A GENERALIZED WINTGEN INEQUALITY FOR QUATERNIONIC CR-SUBMANIFOLDS

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Abstract. The aim of this paper is to extend the classical DDVV inequality to CR-submanifolds of quaternionic Kähler manifolds of constant quaternionic sectional curvature. We first obtain a more general inequality involving the normalized scalar normal curvature ρ_N (defined from the second fundamental form) and then derive a DDVV-type inequality involving the normalized normal scalar curvature $\rho^\perp$ (defined from the normal curvature tensor) for CR-submanifolds in quaternionic ambient space. We also characterize the second fundamental form of those submanifolds for which the equality case holds and give a nontrivial example of submanifold satisfying the equality case identically.

Keywords and phrases: Wintgen inequality, DDVV conjecture, quaternionic Kähler manifold, quaternionic space form, CR-submanifold, totally real submanifold, Lagrangian submanifold.

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1. Introduction

Let M^n be an n-dimensional Riemannian manifold isometrically immersed into an ambient Riemannian space $\bar{M}^{n+m}$. If we denote by $\langle\ ,\ \rangle$ the metric tensor of $\bar{M}$ as well as that induced on M, then the normalized scalar curvature of M, which is an intrinsic invariant of the submanifold M denoted by ρ, and the normalized normal scalar curvature of M, which is an extrinsic invariant of the submanifold denoted by $\rho^\perp$, are defined respectively by

$$\rho = \frac{2\tau}{n(n-1)} \tag{1}$$

and

$$\rho^\perp = \frac{2}{n(n-1)}\sqrt{\sum_{1\le i<j\le n}\ \sum_{1\le r<s\le m}\langle R^\perp(e_i,e_j)\xi_r,\xi_s\rangle^2}, \tag{2}$$

where τ denotes the scalar curvature of M given by

$$\tau = \sum_{1\le i<j\le n} R(e_i,e_j,e_i,e_j),$$

R is the curvature tensor of M, $R^\perp$ is the normal curvature tensor of the immersion, $\{e_1,\dots,e_n\}$ and $\{\xi_1,\dots,\xi_m\}$ are orthonormal bases of the tangent and the normal bundle respectively. We denote by H the mean curvature vector of M, that is

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i),$$

where h is the second fundamental form of the immersion. The squared length of h is

$$||h||^2 = \sum_{i,j=1}^{n} \langle h(e_i, e_j), h(e_i, e_j)\rangle. \tag{3}$$

We also set

$$h_{ij}^r = \langle h(e_i, e_j), \xi_r\rangle,\ \ i, j = 1, \ldots, n,\ \ r = 1, \ldots, m.$$

Then it is well-known that the squared mean curvature of the submanifold M in $\bar{M}$ is defined by

$$\|H\|^2 = \langle H, H\rangle = \frac{1}{n^2}\sum_{r=1}^{m}\left(\sum_{i=1}^{n} h_{ii}^r\right)^2 \tag{4}$$

and the normalized scalar normal curvature is given by

$$\rho_N = \frac{2}{n(n-1)}\sqrt{\sum_{1\le i<j\le n}\ \sum_{1\le r<s\le m}\left(\sum_{k=1}^{n}(h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s)\right)^2}. \tag{5}$$

As a natural generalization of some results of P. Wintgen [55], B.-Y. Chen [17], I.V. Guadalupe and L. Rodriguez [31], it was conjectured by P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [25] that the following inequality holds at every point of a submanifold M^n of a real space form $\bar{M}^{n+m}(c)$ of constant sectional curvature c:

$$\rho \le \|H\|^2 - \rho^{\perp} + c. \tag{6}$$

This conjecture, usually called the *DDVV conjecture*, but also known as the *normal scalar curvature conjecture*, the *DDVV inequality* or the *generalized Wintgen inequality*. It was proved in [27] this conjecture is related to an algebraic inequality of interest in the theory of random matrices (see also [15]). We recall that those submanifolds attaining the equality in the inequality (6) at every point are called *Wintgen ideal submanifolds*. It is known that there are some close connections between such submanifolds and intrinsic pseudo-symmetry conditions [33]. We note that partial results regarding the DDVV conjecture were obtained by many geometers (see, e.g., [23, 24, 28, 50]), but the conjecture in general case was independently proved by J.Q. Ge and Z.Z. Tang [32] and by Z.Q. Lu [35]. There are two main ingredients in proving the DDVV conjecture. The first ingredient is given by [27, Theorem 3.1], which translates the inequality (6) in an algebraic form, as an inequality involving some traceless symmetric matrices. The second ingredient in the proof of the DDVV inequality also with its equality conditions is the following result.

Theorem 1.1. [32, 35] *Let $B_1, ..., B_m$ be $(n \times n)$ real symmetric matrices. Then*

$$\sum_{r,s=1}^{m} ||[B_r, B_s]||^2 \le \left(\sum_{r=1}^{m} ||B_r||^2\right)^2,$$

where $||X||^2$ denotes the sum of squares of entries of the matrix X and

$$[X, Y] = XY - YX$$

is the commutator of the matrices X and Y.

Moreover, the equality sign holds in the above inequality if and only if under some $O(n) \times O(m)$ action all matrices B_r are zero, except 2 matrices which can be written as

$$\begin{pmatrix} 0 & \mu & 0 & ... & 0 \\ \mu & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad \begin{pmatrix} \mu & 0 & 0 & ... & 0 \\ 0 & -\mu & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix},$$

where μ is a real number.

As pointed out in [22], a natural and interesting problem is to use the tools developed in [32, 35] in order to extend the inequality (6) to the case of ambient spaces other than real space forms. Consequently, this inequality has been extensively studied in recent years by many authors in various geometrical contexts. In particular, the DDVV inequality was extended by I. Mihai for Lagrangian and slant submanifolds in complex space forms [39] and for Legendrian submanifolds in Sasakian space forms [40], and also by M.E. Aydin, A. Mihai and I. Mihai for statistical submanifolds in statistical manifolds of constant curvature [7, 8]. On the other hand, J. Roth [45] obtained a DDVV inequality for submanifolds of warped products of the form $I \times_f M^n(c)$, where I is an interval, $f : I \to \mathbb{R}$ is a nowhere-vanishing smooth function and $M^n(c)$ is an n-dimensional real space form of constant sectional curvature c, generalizing some previous results of Q. Chen and Q. Cui [22]. Moreover, very recently, M.N. Boyom, M. Aquib, M.H. Shahid and M. Jamali [14] derived a DDVV-type inequality for Lagrangian submanifolds in holomorphic statistical space forms, C. Murathan and B. Şahin [42] proved a general Wintgen inequality for statistical submanifolds of statistical warped product manifolds, H. Aytimur and C. Özgür [5] derived the generalized Wintgen inequality for submanifolds in statistical manifolds of quasi-constant curvature, while P. Bansal, S. Uddin and M.H. Shahid [9] obtained the DDVV inequality for statistical submanifolds in Kenmotsu statistical ambient space.

The aim of this paper to extend the DDVV conjecture in quaternionic setting, proving an optimal inequality involving the normalized scalar curvature ρ, the normalized normal scalar curvature $\rho^{\perp}$ and the squared length of the mean curvature vector for CR-submanifolds in quaternionic Kähler manifolds of constant quaternionic sectional curvature, generalizing some recent results obtained by G. Macsim and V. Ghişoiu [36] for totally real and Lagrangian submanifolds. In order to derive this inequality, we first apply Theorem 1.1 and obtain a general inequality involving the normalized scalar normal curvature ρ_N for CR-submanifolds in quaternionic Kähler manifolds of constant quaternionic sectional curvature. We recall that the quaternionic Kähler manifolds of dimension greater than 8 are a class of Einstein spaces [34, 46] that are relevant for mathematical physics, with applications in string theory, solitons, gravity and general relativity [30, 38], while a particular class of CR-submanifolds, namely the class of Lagrangian submanifolds, plays a key role in quantum mechanics and Hamiltonian systems, as well as in singularity theory [54].

2. Preliminaries

This section gives several basic definitions and notations for our framework based mainly on [20, 34].

Let M^n be an n-dimensional Riemannian submanifold of a Riemannian manifold $\bar{M}$ and let $\langle\ ,\ \rangle$ be the metric tensor of $\bar{M}$ as well as that induced on M. If $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and ∇ is the covariant differentiation induced on M, then the Gauss and Weingarten formulas are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \forall X,Y \in \Gamma(TM)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \forall X \in \Gamma(TM), \ \ \forall N \in \Gamma(TM^{\perp})$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on the normal bundle and A_N is the shape operator of M with respect to N. If we denote by $\bar{R}$ and R the curvature tensor fields of $\bar{\nabla}$ and ∇, then we have the Gauss equation:

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \langle h(X,W), h(Y,Z)\rangle \\ &\quad -\langle h(X,Z), h(Y,W)\rangle, \end{aligned} \tag{7}$$

for all $X,Y,Z,W \in \Gamma(TM)$, where the sign convention used for the curvature tensor is the following

$$\bar{R}(X,Y,Z,W) = \langle \bar{R}(X,Y)W, Z\rangle,$$

with

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z.$$

Similarly, for normal vector fields ξ and η, the relation

$$\bar{R}(X,Y,\xi,\eta) = R^{\perp}(X,Y,\xi,\eta) + \langle [A_\xi, A_\eta]X, Y\rangle \tag{8}$$

holds, which is called the equation of Ricci, where

$$\langle [A_\xi, A_\eta]X, Y\rangle = \langle A_\xi A_\eta X, Y\rangle - \langle A_\eta A_\xi X, Y\rangle$$

for all $X,Y \in \Gamma(TM)$.

Assume now that $(\bar{M}, \langle\ ,\ \rangle)$ is a Riemannian manifold such that there exists a rank 3-subbundle σ of $End(T\overline{M})$ with local basis $\{J_1, J_2, J_3\}$ satisfying for all $\alpha \in \{1,2,3\}$:

$$\langle J_\alpha\cdot, J_\alpha\cdot\rangle = \langle\cdot,\cdot\rangle$$

and

$$J_\alpha^2 = -\mathrm{Id}, \ \ J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2},$$

where Id denotes the identity tensor field of type (1, 1) on $\bar{M}$ and the indices are taken from $\{1,2,3\}$ modulo 3. Then $(\bar{M}, \sigma, \langle\ ,\ \rangle)$ is said to be an almost quaternionic Hermitian manifold. It is easy to see that such manifold is of dimension $4m$, $m \geq 1$. Moreover, if the bundle σ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of $\langle\ ,\ \rangle$, then $(\bar{M}, \sigma, \langle\ ,\ \rangle)$ is said to be a *quaternionic Kähler manifold.*

Let $(\bar{M}, \sigma, \langle\ ,\ \rangle)$ be a quaternionic Kähler manifold and let X be a non-null vector field on $\bar{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c. It is well-known that a quaternionic Kähler manifold $(\bar{M}, \sigma, \langle\ ,\ \rangle)$ is a quaternionic space form, denoted $\bar{M}(c)$, if and only if its curvature tensor is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c}{4}\{\langle Z,Y\rangle X - \langle X,Z\rangle Y + \sum_{\alpha=1}^{3}[\langle Z, J_\alpha Y\rangle J_\alpha X - \\ &\quad -\langle Z, J_\alpha X\rangle J_\alpha Y + 2\langle X, J_\alpha Y\rangle J_\alpha Z]\} \end{aligned} \tag{9}$$

for all vector fields X, Y, Z on $\bar{M}$ and any local basis $\{J_1, J_2, J_3\}$ of σ.

It is also known that quaternionic space forms are locally congruent to either a quaternionic projective space $\mathbb{H}P^n(c)$ of quaternionic sectional curvature $c > 0$, a quaternionic Euclidean space $\mathbb{H}^n$ of null quaternionic sectional curvature or a quaternionic hyperbolic space $\mathbb{H}H^n(c)$ of quaternionic sectional curvature $c < 0$ [1]. We note that, if $c \neq 0$, then for sake of simplicity only two cases are usually considered, namely $c = 4$ and $c = -4$ (see, e.g., [43]).

Let M^n be a submanifold of a quaternionic Kähler manifold $(\bar{M}^{4m}, \sigma, \langle\ ,\ \rangle)$. Then for any vector field X tangent to the submanifold M, we have the decomposition

$$J_\alpha X = J_\alpha^\top X + J_\alpha^\perp X, \quad \alpha = 1, 2, 3, \tag{10}$$

where $J_\alpha^\top X$ denotes the tangential component of $J_\alpha X$ and $J_\alpha^\perp X$ denotes the normal component of $J_\alpha X$.

We recall that a submanifold M^n of a quaternionic Kähler manifold $(\bar{M}^{4m}, \sigma, \langle\ ,\ \rangle)$ is said to be a totally real submanifold if each 2-plane of M is mapped into a totally real plane in $\bar{M}$, or, equivalently, if each tangent space of M is mapped into the normal space by any local basis $\{J_1, J_2, J_3\}$ of σ (see [29]). Similarly, a submanifold M^n of a quaternionic Kähler manifold $(\bar{M}^{4m}, \sigma, \langle\ ,\ \rangle)$ is said to be a quaternionic submanifold if each tangent space of M is carried into itself by any local basis of σ.

Analogous to Lagrangian submanifolds in an almost (para-)Hermitian manifold (see, e.g., [6, 18, 19, 48, 53]), a totally real submanifold M^n of a quaternionic Kähler manifold $(\bar{M}^{4m}, \sigma, \langle\ ,\ \rangle)$ of maximum dimension $n = m$ is said to be a Lagrangian submanifold [21, 26]. For a deeper study on the geometry of some special submanifolds in quaternionic Kähler ambient space see [2, 3, 37, 47, 49, 51, 52]. We only point out that the most important class of submanifolds in quaternionic geometry is given by the *quaternionic CR-submanifolds.* It is well known that the notion of CR-submanifold was introduced by Bejancu [11] in Kähler ambient space, as generalization of both totally real and holomorphic submanifolds (see also [16]). According to the Bejancu's definition, the tangent bundle of a CR-submanifold splits into two complementary orthogonal distributions, the first one being holomorphic and the second one being totally real. Recall that a distribution S of rank s on a smooth manifold M is an assignment of an s-dimensional subspace S_x of the tangent space T_xM to each point $x \in M$.

Later, Barros, Chen and Urbano [10] extended this concept in quaternionic setting, considering quaternionic CR-submanifolds of quaternionic Kähler manifolds as a generalization of both quaternionic and totally real submanifolds. A submanifold M of a quaternionic Kähler manifold $(\bar{M}, \sigma, \langle\ ,\ \rangle)$ is said to be a quaternionic CR-submanifold if there exists two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M such that $\mathcal{D}$ is invariant under quaternionic structure and $\mathcal{D}^\perp$ is totally real. It is obvious that a quaternionic CR-submanifold of a quaternionic Kähler manifold reduces to a quaternionic submanifold (respectively, to a totally real submanifold) if $\dim \mathcal{D}_x^\perp = 0$, $x \in M$ (respectively, $\dim \mathcal{D}_x = 0$, $x \in M$). We recall that a quaternionic CR-submanifold M is called *proper* if M is neither a quaternionic submanifold nor a totally real submanifold [44]. In general, if M is a quaternionic CR-submanifold of a quaternionic Kähler manifold $(\bar{M}, \sigma, \langle\ ,\ \rangle)$, then it is clear that the real dimension of $\mathcal{D}_x$ is divisible by 4. Next let us denote $\mathrm{rank}_{\mathbb{H}} \mathcal{D} = p$ and $\mathrm{rank}_{\mathbb{R}}\, \mathcal{D}^\perp = q$. Then it is known (see, e.g., [41]) that we can take a local orthonormal frame in $\bar{M}$:

$$\{e_1, ..., e_p, e_{p+1}, ..., e_{p+q}, e_{p+q+1}, ..., e_m, J_1e_1, ..., J_1e_m, J_2e_1, ..., J_2e_m, J_3e_1, ..., J_3e_m\}$$

such that restricted to M, $\{e_1, \dots, e_p, J_1e_1, \dots, J_1e_p, J_2e_1, \dots, J_2e_p, J_3e_1, \dots, J_3e_p\}$ are in $\mathcal{D}$ and $\{e_{p+1}, ..., e_{p+q}\}$ are in $\mathcal{D}^\perp$.

3. Main inequalities

Theorem 3.1. *Let M^n be a quaternionic CR-submanifold of a quaternionic space form $\bar{M}^{4m}(c)$. Then*

$$\rho \le \|H\|^2 - \rho_N + \frac{c}{4} + \frac{9pc}{n(n-1)}, \tag{11}$$

where $p = \mathrm{rank}_{\mathbb{H}}\, \mathcal{D}$.

Moreover, the equality sign holds in the above inequality at some point $x \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of T_xM and an orthonormal basis $\{\xi_1, \dots, \xi_{4m-n}\}$ of $T_x^\perp M$ such that the shape operators $A_r \equiv A_{\xi_r}$, $r = 1, \dots, 4m-n$, take the following forms:

$$A_1 = \begin{pmatrix} a_1 & b & 0 & \dots & 0 \\ b & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{pmatrix}, \tag{12}$$

$$A_2 = \begin{pmatrix} a_2+b & 0 & 0 & \dots & 0 \\ 0 & a_2-b & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_2 \end{pmatrix}, \tag{13}$$

$$A_3 = a_3 I_n, \tag{14}$$

$$A_r = 0,\ \ r = 4, \dots, 4m-n, \tag{15}$$

where a_1, a_2, a_3 and b are real numbers.

Proof. Let $\{e_1, \dots e_{4p}, e_{4p+1}, \dots, e_n\}$ be an orthonormal frame on M such that $\{e_1, \dots, e_{4p}\}$ are in $\mathcal{D}$ and $\{e_{4p+1}, ..., e_n\}$ are in $\mathcal{D}^\perp$, and let $\{\xi_1, \dots, \xi_{4m-n}\}$ be an orthonormal frame in the normal bundle. We consider the symmetric operators

$B_r := A_r - \langle H, \xi_r \rangle \,\mathrm{Id}$, $r = 1, \ldots, 4m-n$, where A_r denotes the shape operator in the direction of ξ_r. Then it follows that

$$\begin{aligned}\sum_{r=1}^{4m-n} ||B_r||^2 &= \sum_{r=1}^{4m-n} ||A_r - \langle H, \xi_r \rangle \,\mathrm{Id}||^2 \\ &= 2\sum_{r=1}^{4m-n} \sum_{1\le i<j\le n} (h_{ij}^r)^2 + \frac{n-1}{n} \sum_{r=1}^{4m-n} \sum_{i=1}^{n} (h_{ii}^r)^2 \\ &\quad -\frac{2}{n} \sum_{r=1}^{4m-n} \sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r \end{aligned} \tag{16}$$

and

$$\begin{aligned}\sum_{r,s=1}^{4m-n} ||[B_r, B_s]||^2 &= \sum_{r,s=1}^{4m-n} ||[A_r - \langle H, \xi_r \rangle, A_s - \langle H, \xi_s \rangle]||^2 \\ &= \sum_{r,s=1}^{4m-n} ||[A_r, A_s]||^2 \\ &= 4 \sum_{1\le r<s\le 4m-n} \sum_{1\le i<j\le n} \left(\sum_{k=1}^{n} (h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s) \right)^2. \end{aligned} \tag{17}$$

Applying Theorem 1.1 for $B_1, \ldots, B_{4m-n}$ and taking account of (16) and (17), we derive the following inequality

$$\begin{aligned} &2\sum_{r=1}^{4m-n} \sum_{1\le i<j\le n} (h_{ij}^r)^2 + \frac{n-1}{n} \sum_{r=1}^{4m-n} \sum_{i=1}^{n} (h_{ii}^r)^2 - \frac{2}{n} \sum_{r=1}^{4m-n} \sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r \\ &\ge 2\sqrt{\sum_{1\le r<s\le 4m-n} \sum_{1\le i<j\le n} \left(\sum_{k=1}^{n} (h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s) \right)^2}. \end{aligned} \tag{18}$$

Using now (4) and (5) in (18), we obtain

$$n^2 \|H\|^2 - n^2 \rho_N \ge \frac{2n}{n-1} \sum_{r=1}^{4m-n} \sum_{1\le i<j\le n} \left[h_{ii}^r h_{jj}^r - \left(h_{ij}^r\right)^2 \right]. \tag{19}$$

Since $\bar{M}(c)$ is a quaternionic space form, from (7), (9) and (10) we derive

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c}{4} \Bigg\{ \langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle + \sum_{\alpha=1}^{3} \Big[\langle J_\alpha^\top X, Z \rangle \langle J_\alpha^\top Y, W \rangle \\ &\quad - \langle J_\alpha^\top X, W \rangle \langle J_\alpha^\top Y, Z \rangle + 2 \langle J_\alpha^\top X, Y \rangle \langle J_\alpha^\top Z, W \rangle \Big] \Bigg\} \\ &\quad - \langle h(X,W), h(Y,Z) \rangle + \langle h(X,Z), h(Y,W) \rangle, \end{aligned} \tag{20}$$

for $X, Y, Z, W \in \Gamma(TM)$.

Taking now $X = Z = e_i$, $Y = W = e_j$ in (20) and summing over i and j from 1 to n, we get

$$\begin{aligned} 2\tau &= \frac{c}{4}\sum_{i,j=1}^{n}\Bigg\{1-\delta_{ij}^2+\sum_{\alpha=1}^{3}\Big[\langle J_\alpha^\top e_i,e_i\rangle\langle J_\alpha^\top e_j,e_j\rangle \\ &\quad -\langle J_\alpha^\top e_i,e_j\rangle\langle J_\alpha^\top e_j,e_i\rangle+2\langle J_\alpha^\top e_i,e_j\rangle\langle J_\alpha^\top e_i,e_j\rangle\Big]\Bigg\} \\ &\quad -\sum_{i,j=1}^{n}\langle h(e_i,e_j),h(e_j,e_i)\rangle+\sum_{i,j=1}^{n}\langle h(e_i,e_i),h(e_j,e_j)\rangle, \\ &= \frac{c}{4}\left[n(n-1)+3\sum_{i=1}^{n}\sum_{\alpha=1}^{3}\langle J_\alpha^\top e_i,J_\alpha^\top e_i\rangle\right] \\ &\quad -\sum_{i,j=1}^{n}\langle h(e_i,e_j),h(e_j,e_i)\rangle+\sum_{i,j=1}^{n}\langle h(e_i,e_i),h(e_j,e_j)\rangle, \end{aligned}$$

where δ_{ij} denotes the Kronecker delta symbol. Taking now into account that $J_\alpha^\top e_i = J_\alpha e_i \in \mathcal{D}$ for $i = 1, \ldots, 4p$, and $J_\alpha^\top e_i = 0$ for $i = 4p+1, \ldots, n$, the above relation implies

$$\begin{aligned} 2\tau &= \frac{n(n-1)c}{4}+9pc-\sum_{i,j=1}^{n}\langle h(e_i,e_j),h(e_j,e_i)\rangle \\ &\quad +\sum_{i,j=1}^{n}\langle h(e_i,e_i),h(e_j,e_j)\rangle. \end{aligned} \tag{21}$$

Using now (3) and (4) in (21), we derive

$$2\tau = n^2\|H\|^2-\|h\|^2+\frac{n(n-1)c}{4}+9pc. \tag{22}$$

Next we can easily check that (1) and (22) imply

$$\rho = \frac{2}{n(n-1)}\sum_{r=1}^{4m-n}\sum_{1\le i<j\le n}\left[h_{ii}^r h_{jj}^r-\left(h_{ij}^r\right)^2\right]+\frac{c}{4}+\frac{9pc}{n(n-1)}. \tag{23}$$

Combining (19) and (23), we obtain

$$n^2\|H\|^2-n^2\rho_N \ge n^2\left[\rho-\frac{c}{4}-\frac{9pc}{n(n-1)}\right]$$

and the inequality (11) follows now easily.

Finally, analyzing the case of equality in (18), we deduce that the equality sign holds in the inequality (11) at some point $x \in M$ if and only if there exists an orthonormal basis of T_xM and an orthonormal basis of $T_x^\perp M$ such that the shape operators take the forms (12)-(15). □

As an immediate consequence of the Theorem 3.1, we deduce the following results.

Corollary 3.2. *Let M^n be a quaternionic CR-submanifold of the quaternionic Euclidean space $\mathbb{H}^m$. Then*

$$\|H\|^2 \ge \rho+\rho_N. \tag{24}$$

Corollary 3.3. *Let M^n be a quaternionic CR-submanifold of the quaternionic projective space $\mathbb{H}P^m$ of constant quaternionic sectional curvature 4. Then*

$$\|H\|^2 \geq \rho + \rho_N - 1 - \frac{36p}{n(n-1)}, \tag{25}$$

where $p = \operatorname{rank}_{\mathbb{H}} \mathcal{D}$.

Corollary 3.4. *Let M^n be a quaternionic CR-submanifold of the quaternionic hyperbolic space $\mathbb{H}H^m$ of constant quaternionic sectional curvature -4. Then*

$$\|H\|^2 \geq \rho + \rho_N + 1 + \frac{36p}{n(n-1)}, \tag{26}$$

where $p = \operatorname{rank}_{\mathbb{H}} \mathcal{D}$.

Remark 3.5. If M^n is a quaternionic CR-submanifold of a quaternionic space form $\bar{M}^{4m}(c)$ with minimal codimension, then we have [10]

$$T^\perp M = J_1\mathcal{D}^\perp \oplus J_2\mathcal{D}^\perp \oplus J_3\mathcal{D}^\perp.$$

Hence, if $n = 4p + q$, where $p = \operatorname{rank}_{\mathbb{H}} \mathcal{D}$ and $q = \operatorname{rank} \mathcal{D}^\perp$, then we derive that $\dim T^\perp M = 3q$, with $p + q = m$. Therefore we can consider an orthonormal frame

$$\{e_1, \ldots, e_p, e_{p+1} = J_1e_1, \ldots, e_{2p} = J_1e_p, e_{2p+1} = J_2e_1, \ldots, e_{3p} = J_2e_p,$$
$$e_{3p+1} = J_3e_1, \ldots, e_{4p} = J_3e_p, e_{4p+1}, \ldots, e_{4p+q}\}$$

on M such that $\{e_1, \ldots, e_{4p}\}$ are in $\mathcal{D}$ and $\{e_{4p+1}, ..., e_{4p+q}\}$ are in $\mathcal{D}^\perp$. Now, because M^n is a quaternionic CR-submanifold of minimal codimension, we deduce that

$$\{\xi_i = J_1e_{4p+i}, \xi_{q+i} = J_2e_{4p+i}, \xi_{2q+i} = J_3e_{4p+i}\}_{i=1,\ldots,q},$$

is an orthonormal frame in the normal bundle.

For a quaternionic CR-submanifold M^n, we can define similarly as in complex setting [4], the partial mean curvature vectors $H_{\mathcal{D}}$ and $H_{\mathcal{D}^\perp}$ of the submanifold by

$$H_{\mathcal{D}} = \frac{1}{4p}\sum_{i=1}^{4p} h(e_i, e_i), \ H_{\mathcal{D}^\perp} = \frac{1}{q}\sum_{i=4p+1}^{4p+q} h(e_i, e_i).$$

Moreover, we can consider the normalized scalar curvature $\rho_{\mathcal{D}^\perp}$ of the totally real distribution $\mathcal{D}^\perp$ as

$$\rho_{\mathcal{D}^\perp} = \frac{2\tau(\mathcal{D}^\perp)}{q(q-1)} = \frac{2}{q(q-1)}\sum_{1\leq i<j\leq q} R(e_{4p+i}, e_{4p+j}, e_{4p+i}, e_{4p+j}),$$

where $\tau(\mathcal{D}^\perp)$ denotes the scalar curvature of the distribution $\mathcal{D}^\perp$.

Using Gauss equation (7), it is easy to see that

$$2\tau(\mathcal{D}^\perp) = \frac{cq(q-1)}{4} + q^2||H_{\mathcal{D}^\perp}||^2 - ||h_{\mathcal{D}^\perp}||^2,$$

where

$$||H_{\mathcal{D}^\perp}||^2 = \langle H_{\mathcal{D}^\perp}, H_{\mathcal{D}^\perp}\rangle$$

and

$$||h_{\mathcal{D}^\perp}||^2 = \sum_{i,j=4p+1}^{4p+q} \langle h(e_i, e_j), h(e_i, e_j)\rangle.$$

Hence we get

$$\rho_{\mathcal{D}^\perp} = \frac{c}{4} + \frac{1}{q(q-1)}\left(q^2||H_{\mathcal{D}^\perp}||^2 - ||h_{\mathcal{D}^\perp}||^2\right). \tag{27}$$

Next, from (8) and (9) we derive

$$\begin{aligned} R^\perp(X,Y,\xi,\eta) &= \frac{c}{4}\sum_{\alpha=1}^{3}[\langle J_\alpha X,\xi\rangle\langle J_\alpha Y,\eta\rangle - \langle J_\alpha X,\eta\rangle\langle J_\alpha Y,\xi\rangle + 2\langle X,J_\alpha Y\rangle\langle J_\alpha\eta,\xi\rangle] \\ &\quad -\langle [A_\xi,A_\eta]X,Y\rangle, \end{aligned} \tag{28}$$

for $X,Y\in\Gamma(TM)$ and $\xi,\eta\in\Gamma(T^\perp M)$.

Now, using (2), (27) and (28), we obtain after some long but straightforward computations that

$$\begin{aligned} \rho^\perp &= \frac{2}{n(n-1)}\sqrt{\sum_{1\le i<j\le 4p+q}\ \sum_{1\le r<s\le 3q}\langle R^\perp(e_i,e_j)\xi_r,\xi_s\rangle^2} \\ &= \frac{2}{n(n-1)}\sqrt{\frac{n^2(n-1)^2}{4}\rho_N^2 + \frac{3q(q-1)c^2}{32} + \frac{cq(q-1)}{4}\Big(\rho_{\mathcal{D}^\perp} - \frac{c}{4}\Big)}. \end{aligned} \tag{29}$$

Hence from (29) we get immediately

$$\rho^\perp = \sqrt{\rho_N^2 + \frac{3q(q-1)c^2}{8n^2(n-1)^2} + \frac{cq(q-1)}{n^2(n-1)^2}\Big(\rho_{\mathcal{D}^\perp} - \frac{c}{4}\Big)}. \tag{30}$$

Now, using Theorem 3.1 and (30), we can establish the next result that is the quaternionic counterpart of the generalized Wintgen inequality (6) for quaternionic CR-submanifolds of minimal codimension.

Theorem 3.6. *Let M be a quaternionic CR-submanifold of a quaternionic space form $\bar{M}(c)$ with minimal codimension. If the dimension of M is $n=4p+q$, where $p=\mathrm{rank}_{\mathbb{H}}\,\mathcal{D}$ and $q=\mathrm{rank}_{\mathbb{R}}\,\mathcal{D}^\perp$, then the following inequality holds true:*

$$\left(\rho^\perp\right)^2 \le \left[\|H\|^2 - \rho + \frac{c}{4} + \frac{9pc}{n(n-1)}\right]^2 + \frac{3q(q-1)c^2}{8n^2(n-1)^2} + \frac{cq(q-1)}{n^2(n-1)^2}\left(\rho_{\mathcal{D}^\perp} - \frac{c}{4}\right).$$

Moreover, the equality sign holds in the above inequality at some point $x\in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_xM and an orthonormal basis $\{\xi_1,\dots,\xi_{3n}\}$ of $T_x^\perp M$ such that the shape operators $A_r\equiv A_{\xi_r}$, $r=1,\dots,3q$, take the forms from Theorem 3.1.

From Theorem 3.6 we derive immediately the following result.

Corollary 3.7. *Let M be a quaternionic CR-submanifold of the quaternionic Euclidean space $\mathbb{H}^n$. If M has minimal codimension, then*

$$\rho^\perp \le |\|H\|^2 - \rho|. \tag{31}$$

4. Example and final remarks

Remark 4.1. Notice that Theorem 3.6 generalizes the main result of [36], namely the generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms. Indeed, if the quaternionic CR-submanifold M reduces to a Lagrangian submanifold, then we have $p=0$, $q=n$ and $\mathcal{D}^\perp=TM$. Therefore Theorem 3.6 leads to the following.

Corollary 4.2. *Let M^n be a Lagrangian submanifold of a quaternionic space form $\bar{M}^{4n}(c)$. Then*

$$(\rho^\perp)^2 \leq \left(\|H\|^2 - \rho + \frac{c}{4}\right)^2 + \frac{3c^2}{8n(n-1)} + \frac{c}{n(n-1)}\left(\rho - \frac{c}{4}\right). \tag{32}$$

Now it is easy to see that Corollary 4.2 is nothing but [36, Theorem 3]. Hence it follows that indeed Theorem 3.6 generalizes the main result of [36].

Example 4.3. The canonical totally geodesic immersion of the $4n$-dimensional quaternionic projective space $\mathbb{H}P^n(c)$ of constant quaternionic sectional curvature c into the $4m$-dimensional quaternionic projective space $\mathbb{H}P^m(c)$ with the same constant quaternionic sectional curvature, where $n \leq m$, provides us a trivial example of quaternionic submanifold satisfying the equality case of the inequality stated in Theorem 3.1. Similarly, the canonical totally geodesic totally real isometric imbedding of the n-dimensional real projective space $\mathbb{R}P^n\left(\frac{c}{4}\right)$ of constant sectional curvature $\frac{c}{4}$ into the $4n$-dimensional quaternionic projective space $\mathbb{H}P^n(c)$ of constant quaternionic sectional curvature c provides us a trivial example of Lagrangian submanifold satisfying the equality case of the inequalities stated in Theorems 3.1 and 3.6.

Example 4.4. Let T^2 be a flat, totally real, minimal surface in the complex projective plane $\mathbb{C}P^2$ of constant holomorphic sectional curvature c. Since $\mathbb{C}P^2$ can be isometrically imbedded in $\mathbb{H}P^2$ as a totally geodesic totally real submanifold, we deduce that the composition immersion gives us a totally real minimal surface in $\mathbb{H}P^2$ (see [21]). Moreover, this immersion provides us a nontrivial example of Lagrangian submanifold satisfying the equality case of the inequalities stated in Theorems 3.1 and 3.6, due to the fact that the shape operators take the desired forms (with $a_1 = a_2 = a_3 = 0$, $b \neq 0$).

Remark 4.5. We note that, besides CR-submanifolds, there is a second class of submanifolds of great interest in quaternionic geometry, namely *QR-submanifolds*. These submanifolds were introduced by Bejancu [13] as a generalization of the real hypersurfaces of a quaternionic Kähler manifold. In fact, a real submanifold M of a quaternionic Kähler manifold $(\bar{M}, \sigma, \langle\ ,\ \rangle)$ is said to be a QR-submanifold if there exists a vector subbundle D of the normal bundle $TM^\perp$ such that $J_\alpha(D_x) = D_x$ and $J_\alpha(D_x^\perp) \subset T_xM$, for all $x \in M$, $\alpha = 1, 2, 3$, and for any local basis $\{J_1, J_2, J_3\}$ of σ, where $D^\perp$ is the complementary orthogonal bundle to D in $TM^\perp$. We recall that between the two classes of QR-submanifolds and quaternionic CR-submanifolds there exists no inclusion relation because a real hypersurface is a QR-submanifold, but it is not a CR-submanifold, while a totally-real submanifold is a quaternionic CR-submanifold and it is not a QR-submanifold [12]. A very natural open problem is to establish a generalized Wintgen inequality for QR-submanifolds of quaternionic space forms, in case of minimal codimension.

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