

Question 1 : (3+2+2)

1. Find the value of C that satisfies the mean value theorem for the definite integral of the function $f(x) = \sqrt[3]{x-1}$ on the interval $[1, 9]$.
2. Find $\frac{dF(x)}{dx}$ if $F(x) = \int_{\cos x}^{2^{3x}} \sqrt{1+t^3} dt$.
3. Find $\frac{dy}{dx}$ if $y = \tanh(\sqrt{x}) + \sinh^{-1}\left(\frac{2}{\sqrt{x}}\right)$.

Question 2 : (3+3+3+3+2)

Compute the following integrals:

1. $\int \frac{dx}{\sqrt{-x^2 - 2x + 8}}$;
3. $\int \frac{3x^2 + x + 8}{x^3 + 4x} dx$;
5. $\int \frac{dx}{(4 \cdot 2^{-x} + 2^x)}$;
2. $\int x \tanh^{-1}(x) dx$;
4. $\int \frac{dx}{(9 + x^2)^{\frac{3}{2}}}$

Question 3 : (2+3+3+3+3+1+4)

1. Evaluate the following limit $\lim_{x \rightarrow +\infty} \frac{x - xe^x}{x + 2xe^x}$.
2. Discuss whether the improper integral $\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 4}$ converges or diverges.
3. Sketch the region bounded by the curves $y = 5x^2$ and $y = 4 + x^2$ and find its area.
4. Sketch the region bounded by the curves $y = x^2$, $y = -x + 2$ and find the volume of the solid generated by the revolving this region about the x -axis.
5. Find the arc length of the curve defined by: $y = \frac{e^x + e^{-x}}{2}$ on the interval $[0, \ln 3]$.
6. Convert the polar equation $\frac{1}{r} = \frac{\sin \theta + 5 \cos \theta}{2}$ into Cartesian equation.
7. Sketch the region inside $r = 1 + \cos \theta$ that in the second quadrant. Then compute its area

Final exam Math 111 - S II

Q1:

$$1) (b-a) f(c) = \int_a^b f(x) dx \quad \checkmark$$

$$8 \sqrt[3]{c-1} = \frac{3}{4} \left[(x-1)^{\frac{4}{3}} \right]_1^9 = 12 \quad \checkmark$$

$$\sqrt[3]{c-1} = \frac{3}{2} \Rightarrow c-1 = \frac{27}{8} \quad \checkmark$$

$$\Rightarrow c = \frac{35}{8} \in (1, 9)$$

$$2) F(x) = \int_{\cos x}^{2^{3x}} \sqrt{1+t^3} dt$$

$$\frac{dF(x)}{dx} = \sqrt{1+2^{9x}} \left(2^{3x} \cdot 3 \ln 2 \right) + \sqrt{1+\cos^3 x} \sin x \quad \checkmark$$

$$3) y = \tanh(\sqrt{x}) + \sinh^{-1}\left(\frac{2}{\sqrt{x}}\right)$$

$$\dot{y} = \frac{1}{2\sqrt{x}} \operatorname{sech}^2(\sqrt{x}) + \frac{-x^{-\frac{3}{2}}}{\sqrt{1+\frac{4}{x}}} \quad \checkmark$$

Q2:

$$1. \int \frac{dx}{\sqrt{-x^2-2x+8}} = \int \frac{dx}{\sqrt{9-(x+1)^2}} = \sin^{-1}\left(\frac{x+1}{3}\right) + C \quad \checkmark$$

$$2. \int x \tan^{-1} x dx$$

$$\tan^{-1} x = u \Rightarrow du = \frac{dx}{1+x^2} \quad \checkmark$$

$$x dx = dv \Rightarrow v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$I = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{1 + x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int (1 - \frac{1}{1+x^2}) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + C$$

$$3. I = \int \frac{3x^2 + x + 8}{x^3 + 4x} dx$$

$$\frac{3x^2 + x + 8}{x^3 + 4x} = \frac{3x^2 + x + 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$3x^2 + x + 8 = A(x^2 + 4) + Bx^2 + Cx$$

$$3x^2 + x + 8 = (A+B)x^2 + Cx + 4A \Rightarrow$$

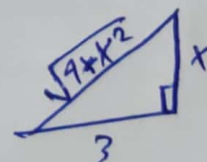
$$\left. \begin{array}{l} A+B=3 \\ C=1 \\ 4A=8 \end{array} \right\} \Rightarrow A=2, B=1, C=1$$

$$I = \int \frac{2}{x} dx + \int \frac{x+1}{x^2+4} dx = \int \frac{2}{x} dx + \int \frac{x}{x^2+4} dx + \int \frac{1}{4+x^2} dx$$

$$= 2 \ln|x| + \frac{1}{2} \ln(x^2+4) + \frac{1}{2} \tan^{-1} \frac{x}{2} + C$$

$$4. I = \int \frac{dx}{(9+x^2)^{\frac{3}{2}}}$$

$$x = 3 \tan \theta \Rightarrow \tan \theta = \frac{x}{3}$$



$$dx = 3 \sec^2 \theta d\theta$$

$$I = \int \frac{3 \sec^2 \theta d\theta}{(9+9 \tan^2 \theta)^{\frac{3}{2}}} = \frac{1}{9} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \frac{1}{9} \int \cos \theta d\theta$$

$$I = \frac{1}{9} \sin \theta + C = \frac{1}{9} \frac{x}{\sqrt{9+x^2}} + C \quad \checkmark$$

$$5. \quad I = \int \frac{dx}{(4 \cdot 2^x + 2^x)} = \int \frac{2^x}{(4 + 2^x)} dx = \int \frac{2^x}{(4 + (2^x)^2)} dx \quad \checkmark$$

$$I = \frac{1}{2 \ln 2} \tan^{-1} \left(\frac{2^x}{2} \right) + C \quad \checkmark$$

Q3

$$1. \quad \lim_{x \rightarrow +\infty} \frac{x - x e^x}{x + 2x e^x} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \rightarrow +\infty} \frac{\cancel{x} e^{(2+x)} \checkmark}{\cancel{x} (4 + 2x) \checkmark} = \lim_{x \rightarrow +\infty} \frac{x+2}{2x+4} = \frac{1}{2}$$

$$2. \quad \int_{-\infty}^{+\infty} \frac{dx}{4+x^2} = \int_{-\infty}^0 \frac{dx}{4+x^2} + \int_0^{+\infty} \frac{dx}{4+x^2}$$

$$= \lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{4+x^2} + \lim_{c \rightarrow +\infty} \int_0^c \frac{dx}{4+x^2} \quad \checkmark$$

$$= \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) \right]_t^0 + \lim_{c \rightarrow +\infty} \left[\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) \right]_0^c \quad \checkmark$$

$$= \lim_{t \rightarrow -\infty} \left[\frac{1}{2} (\tan^{-1}(0) - \tan^{-1}(\frac{t}{2})) \right]$$

$$+ \lim_{c \rightarrow +\infty} \left[\frac{1}{2} (\tan^{-1}(\frac{c}{2}) - \tan^{-1}(0)) \right] \quad \checkmark$$

$$= \frac{1}{2} (0 - \frac{\pi}{2}) + \frac{1}{2} (\frac{\pi}{2} - 0) = \frac{\pi}{2} \quad \text{Ans}$$

3. $y = 5x^2$ & $y = 4 + x^2$

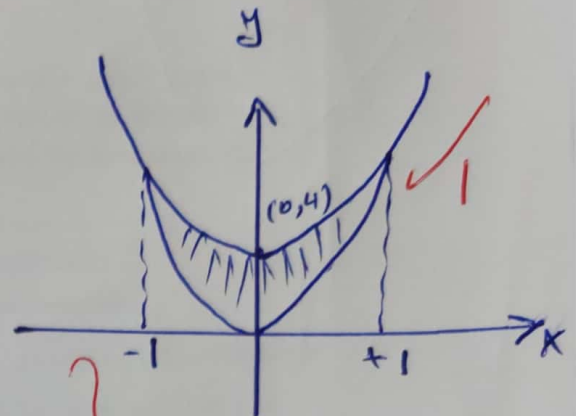
$$5x^2 = 4 + x^2 \Rightarrow 4x^2 = 4 \Rightarrow x^2 = 1$$

$$(x-1)(x+1) = 0 \Rightarrow x = -1, x = +1$$

$$A = \int_{-1}^{+1} [(4+x^2) - 5x^2] dx$$

$$A = \int_{-1}^{+1} [4 - 4x^2] dx = \left[4x - \frac{4}{3}x^3 \right]_{-1}^{+1}$$

$$= \left(4 - \frac{4}{3} \right) - \left(-4 + \frac{4}{3} \right) = \frac{16}{3}$$



4. $y = x^2$ & $y = -x + 2$

$$x^2 = -x + 2 \Rightarrow x^2 + x - 2 = 0$$

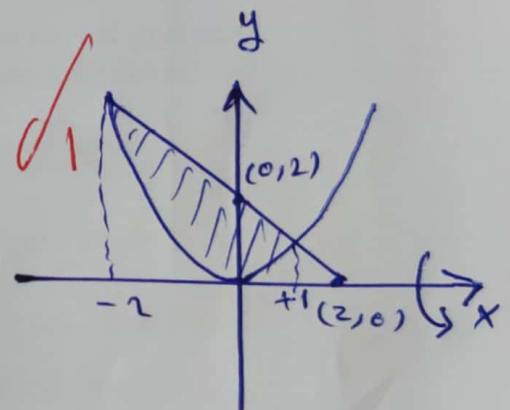
$$(x+2)(x-1) = 0 \Rightarrow x = -2, x = 1$$

$$V = \pi \int_{-2}^{+1} [(2-x)^2 - (x^2)^2] dx$$

$$= \pi \int_{-2}^{+1} [4 - 4x + x^2 - x^4] dx$$

$$= \pi \left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_{-2}^{+1}$$

$$\pi \left[\left(4 - 2 + \frac{1}{3} - \frac{1}{5} \right) - \left(-8 - 8 - \frac{8}{3} + \frac{2^5}{5} \right) \right]$$



$$5. \quad y = \frac{e^x + e^{-x}}{2} = \cosh x \quad \text{on } [0, \ln 3]$$

$$y' = \sinh(x)$$

$$L = \int_0^{\ln 3} \sqrt{1 + (y')^2} dx = \int_0^{\ln 3} \sqrt{1 + \sinh^2 x} dx = \int_0^{\ln 3} \cosh x dx$$

$$= [\sinh x]_0^{\ln 3} = \frac{1}{2} [e^x - e^{-x}]_0^{\ln 3} = \frac{1}{2} [\frac{1}{3} - e^3]$$

$$6. \quad \frac{1}{r} = \frac{\sin \theta + 5 \cos \theta}{2} \Rightarrow 2 = r \sin \theta + 5 r \cos \theta \quad \checkmark 0.5$$

$$2 = y + 5x \Rightarrow y = -5x + 2 \quad \checkmark 0.5$$

$$7. \quad r = 1 + \cos \theta$$

$$\frac{\pi}{2} \leq \theta \leq \pi$$

$$A = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (1 + \cos \theta)^2 d\theta \quad \checkmark 1$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta \quad \checkmark 1$$

$$= \frac{1}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_{\frac{\pi}{2}}^{\pi} \quad \checkmark 1$$

$$= \frac{1}{2} \left[\left(\frac{3\pi}{2} + 0 + 0 \right) - \left(\frac{3\pi}{4} + 2 + 0 \right) \right] = \frac{3\pi}{8}$$

