SEMESTER II FINAL EXAMINATION, 1443

DEPT. MATH., COLLEGE OF SCIENCE, KSU

MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [Marks: 4+4+3=11]

(a) For which values of m will the following linear system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + my + 2z &= 3 \\ 4x + (6+m)y - mz &= 13 - m \\ x + 2(m-1)y + (m+4)z &= m+2 \end{aligned}$$ 4

(b) Let R and S be 3×3 matrices such that $RS + R - 2I = 0$. Find R^{-1} if

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$ 4

(c) Find the value of x so that $|A| = 6$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3x-6 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$ 3

Q2. [Marks: 3+4+3=10]

(a) Find the volume of the parallelepiped (box) having adjacent sides **AB**, **AC**, and **AD**, where $A(1,0,-1)$, $B(1,0,3)$, $C(4,3,2)$, $D(7,1,0)$. 3

(b) Determine whether the following lines l_1 and l_2 are parallel or they intersect. If they intersect, find the point of intersection:

$$l_1 : x = 3+t,\ y = 5-t,\ z = -2+2t; \quad l_2 : x = 2+s,\ y = 3-2s,\ z = -1+3s.$$ 4

(c) Sketch the graph of $z^2 - 4y^2 - 9x^2 = 4$, and identify the surface. 3

Q3. [Marks: 2+4=6]

(a) Find the position vector $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2\mathbf{i} - 4t^3\mathbf{j} + 6\sqrt{t}\,\mathbf{k}$ subject to the condition $\mathbf{r}(0) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$. 2

(b) Find general formula for the tangential and normal components of acceleration and for the curvature of the curve C given by $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{t^3}{3}\mathbf{k}$ 4

Q4. [Marks: 2+3+5+3=13]

(a) Use partial derivative to find $\frac{dy}{dx}$ if $x^2 + 3xy + 2x - y - 10 = 0$ 2

(b) Find an equation of the tangent plane and parametric equations of normal line to the surface $z = e^x(1 + \cos y)$ at the point $P(0,0,2)$. 3

(c) Find the local extrema and saddle points if any of the function $f(x,y) = \frac{4}{3}x^3 - xy^2 + 5y^2 + 24y$ 5

(d) Use Lagrange multipliers to find extrema of $f(x,y,z) = 4x^2 + y^2 + 5z^2$ subject to the constraint $2x + 3y + 4z = 12$. 3

Solution to M 107 Final Exam

Q1 (a)

$$\left[\begin{array}{ccc|c} 1 & m & 2 & 3 \\ 4 & 6+m & -m & 13-m \\ 1 & 2(m-1) & m+4 & m+2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & m & 2 & 3 \\ 0 & 6-3m & -m-8 & 1-m \\ 0 & m-2 & m+2 & m-1 \end{array}\right] \longrightarrow$$

② $$\left[\begin{array}{ccc|c} 1 & m & 2 & 3 \\ 0 & \underbrace{m-2} & m+2 & m-1 \\ 0 & 0 & 2\underbrace{(m-1)} & 2(m-1) \end{array}\right]$$

②
- If $m=2$, the system has no solution
- If $m \neq 1$ and $m \neq 2$, the system has unique solution
- If $m=1$, the system has infinitely many solution

(b) $RS + R = 2I \Rightarrow R(S+I) = 2I \Rightarrow$

④ $$R^{-1} = \frac{1}{2}(S+I) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

(c) $$\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3x-6 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 6$$

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3x-4 & 1 \\ 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 3x-4 & 1 \\ -2 & 2 \end{vmatrix}$$

$|A| = 6x-8+2$

$|A| = 6x-6$

③ $\Rightarrow 6x-6 = 6 \Rightarrow x = 2$

Q2 (a) $\vec{AB} = \langle 0,0,4\rangle$, $\vec{AC} = \langle 3,3,3\rangle$, $\vec{AD} = \langle 6,1,1\rangle$

③ $$V = |(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = \begin{vmatrix} 0 & 0 & 4 \\ 3 & 3 & 3 \\ 6 & 1 & 1 \end{vmatrix} = |4(-15)| = |-60|$$

$V = |-60| =$

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Q_2 (b) $\underline{v}_1 = \langle 1, -1, 2 \rangle$, $\underline{v}_2 = \langle 1, -2, 3 \rangle$

$\frac{1}{1} \neq \frac{-1}{-2}$ the two lines are not parallel.

Let $A(x_0, y_0, z_0)$ be the intersecting point.

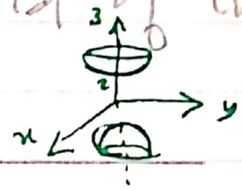
4

$$\begin{cases} 3 + t_0 = 2 + s_0 & (1) \\ 5 - t_0 = 3 - 2s_0 & (2) \\ -2 + 2t_0 = -1 + 3s_0 & (3) \end{cases}$$

From (1) and (2) $t_0 = -4$, $s_0 = -3$, eq. (3) is verified. So the point of intersection is $A(-1, 9, -10)$.

(c) $\frac{z^2}{2^2} - \frac{y^2}{1^2} - \frac{x^2}{\left(\frac{2}{3}\right)^2} = 1$ Hyperboloid of two sheets

3



traces

• $z=0$ no pts

• $y=0$ $\frac{z^2}{1} - \frac{x^2}{(2/3)^2} =$ Hyperbola

• $x=0$ Hyperbola

Q_3 (a) $\underline{r}'(t) = 2\underline{i} - 4t^3\underline{j} + 6\sqrt{t}\,\underline{k}$

2

$\Rightarrow \underline{r}(t) = 2t\underline{i} - t^4\underline{j} + 4t^{3/2}\underline{k} + \underline{c}$

$\underline{r}(0) = \underline{i} + 5\underline{j} + 3\underline{k}$ and $\underline{r}(0) = 0\underline{i} + 0\underline{j} + 0\underline{k} + \underline{c}$

$\Rightarrow \underline{c} = \underline{r}(0)$. Thus, $\underline{r}(t) = (2t+1)\underline{i} + (5 - t^4)\underline{j} + (4t^{3/2} + 3)\underline{k}$

(b) $\underline{r}(t) = 2t\underline{i} + t^2\underline{j} - \frac{t^3}{3}\underline{k}$.

$$\left.\begin{aligned} \underline{r}'(t) &= 2\underline{i} + 2t\underline{j} - t^2\underline{k} \\ \underline{r}''(t) &= 0\underline{i} + 2\underline{j} - 2t\underline{k} \end{aligned}\right\} \Rightarrow$$

Q_3 • $a_T = \dfrac{\underline{r}'(t)\cdot\underline{r}''(t)}{\|\underline{r}'(t)\|}$ (tangential component) of acceleration

(2) $= \dfrac{\langle 2, 2t, -t^2\rangle \cdot \langle 0, 2, -2t\rangle}{\|\langle 2, 2t, -t^2\rangle\|}$

$= \dfrac{0+4t+2t^3}{\sqrt{4+4t^2+t^4}} = \dfrac{4t+2t^3}{\sqrt{4+4t^2+t^4}}$

$$\underline{r}'(t)\times\underline{r}''(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2t & -t^2 \\ 0 & 2 & -2t \end{vmatrix}$$

$= \underline{i}(-4t^2+2t^2) - \underline{j}(-4t) + \underline{k}(4)$

$= -2t^2\underline{i} + 4t\underline{j} + 4\underline{k}$

(2) • $a_N = \dfrac{\|\underline{r}'(t)\times\underline{r}''(t)\|}{\|\underline{r}'(t)\|} = \dfrac{\sqrt{4t^4+16t^2+16}}{\sqrt{4+4t^2+t^4}} = \dfrac{2(t^2+2)}{t^2+2}$

Normal component of acceleration

$= 2$

Curvature: • $K = a_N \cdot \dfrac{1}{\|\underline{r}'(t)\|^2} = 2\cdot\dfrac{1}{(t^2+2)^2} = \dfrac{2}{(t^2+2)^2}$.

(2) Q_4 (a) $F(x,y) = x^2+3xy+2x-y-10 = 0$

$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{2x+3y+2}{3x-1}$

(b) $z = f(x,y) = e^x(\cos y + 1)$

$f_x = e^x(\cos y+1)$, $f_y = -e^x \sin y$ $\Rightarrow$

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Q1 (b) An eq. of the tangent plane at $P(0,0,2)$:

$f_x(0,0)x + f_y(0,0)y - (z-2) = 0$

i.e., $2x - z + 2 = 0$

(3)

Put $F(x,y,z) = f(x,y) - z = 0$

$\nabla F = \langle f_x, f_y, -1 \rangle$

$\Rightarrow \nabla F(0,0,2) = \langle 2, 0, -1 \rangle$

Eq. of the normal line at P:

$x = 2t, \; y = 0, \; z = 2 - t.$

(c) $f(x,y) = \frac{4}{3}x^3 - xy^2 + 5y^2 + 24y$

(5)

Critical Points: $f_x = 4x^2 - y^2$ and $f_y = -2xy + 10y + 24$

$f_x = 0, f_y = 0 \Rightarrow \begin{cases} 4x^2 - y^2 = 0 \\ -2xy + 10y + 24 = 0 \end{cases} \Rightarrow$

$\begin{cases} y = \pm 2x \\ xy - 5y - 12 = 0 \end{cases} \Longrightarrow x = 2 \text{ or } x = 3$

Hence critical points are: $(2,-4), (3,-6)$

Also, we have critical points: $(-1,-2), (6,+12)$

$\Longrightarrow$

4 (c) $f_{xx} = 8x, \quad f_{xy} = -2y$

$f_{yy} = -2x + 10$

At $(2,-4)$: $\det\begin{pmatrix} 16 & +8 \\ +8 & 6 \end{pmatrix} = 32 > 0$ and

$f_{xx}(2,-4) = 16 > 0$

local minimum

For $(3,-6)$: $\det\begin{pmatrix} 24 & +12 \\ +12 & +4 \end{pmatrix} < 0$ saddle point

For other points we get saddle points

(d) $\nabla(4x^2 + y^2 + 5z^2) = \lambda \nabla(2x + 3y + 4z - 12)$

$\Rightarrow 8x\underline{i} + 2y\underline{j} + 10z\underline{k} = \lambda(2\underline{i} + 3\underline{j} + 4\underline{k})$

3

$$\Rightarrow \begin{cases} 8x = 2\lambda & (1) \\ 2y = 3\lambda & (2) \\ 10z = 4\lambda & (3) \\ 2x + 3y + 4z - 12 = 0 & (4) \end{cases}$$

$\Rightarrow \lambda = 4x = \frac{2}{3}y = \frac{5}{2}z$

These conditions $\Rightarrow$

$\Rightarrow y = 6x$ and $z = \frac{8}{5}$

From the last (4), we get

$$2x + 18x + \frac{32}{5}x - 12 = 0$$

$\Rightarrow x = \frac{5}{11}$. Hence $y = 6\left(\frac{5}{11}\right) = \frac{30}{11}$, $z = \frac{8}{11}$

Maximum occur at $\left(\frac{5}{11}, \frac{30}{11}, \frac{8}{11}\right)$