Actuarial Mathematics of Life Insurance

How can calculate premium in life insurance?

The ratemaking of life insurance policies (i.e. calculation premiums) is depending upon three elements, they are:

i) *Mortality rates*: These rates mean probabilities of death and survival of life insurance policies. They can be calculated by ***Mortality table.***

ii) *Interest* (i.e. interest rate that by which the premiums are invested)

iii) *Sum insured* (i.e. the face value of the policy)

Using the foregoing elements, the fundamental relationship between the net premium and the future benefits for any type of life insurance policies may be expressed on the purchase date as follows:

Present value of *net premiums* = present value of *future benefits* (on purchase date)

What is the mortality table?

***The answer*:**

The mortality table is "*a statistical table by means of which the probabilities of death and survival may be measred at any given age for a group of individual"*.
As shown from the following table

Commissioners 1958 standard Ordinary CSO table of mortabity

Age		Number of living (L_x)	Number of dying (d_x)	Probability of death (q_x)	Probability of survival (P_x)
Male	Femal				
0		10000000	70800	0.00708	0.99292
1		9929200	17475	0.00176	0.99824
2		9911725	15066	0.00152	0.99848
3		9896659	14449	0.00146	0.99854
4		9882210	13835	0.0014	0.9986
5		9868375	13322	0.00135	0.99865
6		9855053	12812	0.0013	0.9987

1- Construction The Mortality Table

After study of mortabity table and its contents the following question may be raised:

How can we construct the mortabity table?

The answer:

The mortality table, may be constructed according to the following steps:

- Determining the *radix* of the table.
- Using the foregoing relationships, in particular, the relationships $q_x = 1 - p_x$ and $p_x = 1 - q_x$ for calcutating probabilities of death and survival
- Finding number of living for all ages using the relationships $L_{x+1} = L_x \cdot P_x$

  $, \ L_{x+2} = L_{x+1} \cdot P_{x+1}$ …. and so on
- Finding number of dying for all ages using the relationships $d_x = L_x - L_{x+1}$

  $, \ d_{x+1} = L_{x+1} - L_{x+2}$ …. and so on

2- Expectation of life

What is the expectation of life?

Expectation of life at age (x) is meant," *the average number of year to be lived in future by persons now aged(x)*"

Expectation of life is classified into two types, they are:

i)*Curtate expectation of life (e_x):*

For calculating the curtate expectation (e_x) we assume that:

1) We have a group of individuals (L_x) aged (x) as indicated in the following diagram.

2) All deaths that occur for the individuals in any year take place at the beginning of that year (i.e. fractional parts of years are neglected).

$$e_x = \frac{L_{x+1} + L_{x+2} + L_{x+3} + \dots\dots\dots + L_{w-1}}{L_x}$$

e_x is *The curtate expectation* with negligence of the fractional parts of years.

ii)The complete expectation of life ($\overset{\circ}{e}_x$)

Given that deaths that occur for individuals in any year take place at the end of year, Hence

$$e_x = \frac{L_x + L_{x+1} + \dots\dots\dots + L_{w-1}}{L_x}$$

the complete expectation of life will equal the arithmatic average of formulas

$$\overset{\circ}{e}_x = \frac{1}{2} + e_x = \frac{1}{2} + \frac{1}{L_x} \sum_{t=1}^{w-x-1} L_{w+t}$$

A Notice: *The complete expectation of life* takes into consideration the fractional parts of years. Moreover, it is useful in making comparison between the various of mortality tables.

Solved problems

Example 1:

Given that mortality rates of population in TANTA over the ages 30,31,32,33,34 and 35 as follows:

q_{30} = 0.0155 , q_{31} = 0.0156 , q_{32} = 0.0158

q_{33} = 0.0159 , q_{34} = 0.0160 , q_{35} = 0.0162

Required: Construction of a mortality table in TANTA, if you know L_{30} = 1000,000 individuals

solution

A life table may be constructed according to the following steps.

First step: Finding survival rates by relationship $P_x = 1 - q_x$, $P_x = 1 - q_x$

P_{30} = 1 – 0.0155 = 0.9845 and so on for the next ages as indicated in the following table:

x	L_x	d_x	q_x	P_x
30	1000,000		0.0155	0.9845
31	984500		0.0156	0.9844
32			0.0158	09842
33			0.0159	0.9841
34			0.0160	0.9840
35			0.0162	0.9838

Second step: Finding number of living (L_x) by relationship $L_{x+1} = L_x \cdot P_x$

$L_{31} = L_{30} \cdot P_{30}$

For next age as indicated int the following table:

x	L_x	d_x	q_x	P_x
30	1000,000		0.0155	0.9845
31	984500		0.0156	0.9844
32	696142		0.0158	09842
33	953829		0.0159	0.9841
34	938663		0.0160	0.9840
35	923645		0.0162	0.9838

Third step: Finding number of dying (dx) by relationship $d_x = L_x - L_{x-1}$

$D_{30} = L_{30} - L_{31}$ = 1000,000 – 984500 = 15500 and so on for next ages as indicated in the following table

x	L_x	d_x	q_x	P_x
30	1000,000	15500	0.0155	0.9845
31	984500	15358	0.0156	0.9844
32	696142	15313	0.0158	09842
33	953829	15166	0.0159	0.9841
34	938663	15018	0.0160	0.9840
35	923645	14963	0.0162	0.9838

x	L_x	d_x	q_x	P_x
40		370		
41	99630			
42	99231			
43		454		
44	98350			0.9950

Solution

Using the relashinships that have already studied, we may complete the previous table, age by age, as follows:

1) *age 40*:

$L_x = L_{x+1} + d_x$

$L_{40} = L_{41} + d_{40}$ = 99630 + 370 = 100,000

Also $q_{40} = \mathbf{\frac{d_{40}}{L_{40}} = \frac{370}{100000} = 0.00370}$

So, $P_{40} = 1 - q_{40} = 1 - 0.00370$

2-) *age 41:*

$d_{41} = L_{41} - L_{42}$ = 99630 – 99231 = 399

$q_{41} = \mathbf{\frac{d_{41}}{L_{41}} = \frac{399}{99630} = 0.00400}$

P_{41} = 1 – 0.00400 = 0.99600

$L_{43} = L_{42} - d_{42}$ = 99231 – 427 = 98804

3) *age 42:*

$d_{43} = L_{44} + d_{43}$ = 98350 + 454 = 98804

$q_{42} = L_{41} - L_{43}$ = 99231 – 98804 = 427

q42 = $\mathbf{\frac{d_{42}}{L_{42}} = \frac{427}{99231} = 0.00430}$

4)*age 43 and 44*:

$q_{43} = \frac{d_{43}}{L_{43}} = \frac{454}{98804} = 0.0046$

$P_{43} = 1 - 0.0046 = 0.99540$

$q_{44} = 1 - 0.995 = 0.005$

$d_{44} = L_{44} \times q_{44} = 98350 \times 0.005 = 492$

Hence, the mortality table will be completed as follows:

x	L_x	d_x	q_x	P_x
40	1000000	370	0.0037	0.99630
41	99630	399	0.0040	0.9960
42	99231	427	0.0043	0.9957
43	98804	454	0.0046	0.9954
44	98350	492	0.0050	0.9950

A notice: The preceding table may be constructed by other method by completing L_x first, then d_x, then q_x and p_x .. *try by yourself*.

Example 3: Calculate both the curtate expectation of life and the complete expectation of life for the ages 95 – 100 in the following table:

x	L_x	e_x	$\mathring{e}_x$
95	59		
96	35		
97	12		
98	7		
99	3		
100	1		
101	0		

Solution

The curtate expection of life (e_x) can be constrocted by the following relationship:

$$e_x = \frac{L_x + L_{x+1} + \dots\dots\dots + L_{w-1}}{L_w}$$

Then, *the complete expectation of life* can be constructed by the relationship $\overset{\circ}{e}_x = e_x + \frac{1}{2}$ as shown below:

$$e_{95} = \frac{L_{96} + L_{97} + L_{98} + L_{98} + L_{99} + L_{100} + L_{101}}{L_{95}}$$

$$= \frac{35 + 12 + 7 + 3 + 1 + 0}{59} = 0.9830$$

$$\overset{\circ}{e}_{95} = 0.5 + 0.9830 = 1.4830$$

Also :

$$e_{96} = \frac{L_{97} + L_{98} + L_{98} + L_{99} + L_{100} + L_{101}}{L_{96}}$$

$$= \frac{12 + 7 + 3 + 1 + 0}{35} = 0.6571$$

$$\overset{\circ}{e}_{95} = 0.5 + 0.6571 = 1.1571$$

and so on for the next ages (97 – 100), then the table will become as follows:

x	L_x	e_x	$\overset{\circ}{e}_x$
95	59	0.9830	1.4830
96	35	0.6571	1.1571
97	12	0.61661	1.4166
98	7	0.5714	1.0714
99	3	0.3333	0.8333
100	1	0	0.500
101	0	-	-

3- Probabilities of Death and Survival for a Person

3.1 Probability of death (q_x):

$$q_x = \frac{d_x}{L_x} = \frac{L_x - L_{x+1}}{L_x}$$

$$P_x = \frac{L_{x+1}}{L_x}$$

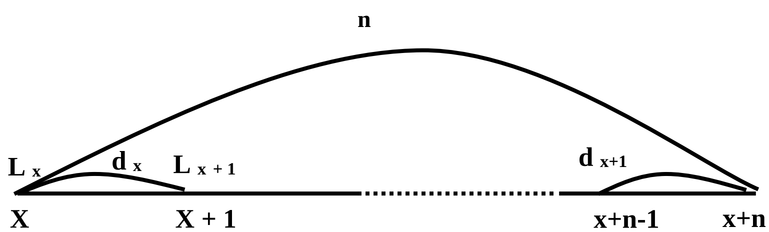
3.2 Probability of survival (P_x):

$$P_x = \frac{L_{x+1}}{L_x}$$

As we have already seen ,the heart of the morality table is, $\mathbf{q_x}$, that is called *probability of death*. This probability has a vital role in calulating *the net premium* for any life insurance policy where, The premium equals the probability (q_x or p_x) multiplied by sum insured.

3.3-Probability of death for a person aged (x) over n year($^n q_x$)

The symbol $^n q_x$ means the probability that a person aged (x) will die before reaching age (x+n). *In other words*, the probability that a person will die over n years. That is between age (x) and age (x+n) as indicated in following diagram



Consquently, $$n_{q_x} = \frac{d_x + d_{x+1} + d_{x+2} + \dots\dots\dots\dots + + d_{x+n-1}}{L_x} = \frac{L_x - L_{x+n}}{L_x} \qquad (6.10)$$

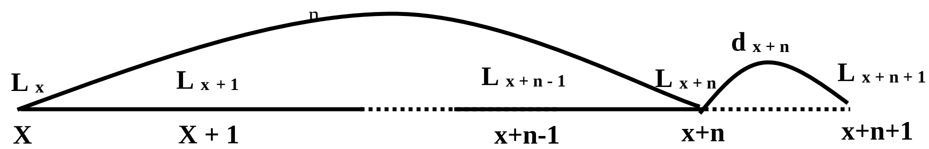
6.5.2-Probability of survival for a person aged (x) n years ($^n p_x$)

The $^n p_x$ means the probability that a person aged (x) will live to reach age (x+n). That is out of the L_x persons alive at age (x) there are L_{x+n} survivors at age (x+n) as indicated in the preceding diagram. Hence,

$$nP_x = \frac{L_{x+n}}{L_x} \qquad (6.11)$$

6.5.3- Probability of survival of a person n years and his death over 1 year ($^n/q_x$)

The sympol $^{n}/q_{x}$ means the probability that a person aged (x) will live to reach age (x+n), then die between age (x+n) and age (x+n+1) as indicated in the following diagram.



Hence,

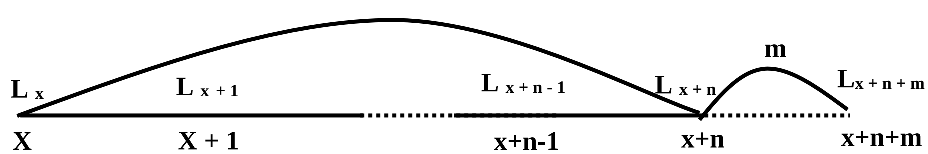
$$^{n}/q_{x} = \frac{L_{x+n} - L_{x+n-1}}{L_{x}} \qquad (6.12)$$

or

$$^{n}/q_{x} = \frac{d_{x+n}}{L_{x}} \qquad (6.12)\text{ repeated}$$

2.5.4-probability of survival of a person n year and his death over m years ($^{n}/^{m}q_{x}$)

This symbol $^{n}/^{m}q_{x}$ means the probability that a person aged (x) will live to reach age (x+n) then die between age (x + n) and age (x + n + m) as indicated in the following diagram:



Hence

$$^{n}/^{m}q_{x} = \frac{d_{x+n} + d_{x+n+1} + d_{x+n+2} + \ldots\ldots + d_{x+n+m-1}}{L_{x}} \qquad (6.13)$$

or

$$^{n}/^{m}q_{x} = \frac{L_{x+n} - L_{x+n+m}}{L_{x}} = {}^{n}p_{x} - {}^{n+m}p_{x} \qquad (6.13)\text{ repeated}$$

In conclusion by contemplating the preceding notation, it should be noted that:

a) The letter ***P*** with the proper subscripts is used to denote the probability of a person living a given period
b) The letter ***q*** is used to denote the probability of a person dying during a given period

Solved problem

Example 4

Interpret in words the following symbols, then calculate their values using the American life table (1958 CSO)

a) q_{25} , P_{60}
b) $^{5}P_{30}$, $^{7}q_{27}$
c) $^{6}/q_{35}$, $^{7/5}q_{25}$

solution

a) **q_{25}**: means a probability that a person aged (25) will die over one year. That is between age (25) and age (26)

$$q_{25} = \frac{L_{25} - L_{26}}{L_{25}} = \frac{9575636 - 9557155}{9575636} = 0.00193$$

or

$$q_{25} = \frac{d_{25}}{L_{25}} = \frac{18981}{9575636} = 0.00193$$

- **P_{60}:** means a probability that a person aged (60) will live to reach age (61)

$$P_{60} = \frac{L_{61}}{L_{60}} = \frac{7542106}{7698698} = 0.9797$$

b) $^{5}p_{30}$: means a probability that a person aged (30) will live to reach age (35)

$$^{5}P_{60} = \frac{L_{35}}{L_{30}} = \frac{9373807}{9480358} = 0.9888$$

- $^{7}q_{25}$: means probability that a person aged (25) will die over seven years. That is between age 25 and 1ge 32

$$ {}^{7}q_{25} = \frac{L_{25} - L_{32}}{L_{25}} = \frac{9575636 - 9439447}{9575636} = 0.0142 $$

c)$^{6}/q_{35}$: means a probability that a person aged (35) will live to reach age (41), then die between age (41) and age (42):

$$ 6/q_{35} = \frac{L_{41} - L_{42}}{L_{35}} = \frac{9208737 - 9173375}{9373807} = 0.0037 $$

or

$$ 6/q_{35} = \frac{d_{41}}{L_{35}} = \frac{35362}{9373807} = 0.0037 $$

- $^{7}/^{5}q_{25}$: means a probability that a person aged (25) will live to reach age (32) then die between age (32) and age (37)

$$ {}^{7}/{}^{5}q_{25} = \frac{L_{32} - L_{37}}{L_{25}} = \frac{9439447 - 9325594}{9575636} = 0.01189 $$

or

$$ {}^{7}/{}^{5}q_{25} = \frac{d_{32} + d_{33} + d_{34} + d_{35} + d_{36}}{L_{25}} $$

$$ = \frac{21239 + 21850 + 22551 + 23528 + 24685}{9575636} $$

$$ = 0.01189 $$