King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2016–2017
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**Final Exam Quantitative Methods in Finance (three pages) Actu. 468 (40%)**

**Monday, January 16, 2016 / Rabi II 18, 1438 (three hours)**

**Problem 1. (8 marks)**
We specify below the basic elements of a financial market with $T$ periods:

- A finite probability space $\Omega = \{\omega_1, \dots, \omega_k\}$ with $k$ elements.
- A probability measure $P$ on $\Omega$, such that $P(\omega) > 0$ for all $\omega \in \Omega$.
- A riskless asset (a saving account) $S_t^0, t \in \{0, 1, 2, \dots T\}$ such that $S_0^0 = 1$ with a constant interest rate $r$.
- A $d$–dimensional price process $S_t$, $t \in \{0, 1, 2, \dots T\}$ where $S_t = (S_t^0, S_t^1, \dots, S_t^d)$ and $S_t^i$ stands for the price of the asset $i$ at time $t$.

Assume that $k = 3$, $d = 1$, and $T = 2$ and consider the following model

| $n$ | $S_n^0$ | $S_n^1$ | | | |
|---|---|---|---|---|---|
| | | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
| 0 | 1 | 5 | 5 | 5 | 5 |
| 1 | $1+r$ | 8 | 8 | 4 | 4 |
| 2 | $(1+r)^2$ | 9 | 6 | 6 | 3 |

(for simplicity of computations take $r = 0$)

1. (**1 mark**) Find the sigma algebras $\mathcal{F}_0$ and $\mathcal{F}_2$.
2. (**1 mark**) Find the sigma algebra $\mathcal{F}_1$.
3. (**1 mark**) Let $Q$ be a probability on $\Omega = \{\omega_1, \dots, \omega_4$, Calculate $E_Q\left[S_2^1 \mid \mathcal{F}_1\right]$
4. (**1 mark**) Is this model a arbitrage free? explain your answer
5. (**1 mark**) Is this model complete? explain your answer
6. (**1 mark**) Consider a put option with strike price 7. Give the payoff of this put. Is is attainable? explain your answer
7. (**1 mark**) Find the price of this option at times 0 and 1. (+ **1 mark bonus**)
8. (**1 mark**) Find the hedging portfolio (+ **1 mark bonus**)

**Problem 2. (8 marks)**
Consider stock with a current price \$100 and a constant annualized volatility $\sigma$ of 20%. The stock does not pay dividends. A risk–less asset is worth \$0.95 today and is worth \$1 in one year maturity.
Consider also European and American put options on the stock with a maturity of **two years** and a strike price of \$110.

1. **(1 mark)** Using the approach discussed in the lectures, construct a **two–step** binomial tree to approximate the stock price dynamics, with each step being **one year**. Build the stock price at each **node** at **one** and **two** years.
2. **(1 mark)** Compute the risk–neutral probability measure.
3. **(1 mark)** Give the binomial tree of the European put option.
4. **(1 mark)** Based on the binomial tree, find the current value of the European put option.
5. **(1 mark)** Compute the delta of the European put option for the first and the second periods.
6. **(1 mark)** Give the binomial tree of the American put option.
7. **(1 mark)** Based on the binomial tree, find the current value of the American put option.
8. **(1 mark)** Compute the delta of the American put option for the first and the second periods.

**Problem 3. (8 marks)**
An XYZ asset is currently priced at \$700 per share and an analyst predicts that the stock will hit 1000 at some point in the next $T = 1.5$ years. Assume that the stock price $S_t$ of XYZ is given by the Black–Scholes formula. The stock pays no dividends, and the risk–free interest rate is $r = 6\%$. Take the rate of return of the stock price to be $\mu = 10\%$ and the volatility to be $\sigma = 30\%$.

1. **(1 mark)** Give the expression of $S_T$ under the historical probability
2. **(1 mark)** Give the distribution of $\ln(\frac{S_T}{S_0})$.
3. **(1 mark)** Let $\mathbf{N}(\cdot)$ be the c.d.f. of the S.N.D. $\mathcal{N}(0,1)$, we know that $\mathbf{N}(x)+\mathbf{N}(-x) = 1$ for all $x$. Calculate the probability that $S_{1.5} > 1000$ in this model assuming that $\mathbf{N}(0.7463) = 0.7722$.
4. **(1 mark)** To take advantage of the expected rise in XYZ stock, you decide to buy a call option with strike 1000 on one share of XYZ. The option expires in 1.5 years and implied volatility is 30%.
   What are the input parameters for the Black–Scholes formula to value this option?
5. **(1 mark)** What are their values?
6. **(1 mark)** Given $\mathbf{N}(-0.5421) = 0.2939$ and $\mathbf{N}(-0.9095) = 0.1815$. What is the Black–Scholes value of your call option at the time of purchase?
7. **(1 mark)** Find the replicating portfolio from the side of the seller of this option today?
8. **(1 mark)** Suppose that the **6 months** price of XYZ stock is 1000. Given $\mathbf{N}(0.35) = 0.6368$ and $\mathbf{N}(0.05) = 0.52$ what would the value of your option be then at this time?

**Problem 4. (8 marks)**
Assume that the market–maker is trading options on an underlying asset with the price process $(S_t)_{t\geq 0}$ with initial price $S_0 = 40$, a volatility of 25% and the risk–free rate $r = 5\%$. Assume that the asset pays a dividend yield $q = 2\%$. The market–maker sells a 42–strike put on $(S_t)_{t\geq 0}$ in three months from now.

1. **(1 mark)** Give the formula of the Black–Scholes price of the above put ?
2. **(1 mark)** Given $\mathbf{N}(0.3928) = 0.6527$ and $\mathbf{N}(0.2678) = 0.6056$, find numerically the put price
3. **(1 mark)** Give the formula of the delta of the above put?
4. **(1 mark)** Find a numerical value of the delta
5. **(1 mark)** The market–maker sells 100 shares of the above put and decides to delta–hedge this position. What investment is required to do so?
6. **(1 mark)** If the next month price of the stock is 42. Find the one month price of the option assuming that $\mathbf{N}(0.002) = 0.5$ and $\mathbf{N}(-0.1) = 0.46$.
7. **(1 mark)** Find the one month delta of 100 shares of the put.
8. **(1 mark)** Find the hedging strategy of the above position.

**Problem 5. (8 marks)**

A financial institution has the following portfolio of over–the–counter options on **sterling (the underlying)**

| Type of option | Position | Delta of option | Gamma of option | Vega of option |
|---|---|---|---|---|
| Call of type 1 | −1000 | 0.5 | 2.2 | 1.8 |
| Call of type 2 | −500 | 0.8 | 0.6 | 0.2 |
| Put of type 3 | −2000 | −0.4 | 1.3 | 0.7 |
| Call of type 4 | −500 | 0.7 | 1.8 | 1.4 |

The sign (−) means the position is short.
A **traded option** (say of type 5) is available in the market with a **delta** of 0.6, a **gamma** of 1.5 and a **vega** of 0.8.

Denote by $\Pi_1$ the value of the above portfolio.

1. **(1 mark)** Calculate the Delta of $\Pi_1$.
2. **(1 mark)** Calculate the Gamma of $\Pi_1$ .
3. **(1 mark)** Consider a new portfolio $\Pi_2$ composed on $\Pi_1$ and $\beta$ position in the **traded option**. Find $\beta$ in such away that **new portfolio** $\Pi_2$ is gamma–neutral.
4. **(1 mark)** What position in the **traded option** would make the **new portfolio** $\Pi_2$ gamma–neutral?
5. **(1 mark)** Consider a new portfolio $\Pi_3$ composed on $\Pi_2$ and $\delta$ position in the **sterling (the stock)**. Find $\delta$ in such away that **new portfolio** $\Pi_3$ is delta–neutral.
6. **(1 mark)** What position in the **traded option** and in **sterling** (the stock) would make the **new portfolio** $\Pi_3$ delta–neutral and gamma–neutral?
7. **(1 mark)** Calculate the Vega of this portfolio.
8. **(1 mark)** What position in the **traded option** and in **sterling** (the stock) would make the **new portfolio both** vega–neutral and delta–neutral? **(+1 mark bonus).**

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Problem 1:

0,5

① $F_0 = \{\phi, \Omega\}$

$F_2 = \{\Omega\} = \mathcal{P}(\Omega)$

② let $B_1 = \{w_1, w_2\}$ and $B_2 = \{w_3, w_4\}$

$F_2 = \{\phi, B_1, B_2, \Omega\}$

③

$E_Q[S_2' \mid F_1]$:

1

$8 = \frac{9q_1 + 6q_2}{q_1 + q_2}$ , $4 = \frac{6q_3 + 3q_4}{q_3 + q_4}$

$5 = 8(q_1 + q_2) + 4(1 - q_2 - q_3)$

$\Rightarrow 1 = 4q_1 + 4q_2 \Rightarrow q_1 + q_2 = \frac{1}{4}$

$\Rightarrow 8 = \frac{9q_1 + 6q_2}{\frac{1}{4}} \Rightarrow 2 = 9q_1 + 6(\frac{1}{4} - q_1)$

$\Rightarrow q_1 = \frac{1}{6} \Rightarrow q_2 = \frac{1}{12}$

$5 = 8(\frac{1}{4}) + 4(q_3 + q_4)$

$q_3 + q_4 = \frac{3}{4}$

$4 = \frac{6q_3 + 3(\frac{3}{4} - q_3)}{\frac{3}{4}} \Rightarrow q_3 = \frac{1}{4} \Rightarrow q_4 = \frac{1}{2}$

$Q = \{\frac{1}{6}, \frac{1}{12}, \frac{1}{4}, \frac{1}{2})$

$\cancel{E_Q[S_2' \mid F_1]} = \cancel{9q_1 + 6q_2}$

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(4) ~~Since there is unique solution for Q~~ the model is arbitrage free, since we found set of Q s.t. $0 < Q < 1$ — 1

(5) Since there is unique solution for Q, the model is complete. — 1

(6) payoff : $(7 - S_T)^+$ — 1

the payoff is attainable since the model is complete

(7) ~~$C_0 = V_0 = 8$~~

| | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 3 | 3 |
| 2 | 0 | 1 | 1 | 4 |

$P_0 = V_0 = 3(q_3 + q_4)$
$= \frac{9}{4}$ — 0

$P_1 V_1 = q_2 + q_3 + 4q_4 = \frac{1}{12} + \frac{1}{4} + 2 = \frac{7}{3} \simeq 2.33$

(8) ~~$x_0 + \Delta_0 q = 0$~~ — 0

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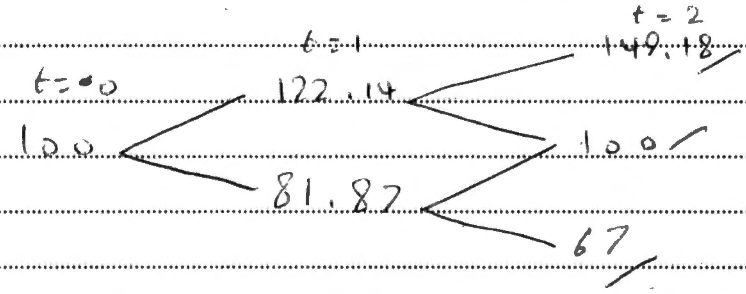
Problem 2;

$\sigma = 0.20$ $S_0 = 100$ $\cancel{0.951} = 0.95(1+r)$

$K = 110$ $\Rightarrow r = 0.053$

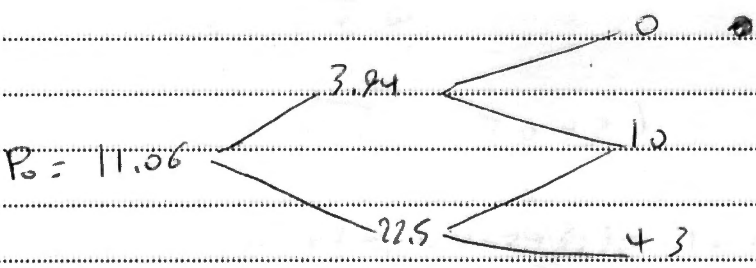
1 ① $u = e^{0.2\sqrt{1}} = e^{0.2}$ , $d = e^{-0.2}$ $q = \frac{e^{rh} - d}{u - d} = 0.585$




1 ② $q = \frac{e^{rh} - d}{u - d} = 0.585$

③ ~~37.18~~

1




1 ④ $P_0 = 11.06$

⑤ $3.94 = 1 + 122.14\,\Delta_0 \Rightarrow \Delta_0 = 0.024$

0 $0 = 1.053 + 149.18\,\Delta_1 \Rightarrow \Delta_1 = -0.0071$

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Problem 3: $S_0 = 700$ $r = 0.06$ $\mu = 10\%$ $\sigma = 0.3$

1 ① $S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}$

1 ② $\ln(\frac{S_T}{S_0}) \sim N((\mu - \frac{\sigma^2}{2})T , \sigma^2 T)$

③ $P(S_{1.5} > 1000)$

$= P(\ln(S_{1.5}) > \ln(1000)) = P(\ln(S_0) + (\mu - \frac{\sigma^2}{2})T + \sigma W_{1.5} > 6.91)$

1 $= P(6.63 + 0.3 W_{1.5} > 6.91) = P(W_{1.5} > 0.93)$

$W_{1.5} = \sqrt{1.5}\, W_1$

$\Rightarrow P(\sqrt{1.5}\, W_1 > 0.93) = P(W_1 > 0.7463)$

$= 1 - P(W_1 < 0.7463) = 1 - 0.7722 = 0.2278$

1 ④ the input values are: $d_+$, $d_-$, $\sigma$, $T$, $S_0$, $K$, $r$

where $d_+ = \dfrac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ and $d_- = d_+ - \sigma\sqrt{T}$

⑤ $\sigma = 0.3$, $r = 0.06$, $T = 1.5$, $S_0 = 700$, $K = 1000$

1 $d_+ = \dfrac{\ln(\frac{700}{1000}) + (0.06 + \frac{0.3^2}{2})1.5}{0.3\sqrt{1.5}} = -0.5421$

$d_- = -0.5421 - 0.3\sqrt{1.5} = -0.9095$

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(6) $C_0 = S_0 N(d_+) - Ke^{-rT} N(d_-)$

1

$= 700(0.2939) - 1000e^{-0.06(1.5)}(0.1815)$

$= 39.85$

(7)

0

(8) $S_{0.5} = 1000$

$d_+ = \dfrac{\ln\left(\frac{1000}{1000}\right) + \left(0.06 + \frac{0.3^2}{2}\right)(1)}{0.3\sqrt{1}} = 0.35$

$d_- = 0.35 - 0.3 = 0.05$

1

$C_0 = 1000N(0.35) - 1000e^{-0.06} N(0.05)$

$= 147.0824$

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Problem 4: $S_0 = 40$ $\sigma = 0.25$ $r = 0.05$
$q = 0.02$ $K = 42$ $T = \frac{3}{12} = \frac{1}{4}$

(1)

$d_+ = \dfrac{\ln\left(\frac{40}{42}\right) + \left(0.05 - 0.02 + \frac{0.25^2}{2}\right)\left(\frac{1}{4}\right)}{0.25\sqrt{\frac{1}{4}}}$

1 Formulas are required.

$d_+ = -0.2678$ , $d_- = -0.2678 - 0.25\sqrt{\frac{1}{4}} = -0.3928$

$P_0 = 42 e^{-0.05(0.25)} N(0.3928) - 40 e^{-0.02(0.25)} N(0.2678)$

You are given numerical values.

1 (2) $P_0 = 2.9697$

1 (3) ~~$\Delta_{put} = N(d_+) - 1 = (1 - N(-d_+)) - 1 = -N(-d_+)$~~

$\Delta_{put} = -e^{-q(T-t)} N(-d_+)$ ~~$= -0.6056$~~

1 (4) $\Delta_{put} =$ ~~$-0.656$~~ $-0.6026$

(5) the value of the short position is $-100 P_0$ and the delta is $-100(-0.6056) = 60.56 \simeq 60$, the investor must hold invest in the stock to hedge his position, let $\beta$ be the number of shares he will hold or sell to compensate the risk ~~in the portfolio~~ delta of the portfolio must be 0

$\Pi_0 = -100 P_0 + \beta S_0 \Rightarrow \Delta \Pi_0 = -100(0.6) + \beta = 0$
$\Rightarrow \beta = -60$

1 He must go short on 60 shares of the stock to hedge his risk.

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(6) $S_{\frac{t}{12}} = 42 \quad d_+ = \dfrac{\ln\left(\frac{42}{42}\right) + \left(0.05 - 0.02 + \frac{0.25^2}{2}\right)\sqrt{\frac{2}{12}}}{0.25\sqrt{\frac{2}{12}}}$

$d_+ = 0.1 \quad , d_- = 0.1 - 0.25\sqrt{\frac{2}{12}} = -0.002$

$P_{0.5} = 42e^{-0.05\left(\frac{2}{12}\right)} N(0.002) - 42\, e^{-0.02\left(\frac{2}{12}\right)} N(-0.1)$

$= 1.40.08$ ???

0.5

0.5

(7) ~~$\Delta put = -N(0.1) = -0.46$~~

$\Delta put = -e^{q\left(\frac{2}{12}\right)} N(-0.1)$

~~for 132 shares →~~ $= -0.46$

1

(8) the value of the short position is $-100 P_{0.5}$

the portfolio is $\Pi_{0.5} = -100 P_{0.5} + \beta S_{0.5}$

$\Delta \Pi_{0.5} = -100(-0.46) + \beta = 0$

$\Rightarrow \beta = -46$

1

the investor must ~~short~~ sell or short on 42 shares to hedge his position neutrally

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Problem 5:

1

1. Delta = $-1000(0.5) - 500(0.8) - 2000(-0.4) - 500(0.7)$
$= \cancel{40}\ -450$

1

2. Gamma = $-1000(2.2) - 500(0.6) - 2000(1.3) - 500(1.8)$
$= -6000$

3. $\Pi_2$ to be Gamma neutral ~~must equal zero~~
$4000(1.5) = 6000$

0

~~$\Gamma_{\Pi_2} = \Gamma_{\Pi_1} + \beta = 0$~~ ~~we must make the ga~~
~~$\Rightarrow -6000 + \beta = 0$~~ $\beta = 6000$
~~$\Rightarrow \beta = 6000$~~

0.5

we choose 4000 traded option to make ~~Gamm~~ the Portfolio Gamma neutral

1

4. we must go long on ~~75~~ 4000 traded option to make the new Portfolio $\Pi_2$ Gamma-neutral

0.5

5. Portfolio $\Pi_3$ to be delta neutral
0 ~~$\Delta_{\Pi_3} = -450 + \beta = 0$~~ $4000(0.6) - 450 = 1950$
~~$\Rightarrow \beta = 450$~~

1

6. we must go short ~~on~~ 1950 in sterling to make the ~~port~~ Portfolio both delta-neutral and Gamma-neutral

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7

$V = -1000(1.8) - 500(0.2) - 2000(0.7) - 500(1.4)$
$= -4000$

1

8 for vega to be neutral

$\to 40 \; 5000(0.8) = 4000$

for delta

~~4000~~ $5000(0.6) - 450 = 2550$

1

we must be long ~~d~~ on vega and short on 2550
in sterling.