Summary :

(I) Let $r(t) = \langle f(t), g(t), h(t) \rangle$ be a vector valued function. Then

(1) derivative $= r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

(2) velocity vector $= v(t) = r'(t)$

(3) Speed $= \| v(t) \| = \| r'(t) \|$

(4) accelaration $= a(t) = r''(t)$

(5) tengent vector $= r'(t)$

(6) Unite tengent vector $= T(t) = \dfrac{r'(t)}{\| r'(t) \|}$

(7) Principal unite normal vector $= N(t) = \dfrac{T'(t)}{\| T'(t) \|}$

(II) Curvature of $r(t)$ is a measur how bent the curve $r(t)$.

(1) Curvature in the Plane :

(1) If $y = f(x)$ then the curvature at Point P

$$= K_p = \frac{\| y''_p \|}{\left[1 + (y'_p)^2\right]^{3/2}}$$

(2) If $x = f(t)$ and $y = g(t)$ then The curvature at Point $P = K_p = \dfrac{| f'_p g''_p - f''_p g'_p |}{\left[f'^2_p + g'^2_p \right]^{3/2}}$

(3) radius of curvature $= \rho = \dfrac{1}{k}$

(4) Center of Curvature $= (h, k)$ Such that

$h = x - \dfrac{y'(1 + y'^2)}{y''}$ and $k = y + \dfrac{1 + y'^2}{y''}$

(2) Curvature in space where $r(t)$ is vector valued function :

(1) $$K = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^2}$$

(3) Components of accelaration :

(1) tangential component $= a_T = \dfrac{r'(t) \cdot r''(t)}{\| r'(t) \|}$

(2) Normal Component $= a_N = \dfrac{\| r'(t) \times r''(t) \|}{\| r'(t) \|}$

Examples :-

① Find the path of curve where the accelaration of a particle moves along the curve is

$a(t) = -2\cos t\,\vec{i} - 2\sin t\,\vec{j} + 2k$. The ~~init~~ initian velocity of particle is $2\vec{j}$ and it starts from the point $(2, 0, 0)$.

Solution : $a(t) = \langle -2\cos t, -2\sin t, 2 \rangle$ where $v(0) = \langle 0, 2, 0 \rangle$.

Now $\boxed{v(t) = \int a(t) + C}$ where C is constant.

$= \langle -2\sin t, 2\cos t, 2t \rangle$

as $v(0) = \langle 0, 2, 0 \rangle$ then

$\langle -2\sin(0), 2\cos(0), 2(0) \rangle + \langle a_1, a_2, a_3 \rangle = \langle 0, 2, 0 \rangle$

$\Rightarrow C = \langle a_1, a_2, a_3 \rangle = 0$

Now, $r(t) = \int v(t) + C$

$= \int \langle -2\sin t, 2\cos t, 2t\rangle + \langle b_1, b_2, b_3\rangle$

$= \langle 2\cos t, 2\sin t, t^2\rangle + \langle b_1, b_2, b_3\rangle$

Now

$r(0) = (2, 0, 0)$ (by assumption)

So, $\langle 2\cos(0), 2\sin(0), (0)^2\rangle + \langle b_1, b_2, b_3\rangle = \langle 2, 0, 0\rangle$

$\Rightarrow \langle b_1, b_2, b_3\rangle = \langle 0, 0, 0\rangle$

$\Rightarrow r(t) = \langle 2\cos t, 2\sin t, t^2\rangle$. $\boxtimes$

Example 2 find the unit tangent vector and principal normal vector of the curve:

$$r(t) = \langle \cos t, \sin t, t\rangle$$

Solution $r'(t) = \langle -\sin t, \cos t, 1\rangle$

$\|r'(t)\| = \sqrt{\underbrace{\sin^2 t + \cos^2 t} + 1}$

$= \sqrt{1+1} = \sqrt{2}$

So, the tangent unit vector $= T(t) = \dfrac{r'(t)}{\|r'(t)\|}$

$= \dfrac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$

$= \langle -\dfrac{1}{\sqrt{2}}\sin t, \dfrac{1}{\sqrt{2}}\cos t, \dfrac{1}{\sqrt{2}}\rangle$

Now $T'(t) = \langle \dfrac{-1}{\sqrt{2}}\cos t, -\dfrac{1}{\sqrt{2}}\sin t, 0\rangle$

$\therefore \|T'(t)\| = \sqrt{\dfrac{\cos^2 t}{2} + \dfrac{\sin^2}{2} + 0} = \dfrac{1}{\sqrt{2}}$

$\therefore N(t) = \dfrac{T'(t)}{\|T'(t)\|} = \dfrac{1}{\sqrt{2}}\langle \dfrac{-1}{\sqrt{2}}\cos t, \dfrac{-1}{\sqrt{2}}\sin t,$

Example : Find the curvature of $r(t) = \langle 2t, t^2, \frac{-1}{3}t^3 \rangle$.

Solution : Notice that the curve $r(t)$ is in $\mathbb{R}^3$ (space)

so, $r'(t) = \langle 2, 2t, -t^2 \rangle$

$$\| r'(t) \| = t^2 + 2$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 2, 2t, -t^2 \rangle}{t^2 + 2}$$

$$T'(t) = \frac{t^2+2 \langle 0, 2, -2t \rangle + 2t \langle 2, 2t, -t^2 \rangle}{(t^2+2)^2}$$

$$= \frac{\langle -4t, \quad 4-2t^2, \ -4t \rangle}{(t^2+2)^2}$$

$$\| T'(t) \| = \frac{2(t^2+2)}{(t^2+2)^2} = \frac{2}{t^2+2}$$

So, The curvatur $= K = \dfrac{\| T'(t) \|}{\| r'(t) \|} = \dfrac{2}{(t^2+2)^2}$ $\boxtimes$

Example : Prove that the curvature of the circle

~~given~~ $x = a \cos t$

$y = a \sin t$

is $K = \frac{1}{a}$?

Solution: $r(t) = \langle a \cos t, a \sin t, 0 \rangle$

$r'(t) = \langle -a \sin t, a \cos t, 0 \rangle$

$$| r'(t) | = a$$

$$T(t) = \frac{r'(t)}{| r'(t) |} = \langle -\sin t, \cos t, 0 \rangle$$

$$T'(t) = \langle \cos t, -\sin t, 0 \rangle$$

$$K = \frac{\| T'(t) \|}{\| r'(t) \|} = \frac{1}{a} \quad \boxtimes$$

Example Find the curvature of curve $y = x^4$ at Point $(1,1)$.

Solution: Notice that $y = f(x)$ in plan:

So, $y = x^4$

$y' = 4x^3 \longrightarrow y'_p$ (i.e. $x = 1$) $= 4$

$y'' = 12x^2 \longrightarrow y''_p = 12$

Therefore, the curvature at $(1,1)$ is,

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{12}{(17)^{3/2}}$$

Example Find the curvature of $x = \cos^3 t$, $y = \sin^3 t$ at Point $\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$.

Solution: Notice that the curve is in xy-plane.

$f'(t) = -3\cos^2 t \sin t$

$f''(t) = 6 \cos t \sin^2 t - 3\cos^3 t$

$g'(t) = 3 \sin^2 t \cos t$

$g''(t) = 6 \sin t \cos^2 t - 3 \sin^3 t$

Notice that $r(t) = \langle \cos^3 t, \sin^3 t \rangle$

So, our goal to find t that satisfies that

$$\left\langle \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} \right\rangle = \langle \cos^3 t, \sin^3 t \rangle$$

$$\Rightarrow \boxed{t = \frac{\pi}{4}}$$

So, $f'\left(\frac{\pi}{4}\right) = 3\cos^2 \frac{\pi}{4} \sin \frac{\pi}{4} = \frac{3}{2\sqrt{2}}$

$g'\left(\frac{\pi}{4}\right) = \frac{-3}{2\sqrt{2}}$

$f''(\pi/4) = \frac{3}{2\sqrt{2}}$ & $g''(\pi/4) = \frac{3}{2\sqrt{2}}$

So, The curvature at $t = \pi/4$ is

$$\kappa = \frac{|f'(t)\, g''(t) - g'(t)\, f''(t)|}{|f'^2(t) + g'^2(t)|^{3/2}}$$

$$= \frac{2}{3}$$

** See Example 17 / Page 28 in the file Part 6.

See Example Page 29 in the file Part 6.