King Saud University
College of Sciences
Department of Mathematics
Math-244 (Linear Algebra); Mid-term Exam; Semester 1 (1443H)
Max. Marks: 30 **Time: 2 hours**

Note: *Attempt all the five questions*!

Question 1 [Marks: 4+2]:

a) **Let** A= $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. **Find** A^{-1} and $A^{-1}adj(A^{-1})$.

b) **Let** A = $\begin{bmatrix} 1 & -1 & 1 \\ \frac{1}{2} & \alpha & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \alpha \end{bmatrix}$. **Find** the *values* of α for which the *matrix* **A** is *not invertible*.

Question 2 [Marks: 2+3×1]:

Let $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & s & 1 \\ -1 & -2 & 0 & -1 \\ s & 0 & 8 & 2 \end{array}\right]$ be the *augmented matrix* of a *system of linear equations.* **Find** the values of s such that the system has:

(i) **unique solution.**
(ii) **infinitely many solutions.**
(iii) **no solution.**

Question 3: [Marks: 3+3]

a) **Let** P_2 denote the *real vector space* of all *polynomials in variable x* of *degree* $\leq$ **2.** **Show** that the set $S = \{v_1 = 1 + 2x,\ v_2 = 2 - x,\ v_3 = 5 + x^2\}$ is a *basis* of P_2.
b) **Let F** be the *vector subspace* of $\mathbb{R}^4$ *spanned by* the *vectors* $v_1 = (1, -2, 0, 1)$, $v_2 = (0, 1, 3, 0)$, $v_3 = (-1, -1, 0, -1)$, $v_4 = (0, 1, -1, 0)$. **Find** a *basis* of **F.**

Question 4: [Marks: 3+1+2]

Let $\boldsymbol{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ and $\boldsymbol{C} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ be *two bases* of a *vector space* $\boldsymbol{V}$

and $_C P_B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ -1 & 6 & 0 & 1 \\ 0 & -10 & 1 & -3 \\ 1 & 4 & 3 & 5 \end{bmatrix}$ is the *transition matrix* from $\boldsymbol{B}$ to $\boldsymbol{C}$. **Find:**

(*i*) $[\frac{1}{2}\boldsymbol{v}_2]_C$

(*ii*) $[\boldsymbol{v}_1 - \boldsymbol{v}_3 + \boldsymbol{v}_4]_C$.

Question 5: [Marks: 3+4]

a) **Find** a *basis* of the *solution space* of the following *system of linear equations*:

$$\begin{aligned} \boldsymbol{x_1} - \boldsymbol{3x_2} + \boldsymbol{x_3} &= \boldsymbol{0} \\ \boldsymbol{2x_1} - \boldsymbol{6x_2} + \boldsymbol{2x_3} &= \boldsymbol{0} \\ \boldsymbol{4x_1} - \boldsymbol{12x_2} + \boldsymbol{4x_3} &= \boldsymbol{0}. \end{aligned}$$

b) **Let A** = $\begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$. **Find:**

(i) a *basis* of ***row*(A)** (ii) ***rank*(A)** (iii) ***nullity*(A)**.

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SOLUTION KEY: 1443/Semester-1/Math-244/Midterm Exam

Solution of Question 1:

a) $A^{-1}=\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$ (2 marks)

and $A^{-1}adj(A^{-1}) = A^{-1}\ |A|^{-1}A = I$ because $|A| = 1$. (2 marks)

b) The given matrix **A** is non-invertible *iff* $\alpha^2 - \frac{1}{4} = |A| = 0$ *iff* $\alpha = \pm\frac{1}{2}$. (2 marks)

Solution of Question 2:

$$[A{:}B]=\left[\begin{array}{ccc|c}1 & 1 & s & 1\\ -1 & -2 & 0 & -1\\ s & 0 & 8 & 2\end{array}\right]\sim\begin{bmatrix}1 & 1 & s & : & 1\\ 0 & 1 & -s & : & 0\\ 0 & 0 & 8-2s^2 & : & 2-s\end{bmatrix}.$$ (2 marks)

Hence, the linear system has:

- (i) unique solution if $s \in \mathbb{R} \setminus \{-2,2\}$; (1 mark)
- (ii) infinitely many solutions if $s = 2$; (1 mark)
- (iii) no solution if $s = -2$. (1 mark)

Solution of Question 3:

a) $\alpha(1+2x)+\beta(2-x)+\gamma(5+x^2) = \alpha v_1 + \beta v_2 + \gamma v_3 = 0$ implies $\alpha=\beta=\gamma=0$. (1 mark)

This means the given set S is linearly independent.

However, $\dim P_2 = 3$. (1 mark)

Hence, $S = \{v_1 = 1+2x,\ v_2 = 2-x,\ v_3 = 5+x^2\}$ is a *basis* of P_2. (1 mark)

b) $[v_1\ v_2\ v_3\ v_4] = \begin{bmatrix}1 & 0 & -1 & 0\\ -2 & 1 & -1 & 1\\ 0 & 3 & 0 & -1\\ 1 & 0 & -1 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -3 & 1\\ 0 & 0 & 1 & -\frac{1}{9}\\ 0 & 0 & 0 & 0\end{bmatrix}$ (R.E.F.). (2 marks)

Hence, $\{v_1, v_2, v_3\}$ is a basis of the space F. (1 mark)

[Note: Some students may use the matrix consisting of rows v_1, v_2, v_3, v_4.]

Solution of Question 4:

From the given transition matrix ${}_BP_C$, we get:

$$[v_1]_C = \begin{bmatrix}1\\-1\\0\\1\end{bmatrix},\ [v_2]_C = \begin{bmatrix}2\\6\\-10\\4\end{bmatrix},\ [v_3]_C = \begin{bmatrix}3\\0\\1\\3\end{bmatrix} \text{ and } [v_4]_C = \begin{bmatrix}2\\1\\-3\\5\end{bmatrix}.$$ (2 marks)

Hence: (i) $[\frac{1}{2}v_2]_C = \begin{bmatrix}1\\3\\-5\\2\end{bmatrix}$ (ii) $[v_1 - v_3 + v_4]_C = \begin{bmatrix}0\\0\\-4\\3\end{bmatrix}$. (2+2 marks)

Solution of Question 5:

a) Solution space of the given linear system = $\{(3s - t, s, t) \mid s, t \in \mathbb{R}\}$. (2 marks)

Hence, $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space. (1 mark)

b) $A = \begin{bmatrix}0 & 2 & 1 & -1\\ 1 & -1 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}\sim\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 0 & 0\end{bmatrix}$ (R.E.F.). (1 mark)

Hence:

- (i) $\{(1,-1,0,1), (0,1,\frac{1}{2},-\frac{1}{2})\}$ is a basis of *row*(**A**) (1 mark)
- (ii) *rank*(**A**) = dim **A** = **2** (1 mark)
- (iii) *nullity*(**A**) = **4** - *rank*(**A**) = **2**. (1 mark)

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