = 4 - \sqrt{x^2 + y^2}$, $z = \sqrt{x^2 + y^2}$. [4]

(d) Verify Stokes's theorem for the surface that is the portion of the paraboloid $z = x^2 + y^2$ cut off by the plane $z = 1$ and the force [4]

$$\vec{F} = (x^2 - 1)\,\vec{i} + (y^2 - 2)\,\vec{j} + z\,\vec{k}.$$

M- 203

Final Exam. (I semester 1441/1442)

Time: 3 Hours Max. Marks: 40

Q#1(a). Determine whether the sequence $\left\{\left(1+\frac{3}{n}\right)^{2n}\right\}$ converges or diverges and if it converges, find its limit. [Marks: 3]

Soln. Let $y=\left(1+\frac{3}{n}\right)^{2n} \Rightarrow \lim_{n\to\infty} y = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}$ 1^{∞} form

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} 2n \ln\left(1+\frac{3}{n}\right)$: $0.\infty$ form

$$= 2\lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{n}} ; \ \frac{0}{0} \text{ form} \quad (1)$$

$$= 2\lim_{n\to\infty} \frac{1}{1+\frac{3}{n}}\left(+\frac{3}{n^2}\right)\Big/\left(+\frac{1}{n^2}\right)$$

$$= 6\lim_{n\to\infty} \frac{1}{1+\frac{3}{n}} = 6 \quad (1)$$

$\therefore \lim_{n\to\infty} y = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n} = e^{6}$; (1) convergent.

(b) Find the interval of convergence and radius of Conv. for the Power Series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-4)^n}{2^{n+1}(n+3)}$. [Marks: 5]

Soln: Applying absolute Ratio-test, we have

$$\lim_{n\to\infty} \frac{(x-4)^{n+1}}{2^{n+2}(n+4)} \times \frac{2^{n+1}}{(x-4)^n}.(n+3) = \frac{1}{2}|x-4|$$

Abs. conv. $\Leftrightarrow \frac{1}{2}|x-4|<1 \Leftrightarrow |x-4|<2$

$\Leftrightarrow -2<x-4<2 \Rightarrow 2<x<6$ (2)

At $x=2$, we have $\sum_{n=1}^{\infty}(-1)^n \frac{(-2)^n}{2^{n+1}(n+3)} = \sum_{n=1}^{\infty} \frac{1}{2(n+3)}$; Divg. (1)

At $x=6$ we have $\sum_{n=1}^{\infty} (-1)^n \frac{(+2)^n}{2^{n+1}(n+3)} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2(n+3)}$

which is cong. by AST. (1)

Hence interval of convg: $(2, 6]$ and radius of cong. $\frac{6-2}{2} = 2$ (1)

(c) Find the Power Series representation for the function $f(x) = \tan^{-1}x$ and approximate $\tan^{-1}(0.1)$ using two non-zero terms. [Marks: 4]

Soln. We know $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots ; |x|<1$ (1)

$\therefore \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots ; |x|<1$

$\therefore \tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\,dt$ (1)

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ (1)

$\therefore \tan^{-1}(0.1) = (0.1) - \frac{0.001}{3} + \cdots \approx 0.1 - 0.00033$

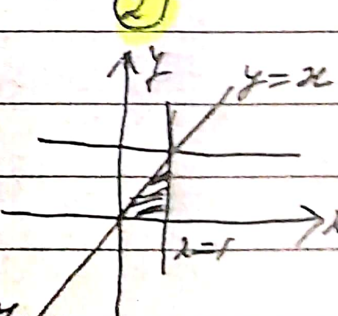
$= 0.0997$ (1)

Q#2(a) Evaluate the double integral $\int_0^1 \int_y^1 (2x^2) e^{xy}\,dx\,dy$ [Marks: 4]

Soln. Reversing the integral, we get $\int_0^1 \int_0^x (2x^2) e^{xy}\,dy\,dx$. (2)

$= \int_0^1 \left[2x^2 \cdot \frac{1}{x} e^{xy}\right]_0^x dx = \int_0^1 2x \cdot e^{x^2}\,dx$ (1)

$= \left[e^{x^2}\right]_0^1 = e - 1$ (1)

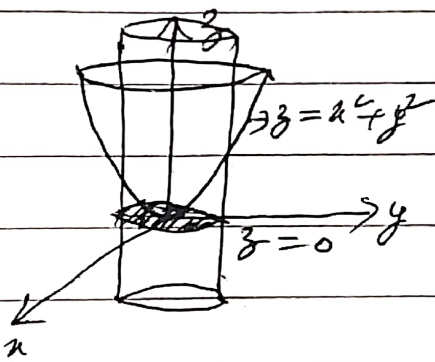


(b) Use triple integral to find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$ [Marks: 4]

Soln. Volume $V = \iiint_Q dV = \int_0^{2\pi}\int_0^{1}\int_0^{r^2} r\,dz\,dr\,d\theta$ (3)

$= \int_0^{2\pi}\int_0^{1} r\cdot r^{2}\,dr\,d\theta$

$= 2\pi\left[\frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}$ (1)

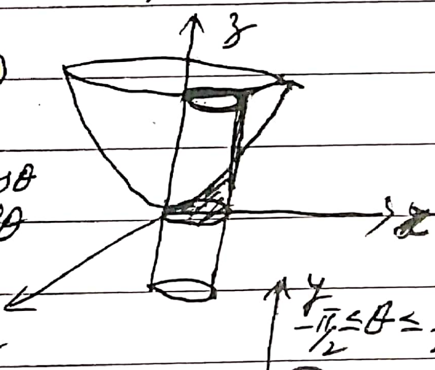


(c) Find the mass of the solid that lies under $z=\sqrt{4x^2+4y^2}$, inside $x^2+y^2=x$ and above $z=0$, where density is given by $\delta=\sqrt{x^2+y^2}$. [Marks: 4]

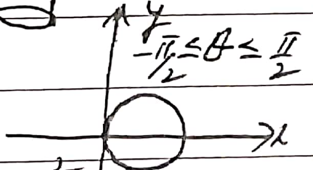
Soln. mass: $m=\iiint \delta\,dV$; $z=\sqrt{4x^2+4y^2}=2\sqrt{x^2+y^2}=2r$

$\delta=\sqrt{x^2+y^2}=r$

$m=\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{2r} r\cdot r\,dz\,dr\,d\theta$ (3)



$=\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} 2r^3\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\left[\frac{2}{4}r^4\right]_0^{\cos\theta}d\theta$



$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta=$

$\cos^4\theta=\left(\frac{1+\cos 2\theta}{2}\right)^2$ $\qquad r=\cos\theta$

$=\int_0^{\pi/2}\left(\frac{1}{4}+\frac{1}{2}\cos2\theta+\frac{1}{2}\cos4\theta\right)d\theta$ $\qquad =\frac{1}{4}\left(1+2\cos2\theta+\cos^2 2\theta\right)$

$=\frac{1}{4}+\frac{1}{2}\cos2\theta+\frac{1}{4}\left(\frac{1+\cos4\theta}{2}\right)$

$=\left[\frac{3}{4}\theta+\frac{1}{2}\frac{\sin2\theta}{2}+\frac{1}{8}\sin4\theta\right]_0^{\pi/2}$

$=\frac{3\pi}{8}+0+0=\frac{3\pi}{8}$ (1)

Q#3(a) Show that the line integral $\int_C (y-1)\,dx + x\,dy$ is independent of path and evaluate the integral

$$\int_{(1,1)}^{(2,3)} (y-1)\,dx + x\,dy.$$

[Marks: 4]

Sol. We have $\vec{F}(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j} = (y-1)\,\vec{i} + x\,\vec{j}$

Equating, we get $f_x(x,y) = y-1$ and $f_y(x,y) = x$

It follows that $f(x,y) = xy - x$. (2)

Therefore, we have $\displaystyle\int_{(1,1)}^{(2,3)} (y-1)\,dx + x\,dy = \Big[xy - x\Big]_{(1,1)}^{(2,3)} = \underline{4}$ (2)

Alternate for independence: $M = y-1$ and $N = x$ $\quad \frac{\partial N}{\partial x} = 1$ and $\frac{\partial M}{\partial y} = 1 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$

(b) Use Green's theorem to find the area bounded by the circle $x^2 + y^2 = 9$ [Marks: 4]

Sol. $C:\ x = 3\cos t;\ y = 3\sin t;\ 0 \le t \le 2\pi$ (2)

$$\text{Area } A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (3\cos t)(3\cos t)\,dt - (3\sin t)(3\sin t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 9(\cos^2 t + \sin^2 t)\,dt = \frac{1}{2}\times 9\,[2\pi] = \underline{9\pi}$$ (2)

(c) Use Divergence theorem to find the flux of the force $\vec{F} = 2x\,\vec{i} - y\,\vec{j} + z\,\vec{k}$ through the surface of the solid bounded by $z = 4 - \sqrt{x^2+y^2}$, $z = \sqrt{x^2+y^2}$ [Marks: 4]

Sol. We have $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV$

$$= 2\int_0^{2\pi}\int_0^{2}\int_r^{4-r} r\,dz\,dr\,d\theta = 2\int_0^{2\pi}\int_0^{2} r(4-2r)\,dr\,d\theta$$ (2)

$$= 2\int_0^{2\pi} \left[2r^2 - \frac{2}{3}r^3\right]_0^2 d\theta = 2\left[8 - \frac{16}{3}\right](2\pi) = \underline{\frac{32}{3}\pi}$$

(d) Verify Stokes's theorem for the surface that is the portion of the paraboloid $z = x^2 + y^2$ cut off by the plane $z = 1$ and the force $\vec{F} = (x^2-1)\vec{i} + (y^2-2)\vec{j} + z\vec{k}$.

[Marks: 4]

Soln. We have the Stokes's theorem as

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot \vec{n}\,dS} \quad \text{---}(1)$$

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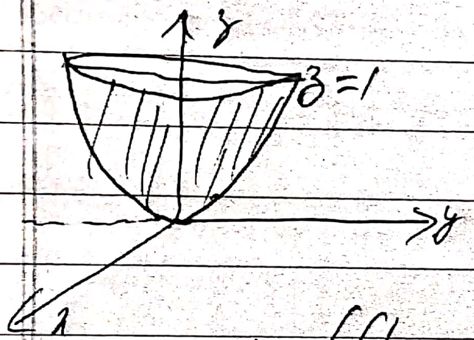
$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$



$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-1 & y^2-2 & z \end{vmatrix} = 0$$

$$\therefore \iint_S (\text{curl}\,\vec{F})\cdot \vec{n}\,dS = 0$$

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Now, we find $\oint_C \vec{F}\cdot d\vec{r} = \oint_C (x^2-1)dx + (y^2-2)dy + z\,dz$

We have C: $x = \cos t$, $y = \sin t$, $z = 1$; $0 \le t \le 2\pi$

$dx = -\sin t\,dt$; $dy = \cos t\,dt$; $dz = 0$

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$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_C (x^2-1)dx + (y^2-2)dy + z\,dz$$

$$= \int_0^{2\pi} (\cos^2 t - 1)(-\sin t)\,dt + (\sin^2 t - 2)\cos t\,dt + 0$$

$$= \int_0^{2\pi} (-\cos^2 t \sin t + \sin t + \sin^2 t \cos t - 2\cos t)\,dt$$

$$= \left[\frac{\cos^3 t}{3} + \cos t + \frac{\sin^3 t}{3} - 2\sin t\right]_0^{2\pi} = 0$$

Hence, $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$: verified.

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