King Saud University, Department of Mathematics
Math 204 (1H30mn), Mid Term Examination 2, S1, 2024-25

Question 1[5,4]

a) Find the largest interval for which the following initial value problem admits a unique solution

$$\begin{cases} \sqrt{x^2-1}\,y'' + \frac{2x}{x^2+1}y' + \frac{\ln(4-x)}{4+\ln(x-1)}y = 0 \\ y(3) = 2, \quad y'(3) = 1. \end{cases}$$

b) Show that the functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = e^x$$

are linearly independent on $(-\infty, +\infty)$.

Question 2[4,4]

a) Solve the differential equation, using the method of variation of parameters

$$3y'' - 9y' + 6y = e^{2x}.$$

b) If $y_1 = x - 1$ is a solution of the differential equation

$$(x^2 - 2x - 1)y'' - 2(x-1)y' + 2y = 0.$$

then obtain its general solution.

Question 3 [4,4]

a) Determine a linear differential equation with constant coefficients having the solutions

$$y_1 = 5,\ y_2 = 6e^{-x}\cos(2x),\ y_3 = e^{-x}\sin(2x),\ y_4 = 3x,\ y_5 = 7x^2.$$

b) Solve the initial value problem

$$\begin{cases} 9y'' + y = 5x + \sin(\frac{x}{3}) \\ y(0) = 0, \ y'(0) = 0 \end{cases}$$

Answer Sheet

Mid 2 S1 2024/2025

Q_1 a) $a_2(x) = \sqrt{x^2-1}$ is continuous for $|x| \geq 1$ and $a_2(x) \neq 0$ for $x \neq \pm 1$.

$a_1(x) = \dfrac{2x}{x^2+1}$ is continuous on $\mathbb{R}$.

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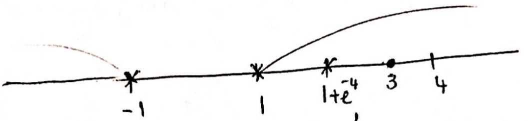
$a_0(x) = \dfrac{\ln(4-x)}{4+\ln(x-1)}$ is cont for $x < 4$. and $x > 1$, and $4+\ln(x-1) \neq 0$

We have $4+\ln(x-1) = 0 \Rightarrow x = 1+e^{-4}$.

So $a_0(x)$ is continuous on $\{x : x>1,\ x<4,\ x \neq 1+e^{-4}\}$ 1

$= (1, 1+e^{-4}) \cup (1+e^{-4}, 4)$.



1

The functions a_0, a_1, a_2 with $a_2(x) \neq 0$ are continuous on

$(1, 1+e^{-4}) \cup (1+e^{-4}, 4)$

Now since $x_0 = 3 \in (1+e^{-4}, 4)$, then the given IVP has 2

a unique solution on the largest interval $(1+e^{-4}, 4)$.

Q_1 b) $c_1 \sin x + c_2 \cos x + c_3 e^{x} = 0$

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$x = 0 \Rightarrow \quad c_2 + c_3 = 0 \longrightarrow (1)$

$x = \frac{\pi}{2} \Rightarrow \quad c_1 + c_3 e^{\frac{\pi}{2}} = 0 \longrightarrow (2)$

$x = +\pi \Rightarrow \quad -c_2 + c_3 e^{\pi} = 0 \longrightarrow (3)$

Eqs (1) and (3) gives $(1+e^{\pi}) c_3 = 0 \Rightarrow c_3 = 0$. Then Eq(1) implies that $c_2 = 0$ and (2) that $c_1 = 0$.

Hence the given functions are linearly indep on $\mathbb{R}$.

Remark: We can also use the Wronskian, and we can find that $W[f_1, f_2, f_3](x) = -2e^{x} \neq 0 \quad \forall x \in \mathbb{R}$, which implies that f_1, f_2, f_3 are linearly independent on $\mathbb{R}$.

Q_2 a): $3y'' - 9y' + 6y = e^{2x}$

HE: $3y'' - 9y' + 6y = 0$

ch Eq $3m^2 - 9m + 6 = 0 \Rightarrow m^2 - 3m + 2 = 0$, $m_1 = 1$, $m_2 = 2$

$y_{gh} = C_1 e^{x} + C_2 e^{2x}$. 1

$y_p(x) = C_1(x) e^{x} + C_2(x) e^{2x}$, where $C_1(x)$, $C_2(x)$ satisfy

$$\begin{cases} C_1(x) e^{x} + C_2'(x) e^{2x} = 0 \\ C_1'(x) e^{x} + 2C_2'(x) e^{2x} = \dfrac{e^{2x}}{3} \end{cases}$$

1

$$W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = e^{3x}$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & e^{2x} \\ \frac{e^{2x}}{3} & 2e^{2x} \end{vmatrix}}{e^{3x}} = -\frac{1}{3}e^{x} \Rightarrow C_1(x) = -\frac{1}{3}e^{x}$$

$$C_2'(x) = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{e^{2x}}{3} \end{vmatrix}}{e^{3x}} = \frac{1}{3} \Rightarrow C_2(x) = \frac{x}{3}.$$

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Thus $y_p = \left(-\frac{1}{3}e^{x}\right)e^{x} + \left(\frac{x}{3}\right)e^{2x} = -\frac{1}{3}e^{2x} + \frac{x}{3}e^{2x}$

Hence $y_g = C_1 e^{x} + C_2 e^{2x} + \frac{e^{2x}}{3}(x-1)$

Q_2 b) $\quad y_1 = x-1$

$$y_2 = (x-1)\int \frac{e^{\int \frac{2x-2}{x^2-2x-1}dx}}{(x-1)^2}\,dx$$

$$= (x-1)\int \frac{e^{\ln(x^2-2x-1)}}{(x-1)^2}\,dx$$

By formula:

$$= (x-1)\int \frac{x^2-2x-1}{(x-1)^2}\,dx$$

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$$= (x-1)\int \frac{x^2-2x+1-2}{(x-1)^2}\,dx$$

$$= (x-1)\int \left[1 - \frac{2}{(x-1)^2}\right] dx$$

$$= (x-1)\left[x + \frac{2}{x-1}\right] = x^2 - x + 2$$

$$y_{gh} = C_1(x-1) + C_2(x^2-x+2)$$

Remark: we can also use reduction of order.

$(Q_3\ a)$

$y_1=5,\ y_2=6e^{-x}\cos(2x),\ y_3=e^{-x}\sin(2x),\ y_4=3x,\ y_5=7x^2$

$y_1=5 \longrightarrow m_1=0$

$y_2=6e^{-x}\cos(2x) \longrightarrow m_2=-1+2i$

$y_3=e^{-x}\sin(2x) \longrightarrow m_3=-1-2i$

2

$y_4=3x \longrightarrow m_4=0$

$y_5=7x^2 \longrightarrow m_5=0.$

The characteristic equation is:

$$m^3(m+1-2i)(m+1+2i)=0$$

$$m^3\left((m+1)^2+4\right)=0$$

2

$$m^5+2m^4+5m^3=0$$

The DE: $y^{(5)}+2y^{(4)}+5y^{(3)}=0.$

$(Q_3\ b)$ $9y''+y=5x+\sin\left(\frac{x}{3}\right)$

ChEq: $9m^2+1=0 \Rightarrow m=\pm\frac{i}{3} \Rightarrow y_{gh}=C_1\cos\left(\frac{x}{3}\right)+C_2\sin\left(\frac{x}{3}\right)$

$y_p=Ax+B+x\left(C\cos\left(\frac{x}{3}\right)+D\sin\left(\frac{x}{3}\right)\right)$ 1

$y_p'=A+C\cos\left(\frac{x}{3}\right)+D\sin\left(\frac{x}{3}\right)+x\left(-\frac{C}{3}\sin\left(\frac{x}{3}\right)+\frac{D}{3}\cos\left(\frac{x}{3}\right)\right)$

$=A+\left(C+x\frac{D}{3}\right)\cos\left(\frac{x}{3}\right)+\left(D-\frac{C}{3}\right)\sin\left(\frac{x}{3}\right)$

$y_p''=\frac{D}{3}\cos\left(\frac{x}{3}\right)-\frac{1}{3}\left(C+x\frac{D}{3}\right)\sin\left(\frac{x}{3}\right)+\frac{1}{3}\left(D-\frac{C}{3}\right)\cos\left(\frac{x}{3}\right)$

Then $3D\cos\left(\frac{x}{3}\right)-3\left(C+x\frac{D}{3}\right)\sin\left(\frac{x}{3}\right)+3\left(D-\frac{C}{3}\right)\cos\left(\frac{x}{3}\right)$ 2

$+Ax+B+x\left(C\cos\left(\frac{x}{3}\right)+D\sin\left(\frac{x}{3}\right)\right)=5x+\sin\left(\frac{x}{3}\right)$

Hence $A=5,\ B=0,\ C=-\frac{1}{6},\ D=0 \Rightarrow y_p=5x-\frac{x}{6}\cos\left(\frac{x}{3}\right)$

$y_g=C_1\cos\left(\frac{x}{3}\right)+C_2\sin\left(\frac{x}{3}\right)+5x-\frac{x}{6}\cos\left(\frac{x}{3}\right)$ $\longrightarrow$ next page

$$y(0) = c_1 = 0$$

$$y' = -\frac{c_1}{3}\sin\left(\frac{x}{3}\right) + \frac{c_2}{3}\cos\left(\frac{x}{3}\right) + 5 - \frac{1}{6}\cos\left(\frac{x}{3}\right) + \frac{x}{18}\sin\left(\frac{x}{3}\right)$$

$$y'(0) = \frac{c_2}{3} + 5 - \frac{1}{6} = 0 \Rightarrow c_2 = -\frac{29}{2}$$

Thus $y = -\frac{29}{2}\sin\left(\frac{x}{3}\right) + 5x - \frac{x}{6}\cos\left(\frac{x}{3}\right)$