King Saud University

Department of Mathematics

Final Examination

ACTU 361 - Mathematics of Finance (1)

(20/3/1437 H, Time 3H)

Exercise 1. [9]

We consider an annuity for n periods in which payments are being made continuously at the rate $f(t)$ at exact every moment t and the interest rate is variable with a variable force of interest δ_t.

1) Give the formula of the present value of this annuity.
2) We suppose that $f(t) = t$ and $\delta_t = \delta$ (constant).
   a) Prove that the present value of this annuity is equal to:
   $$(\bar{I}\bar{a})_n = \frac{\bar{a}_n - nv^n}{\delta}$$
   Where $\bar{a}_n$ is the present value of an annuity payable continuously for n periods.

   b) Deduce the present value of the corresponding perpetuity.
3) Deduce that if the rate of payment at time t is $f(t) = n - t$ and $\delta_t = \delta$ constant, then:
   a) The present value of this annuity is equal to:
   $$(\bar{D}\bar{a})_n = \frac{n - \bar{a}_n}{\delta}$$

   b) The accumulated value of this annuity is equal to:
   $$(\bar{D}\bar{s})_n = \frac{n(1+i)^n - \bar{s}_n}{\delta}$$
4) Determine the present value and the accumulated value that pays a rate of $10 - t$ with a force of interest $\delta = 6\%$.

Exercise 2. [8]

Consider two 30-years bonds each has an annual coupon rate of 5 % paid semiannually and a par value of 1000.

1) The first bond has an annual nominal yield rate of 5% compounded semiannually, and a redemption value of 1200. Calculate the price of this bond.
2) The second bond has an annual nominal yield rate of j compounded semiannually and a redemption value of 800. Give the expression of the price of this bond in term of j.
3) Suppose that the two bonds have the same price. Calculate j.
   (Hint. $j = 4.4\%$)

Exercise 3. [8]

1) Project P requires an investment of 2000 today. The investment pays 1000 one year from today and 2000 two years from today. Calculate the net present value of the project P with an annual effective interest rate equals to 10%.
2) Project Q requires an investment of X one year from today. The investment pays 1000 today and 2000 two years from today. Calculate in term of X the net present value of the project Q with an annual effective interest rate equals to 5%.
3) We suppose that the net present values of the two projects P and Q are equal. Calculate X.

Exercise 4. [8]

A loan of 10,000 is repaid with a payment made at the end of each year for 20 years. The payments are 100, 200, 300, 400, and 500 in year 1 through 5, respectively. In the subsequent 15 years, equal annual payment of X are made. The annual effective interest rate is 5%.

1) Calculate the present value at t=0 of the first 5 years.
2) Find in term of X the present value at t=0 of the last 15 years.
3) Deduce X.

Question 5. [7]

We consider:

- Perpetuity A that has the following sequence of annual payments 1,3,5,7,....
- Perpetuity B of 1 per year.
- Perpetuity C that has the following sequence of annual payment :

$$1, \ 1+r, \ (1+r)^2, \ldots.$$

We suppose that all the three perpetuities started from t=0 with the same annual interest rate of i.

1) Calculate in term of i the present value of every perpetuity.
2) We suppose that the sum of the present values of the Perpetuity A and the Perpetuity B is 25 times large as the present value of perpetuity C, and the present value of Perpetuity B is equal to the present value of Perpetuity C.
   a) Find the value of i
   b) Deduce the value of r

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1) ~~[illegible]~~ $PV = \bar{a}_{\overline{n}|}$

$PV = \left(\frac{1 - v^n}{\delta}\right)$

2) (a) $\int_0^n f(t)\, e^{-\int_0^t \delta_r\, dr}\, dt$

$= \int_0^n t\, e^{-\int_0^t \delta_r\, dr}\, dt$

$= \int_0^n t\, e^{-\delta t}\, dt$

$u = t \qquad du = dt$

$v = e^{-\delta t}\, dt \qquad dv = -\frac{1}{\delta} e^{-\delta t}$

$= t e^{-\delta t}\Big|_0^n - \int_0^n e^{-\delta t}\, dt$

$= n e^{-\delta n} + \frac{1}{\delta} e^{-\delta t}\Big|_0^n$

$= n e^{-\delta n} + \frac{e^{-\delta n}}{\delta} - \frac{1}{\delta}$

we recall that $v = e^{-\delta}$ and $\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$

$= n v^n + \left(\frac{1 - e^{-\delta n}}{-\delta}\right)$

$= \frac{1 - e^{-\delta n}}{\delta} - n v^n$

$= \frac{1 - v^n}{\delta} - n v^n = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}$

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1) $PV = \bar{a}_{\overline{n}|} = \left(\frac{1-v^n}{\delta}\right)$

(2) (a) $\int_0^n t\, e^{-\int_0^t \delta_r dr} = \int_0^n t\, e^{-\delta t}\, dt$

$u = t \qquad du = dt$

$v = \frac{-1}{\delta} e^{-\delta t} \qquad dv = e^{-\delta t}\, dt$

$= \frac{-t}{\delta} e^{-\delta t}\Big|_0^n + \frac{1}{\delta}\int_0^n e^{-\delta t}\, dt$

$= \frac{-t}{\delta} e^{-\delta t}\Big|_0^n - \frac{1}{\delta^2} e^{-\delta t}\Big|_0^n$

$= \frac{1}{\delta}\left(-n e^{-\delta n} - \frac{e^{-\delta n}}{\delta} - \frac{1}{\delta}\right)$

we recall that $v^n = e^{-\delta}$ and $\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$

so $= \frac{1}{\delta}\left(-nv^n - \left(\frac{v^n - 1}{\delta}\right)\right)$

$= \frac{1}{\delta}\left(\bar{a}_{\overline{n}|} - nv^n\right) = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = (\bar{I}\bar{a})_{\overline{n}|}$

(b) $(\bar{I}\bar{a})_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$

$\lim_{n\to\infty} nv^n$
$= \lim_{x\to\infty} \frac{x}{(1+i)^x}$
$= \lim_{x\to\infty} \frac{1}{\ln(1+x)(1+x)^x}$
$= \frac{1}{\infty} = 0$

since $v^n < 1$

$\Rightarrow \lim_{n\to\infty} \frac{\bar{a}_{\overline{n}|} - 0}{\delta} = \lim_{n\to\infty} \frac{\bar{a}_{\overline{n}|}}{\delta}$

$\lim_{n\to\infty} \bar{a}_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{\delta} = \frac{1}{\delta}$

So $\lim_{n\to\infty} \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \frac{\frac{1}{\delta}}{\delta} = \frac{1}{\delta^2}$

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(3) $f(t) = n - t \qquad \delta_t = \delta$

(a) $(\bar{D}\bar{a})_{\bar{n}|} = n\bar{a}_{\bar{n}|} - \left(\frac{\bar{a}_{\bar{n}|} - nv^n}{\delta}\right)$

$= \cancel{}\; n\left(\frac{1-v^n}{\delta}\right) - \frac{\bar{a}_{\bar{n}|} - nv^n}{\delta}$

(9)

$= \frac{n - nv^n}{\delta} - \frac{\bar{a}_{\bar{n}|} - nv^n}{\delta} = \frac{n - \bar{a}_{\bar{n}|}}{\delta} = (\bar{D}\bar{a})_{\bar{n}|}$

(b) $(\bar{D}\bar{s})_{\bar{n}|} = (1+i)^n (\bar{D}\bar{a})_{\bar{n}|}$

$= (1+i)^n \left(\frac{n - \bar{a}_{\bar{n}|}}{\delta}\right) = \frac{n(1+i)^n - \bar{s}_{\bar{n}|}}{\delta} = (\bar{D}\bar{s})_{\bar{n}|}$

(N)

(4) $PV = 10\,\bar{a}_{\bar{n}|} - \left(\frac{\bar{a}_{\bar{n}|} - nv^n}{\delta}\right) \qquad \delta = 6\%$

$v^n = e^{-0.06 \cdot 10}$

$PV = \frac{10 - 10v^n}{\delta} - \frac{\bar{a}_{\bar{n}|} - 10v^n}{\delta} \qquad n = 10$

$= \frac{10 - \bar{a}_{\overline{10}|}}{\delta} \qquad \cancel{10 - (1 - e^{-}}$

$v^{10} = e^{-0.06 \cdot 10}$

$\frac{1}{v^{10}} = e^{0.06 \cdot 10}$

(7)

$= \frac{10 - \left(\frac{1 - e^{-0.06 \cdot 10}}{0.06}\right)}{0.06} = 41.34$

$\text{Acc Value} = \frac{\cancel{10e^{-}}\; 10\, e^{0.06 \cdot 10} - \left(\frac{e^{0.06 \cdot 10} - 1}{0.06}\right)}{0.06}$

$= 75.32$

1) $PV =$

(2) (a) $\int_0^n t\, e$

$u = t$

$v = -\frac{1}{\delta} e^{-\delta}$

$= \frac{-t}{\delta} e$

$= \frac{-t}{\delta} e$

$= \frac{1}{\delta}($

we recall

$so = \frac{1}{\delta}($

$= \frac{1}{\delta}($

(b) $(\bar{I}\bar{a}$

since v^n

$\Rightarrow \lim_{n \to \infty} \bar{a}_{\bar{n}|}$

$\Rightarrow \lim_{n \to \infty} \bar{a}_{\bar{n}|} =$

$so \lim_{n \to \infty}$

[redacted]

$+\frac{2000}{1.1^2}$

3

$\frac{2000}{(1.05)^2}$

3

561.98

, 0

$5)^2$

8

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5% coupon

Ex 2:

$n = 30$ Semi-annually Par = 1000

$n = 60$

1) $i^{(p)} = 0.05$ $i_{[p]} = 0.025$

Coupon = $0.025 \times 1000 = 25$

Price of bond = $25\left(\frac{1-(1.025)^{-60}}{0.025}\right) + 1200(1.025)^{-60}$

$= 1045.46$

2

2) $i^{(p)} = \cancel{t}\, j^{(p)}$ $i_{[p]} = \frac{j^{(p)}}{2} = \cancel{t}\, j_{[p]}$

Price of bond = $25\left(\frac{1-(1+\frac{j^{(p)}}{2})^{-60}}{\frac{j^{(p)}}{2}}\right) + 800(1+\frac{j^{(p)}}{2})^{-60}$

2

3 ~~1045.46~~ $i^{(p)} = j^{(p)} \Rightarrow 1+j = (1+\frac{j^{(p)}}{2})^2$

Coupon yearly = 50 $j^{(p)} = 2((1+j)^{\frac{1}{2}}-1)$

$1045.46 = 50\left(\frac{1-(1+j)^{-30}}{2((1+j)^{\frac{1}{2}}-1)}\right) + 800(1+j)^{-30}$

$x_* = j$

$(x_1, y_1) = (0.05, 963.22)$

$(x_2, y_2) = (0.04, 1119.82)$

$x_* = x_1 + (y_* - y_1)\frac{x_2 - x_1}{y_2 - y_1}$

$x_* = 0.05 + (1045.46 - 963.22)\frac{0.04 - 0.05}{1119.82 - 963.22}$

$x_* = 0.0448 = j$

$j \approx 0.044 = 4.4\%$

2

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Ex 3: ① $NPV_P = -2000 + \frac{1000}{(1.1)} + \frac{2000}{1.1^2}$

$= 561.98$

3

② $NPV_Q = 1000 + \left(\frac{-X}{1.05}\right) + \frac{2000}{(1.05)^2}$

3

③ $1000 + \left(\frac{-x}{1.05}\right) + \frac{2000}{(1.05)^2} = 561.98$

$\frac{-x}{1.05} = 561.98 - 1000 - \frac{2000}{(1.05)^2}$

$\frac{-x}{1.05} = -2252.08$

$-x = -2364.68$

$x = 2364.68$

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Ex 2:

1) $i^{(p)} = 0$...

Coupon = ...

Price of bond ...

2) $i^{(p)} = $ ...

Price of bond ...

③

Coupon yearly = ...

1045.46 = ...

$x_* = j$

$(x_1, y_1) = ($...

$(x_2, y_2) = (0$...

$x_* = x$...

$x_* = x_0$...

$x_* = 0.$...

$j \approx 0.$...

[redacted]

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3396.17

8.88

Ex 4: $L = 10000$

$100 \quad 200 \quad 300 \quad 400 \quad 500 \quad X \quad X$

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 20$

1) $(I\ddot{a})_{\overline{5}|} = 100\left(\frac{\ddot{a}_{\overline{5}|} - n v^n}{i}\right)$

$= 100\left(\frac{\left(\frac{1-(1.05)^{-5}}{0.05}(1.05)\right) - 5(1.05)^{-5}}{0.05}\right)$

$= 1256.64$ at $t=0$

3

(2) ~~= 1603.83~~

$Pv = v^5 X \left(\frac{1-(1.05)^{-15}}{0.05}\right)$

3

$\cancel{1256.64 = \frac{X}{(1.05)^5}\left(\frac{1-(1.05)^{-15}}{0.05}\right)}$

$\cancel{1256.64 = \frac{X}{(1.05)^5}(10.38)}$

$\cancel{1603.64 = 10.38X}$

$\cancel{X = 154.49}$

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(3) Remaining balance at $t=5$

$$= 10000 - (1+0.05)^{5}\, 1256.64 = 8396.17$$

So

$$8396.17 = X\left(\frac{1-(1.05)^{-15}}{0.05}\right)$$

$$\frac{8396.17}{10.38} = X \Rightarrow X = 808.88$$

Ex 4: L =

1) $(Ia)_{\overline{5|}} = 1$

$= 100\,\Big($

$= 1256.6$

(2) ~~Pv at~~

$Pv = v^{5} X\,($

~~1256.64~~

~~1256.64~~

~~1603.64~~

~~X =~~

[redacted]

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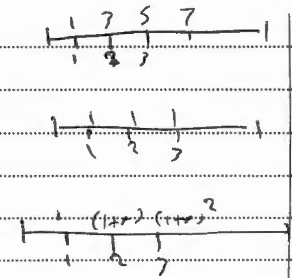
Q5! Interest rate $= i$



1) $PV\ A = 2\left[\frac{1}{i} + \frac{2}{i^2}\right]$

$PV\ B = \frac{1}{i}$ ✓

$PV\ C = \frac{1}{i-r}$ ✓

$PV\ B = PV\ C$

2) $PV\ A + PV\ B = 25\ PV\ C$

$PV\ A = 24\ PV\ C \Rightarrow PV\ A = 24\ PV\ B$

$\cancel{\frac{1}{i} + \frac{2}{i^2} = \frac{1}{i-r}}$ $\quad PV\ A = \frac{24}{i}$

[illegible]

696

6%

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a) $PvA = \frac{24}{i} = 24\, PvB$

$$\frac{1}{i} + \frac{2}{i^2} = \frac{24}{i}$$

$$\frac{i^2 + 2i}{i^3} = \frac{24}{i}$$

$$24 i^3 = i^3 + 2i^2$$

$$23 i^3 - 2i^2 = 0$$

$$i^2(23i - 2) = 0$$

$$i \neq 0 \quad \text{or} \quad 23i - 2 = 0$$

$$23i = 2$$

$$i = \frac{2}{23}$$

$$i = 0.08696$$

$$\underline{i = 8.696\%}$$

$\cancel{PvB = PvC}$

$\cancel{\frac{1}{0.08696} = \frac{1}{0.08696 - r}}$

$\cancel{11.4499 = \frac{1}{0.08696 - r}}$

$\cancel{11.499(0.08696) - 11.499r = 1}$

$\cancel{11.499r = 1 - 0.99995}$

Q5:

1) $PvA =$

$PvB =$

$PvC =$

2 $PvA +$

$PvA =$

[redacted]

$e^{-\delta t}\, dt$

$\cancel{\int e^{-\delta t}}\, dt$

$-\left(\frac{e^{-\delta n}}{\delta} - \frac{1}{\delta}\right)$

$v^n - \frac{1}{\delta}\, \bar{a}_{\overline{n}|}$

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(12) $P_V A = 24 P_V C$

$$\frac{1}{0.08696} + \frac{1}{0.08696^2} = \frac{24}{0.08696 - r}$$

$$143.74 = \frac{24}{0.08696 - r}$$

$$12.499 - 143.74r = 24$$

$$143.74r = -11.500$$

$$r = \frac{-11.500}{143.74}$$

$$r = -0.0800$$

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