

1) Prove that $1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$; $\forall n \geq 1$.

B.S: $1^3 = \frac{f(1)^2}{2} = 1$ true.

$\Rightarrow P(1)$ is true.

I.S: Let $k \geq 1$; we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.

$P(k)$ is true $\Rightarrow 1^3 + 2^3 + \dots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$. I.H.

$P(k+1): 1^3 + 2^3 + \dots + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$? we should prove this.

$$1^3 + 2^3 + \dots + (k+1)^3 \stackrel{\text{I.H.}}{=} 1^3 + 2^3 + \dots + k^3 + (k+1)^3$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2 \left(\frac{k^2}{4} + k+1\right)$$

$$= (k+1)^2 \left(\frac{k^2}{4} + 4k+4\right)$$

$$= \frac{(k+1)^2 \left(\frac{k^2}{4} + 4k+4\right)}{2^2} = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 1$; $P(n)$ is true.

2) ... Let $P(n)$ be the statement that: $1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$. We prove that $P(n)$ is true for all non-negative integer n .

B.S: $P(0): 1^2 = 1 = \frac{1(1)(3)}{3} = 1 \Rightarrow P(0)$ is true.

I.S: Let $k \geq 0$; we assume that $P(k)$ is true, and we prove that $P(k+1)$ is true.

$P(k)$ true $\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = (k+1)(2k+1)(2k+3)$ I.H.

$P(k+1): 1^2 + 3^2 + 5^2 + \dots + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$? we prove this.

$$\underbrace{1^2 + 3^2 + 5^2 + \dots + (2k+1)^2}_{\text{I.H.}} + (2k+3)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + \frac{(2k+3)^2 \times 3}{3}$$

$$= \frac{(2k+3)}{3} ((k+1)(2k+1) + 3(2k+3))$$

$$= \frac{(2k+3)}{3} (2k^2 + k + 2k + 1 + 6k + 9)$$

$$= \frac{(2k+3)}{3} (2k^2 + 9k + 10)$$

$$\text{we have } P(k+2) \cdot (2k+5) = 2k^2 + 5k + 4k + 10 \\ = 2k^2 + 9k + 10.$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+3)^2 = \frac{P(k+2)(2k+3)(2k+5)}{3}.$$

$\Rightarrow P(k+1)$ is true $\Rightarrow \forall n \geq 0, P(n)$ is true.

3) Let $P(n)$ be the statement that: $3 + 3 \times 5 + 3 \times 5^2 + \dots + 3 \times 5^n = \frac{3(5^{n+1}-1)}{4}$.
we prove that $P(n)$ is true $\forall n \geq 0$.

B.S: $P(0)$: L.H.S: $3 \times 5^0 = 3$.

$$\text{R.H.S: } \frac{3(5-1)}{4} = 3 \quad \left\{ \right.$$

$$\text{L.H.S} = \text{R.H.S} \Rightarrow P(0) \text{ is true.}$$

I.S: Let $k \geq 0$; we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.
 $P(k)$ is true $\Rightarrow 3 + 3 \times 5 + 3 \times 5^2 + \dots + 3 \times 5^k = \frac{3(5^{k+1}-1)}{4}$, I.H.

$$P(k+1): 3 + 3 \times 5 + 3 \times 5^2 + \dots + 3 \times 5^k + 3 \times 5^{k+1} = \frac{3(5^{k+2}-1)}{4},$$

$$\underbrace{3 + 3 \times 5 + 3 \times 5^2 + \dots + 3 \times 5^k}_{\text{I.H.}} + 3 \times 5^{k+1} = \frac{3(5^{k+1}-1)}{4} + 3 \times 5^{k+1}.$$

$$= \frac{3 \times 5^{k+1} - 3 + 12 \times 5^{k+1}}{4}$$

$$= \frac{15 \times 5^{k+1} - 3}{4} = \frac{3(5 \times 5^{k+1} - 1)}{4}$$

$$= \frac{3(5^{k+2}-1)}{4}$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 0, P(n)$ is true.

4) $P(n): \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$.

We prove that $P(n)$ is true $\forall n \geq 1$.

B.S: $P(1): \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$ is true.

I.S: Let $k \geq 1$; we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.

$$P(k) \text{ true} \Rightarrow \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

$$P(k+1): \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}} ??$$

$$\begin{aligned}
 \underbrace{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}}_{\text{sum of first } k \text{ terms}} + \frac{1}{2^{k+1}} &= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\
 &= \frac{2(2^k - 1) + 1}{2^{k+1}} \\
 &= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\
 &= \frac{2^{k+1} - 1}{2^{k+1}}
 \end{aligned}$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 1$; $P(n)$ true.

5) $P(n): 3^n > 2^{n+1}, \forall n \geq 2$.

B.S.P(2): $3^2 = 9 > 2^3 = 8$. true.

I.S: Let $k \geq 2$; we assume that $P(k)$ is true and we prove that $P(k+1)$ is true.

$P(k)$ true $\Rightarrow 3^k > 2^{k+1}$. I.H.

$P(k+1): 3^{k+1} > 2^{k+2}$. ??

$$\begin{aligned}
 3^{k+1} &= 3 \times 3^k > 3 \times 2^{k+1} \quad \text{I.H.} \\
 &> 2 \times 2^{k+1} \quad |3 > 2 \\
 &= 2^{k+2}
 \end{aligned}$$

$\Rightarrow P(k+1)$ is true $\Rightarrow \forall n \geq 2$, $P(n)$ is true.

Strong induction

$$\begin{cases} a_1 = 8 \\ a_2 = 4 \\ a_n = a_{n-1} + a_{n-2}, n \geq 3 \end{cases}$$

Prove that a_n is even $\forall n \geq 1$: $P(n): a_n$ is even.

B.S: $P(1): a_1 = 8$ is even. true

$P(2): a_2 = 4$ is even. true.

I.S: Let $k \geq 2$; we assume that $P(1), P(2), \dots, P(k)$ are true and we prove that $P(k+1)$ is true.

$P(k+1): a_{k+1}$ is even.

$$a_{h+1} = a_h + a_{h-1}.$$

$P(h)$ true $\Rightarrow a_h$ is even $\Rightarrow a_h = 2t; t \in \mathbb{Z}$.

$P(h-1)$ true $\Rightarrow a_{h-1}$ is even $\Rightarrow a_{h-1} = 2s; s \in \mathbb{Z}$.

$$\Rightarrow a_{h+1} = 2t + 2s = 2(t+s) \text{ is even.}$$

$\Rightarrow P(h+1)$ is true $\Rightarrow \forall n \geq 1; P(n)$ is true.

$$2) \begin{cases} a_1 = -1 \\ a_2 = -2 \\ a_3 = -5 \\ a_n = a_{n-1} a_{n-2} a_{n-3} \end{cases} \quad \forall n \geq 4$$

$P(m)$: " $a_n \leq 0$." we prove that $P(m)$ is true $\forall m \geq 1$.

B.S.: $P(1)$: $a_1 = -1 \leq 0$.

$P(2)$: $a_2 = -2 \leq 0$.

$P(3)$: $a_3 = -5 \leq 0$.

I.S.: ~~Let~~ Let $h \geq 3$; we assume that $P(1), P(2), \dots, P(h)$ are true
and we prove that $P(h+1)$ is true.

$P(h+1)$: $a_{h+1} \leq 0 ??$

$$a_{h+1} = a_h a_{h-1} a_{h-2}.$$

$P(h)$ true $\Rightarrow a_h \leq 0$.

$P(h-1)$ true $\Rightarrow a_{h-1} \leq 0$ } $\Rightarrow a_{h+1} \leq 0 \Rightarrow P(h+1)$ is true
 $P(h-2)$ true $\Rightarrow a_{h-2} \leq 0$ } $\Rightarrow \forall n \geq 2; P(n)$ true.

$$3) \begin{cases} a_0 = 0 \\ a_1 = 2 \end{cases}$$

$$\{ a_n = 4a_{n-1} - 3a_{n-2}; n \geq 2$$

$P(n)$: $a_n = 3^n - 1$; we prove that $P(n)$ true $\forall n \geq 0$.

B.S.: $P(0)$: $a_0 = 1 - 1 = 0$

$P(1)$: $a_1 = 3 - 1 = 2$) true \Rightarrow

I.S.: We assume that $P(h)$ is true and we prove that $P(h+1)$ is true.

$$P(h+1): ? \quad a_{h+1} = 4a_h - 3a_{h-1} ; a_{h+1} = 4a_h - 3a_{h-1}$$

$P(h)$ true $\Rightarrow a_h = 3^h - 1$

$$P(h-1)$$
 true $\Rightarrow a_{h-1} = 3^{h-1} - 1 \quad \Rightarrow a_{h+1} = 4 \times 3^h - 4 - 3 \times 3^{h-1} + 3$

$$= 4 \times 3^h - 3^h - 1 = 3 \times 3^{h-1} = 3^{h+1} - 1$$

$\Rightarrow P(h+1)$ is true $\Rightarrow \forall n \geq 0; P(n)$ true.