PHYS 454
HANDOUT 3-*Operators in QM-The Uncertainty Principle*

1. What is the angular momentum operator in Cartesian coordinates?

2. Show that $\left[\hat{x}, \hat{p}_x\right] = i\hbar$.

3. Show that the translation operator, which is defined as $\hat{T}f(x) = f(x+\delta)$, is linear.

4. Show that the parity operator is linear.

5. Show that the functions e^{ikx}, e^{-ikx} are eigenfunctions of the momentum operator. What about the function $e^{ikx} + e^{-ikx}$?

6. Find the eigenfunctions of the operator $\hat{A} = \hat{x} - \frac{\hat{p}}{i\hbar}$.

7. If $\left|\psi_1\right\rangle = \hat{A}\left|\phi_1\right\rangle$ and $\left|\psi_2\right\rangle = \hat{B}\left|\phi_2\right\rangle$ find $\left\langle\psi_2\middle|\psi_1\right\rangle$.

8. If $\left|\psi\right\rangle = \sum_n c_n\left|\psi_n\right\rangle$ and $\left|\phi\right\rangle = \sum_n b_n\left|\psi_n\right\rangle$ find $\left\langle\phi\middle|\psi\right\rangle$.

9. If $\left|\psi\right\rangle = c_1\left|\psi_1\right\rangle + c_2\left|\psi_2\right\rangle$ find $\left\langle\psi\middle|\psi\right\rangle$. Discuss the result.

10. If $\left|\psi\right\rangle = \sum_n c_n\left|\psi_n\right\rangle$ with $\left|\psi_n\right\rangle$ forming an orthonormal basis find $\left\langle\psi\middle|\psi\right\rangle$.

11. Let the state $\left|\psi\right\rangle = N\left|\psi_1\right\rangle - N\left|\psi_2\right\rangle$ with $\left|\psi_1\right\rangle$ and $\left|\psi_2\right\rangle$ orthonormal eigenvectors of a physical quantity. Find N so $\left|\psi\right\rangle$ to be normalized.

12. Let the state $\left|\psi\right\rangle = N\left|\psi_1\right\rangle + 2iN\left|\psi_2\right\rangle + iN\left|\psi_3\right\rangle$ with $\left|\psi_1\right\rangle$, $\left|\psi_2\right\rangle$ and $\left|\psi_3\right\rangle$ orthonormal eigenvectors of a physical quantity. Find N so $\left|\psi\right\rangle$ to be normalized.

13. You are given the physical quantity A which corresponds to an operator: $\hat{A} = -\frac{(\hat{x}^2+1)\hat{p}}{\hbar} + i\hat{x}$. Find the state which corresponds to $\hat{A}\psi(x) = 0$ and normalize it. You are given that $\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \pi$.

14. Show that any operator $\hat{a}$ can be written in the form $\hat{a} = \hat{Q} + i\hat{P}$, where $\hat{Q}$, $\hat{P}$ are hermitian operators.

15. Show that the product of two projection operators is a projection operator as well.

16. Show that the parity operator has eigenvalues ± 1.

17. Find the parity of the following functions (i) cos$x$, (ii) sin$x$ and (iii) cos$x$+sin$x$.

18. Prove the Heisenberg's uncertainty principle.

19. Prove the Ehrenfest's Theorem.

20. Prove the energy-time uncertainty relation.

21. Show that when a system is in an eigenfunction of an operator $\hat{A}$, then $\Delta A = 0$.

22. Show that if two physical quantities can be measured with full precision simultaneously then their operators commute.

23. Show that for the components ℓ_x, ℓ_y and ℓ_z of the angular momentum we have the following commutation relations:

$$\left[\ell_x, \ell_y\right] = i\hbar\ell_z, \; \left[\ell_y, \ell_z\right] = i\hbar\ell_x, \; \left[\ell_z, \ell_x\right] = i\hbar\ell_y.$$

$$\left[\ell_z, \ell^2\right] = 0.$$

24. Starting from the relation that $\left[x, p\right] = i\hbar$, prove the relations:

$$\left[x, p^2\right] = 2i\hbar p, \quad \left[x, p^3\right] = 3i\hbar p^2 \ldots \quad \left[x, p^n\right] = in\hbar p^{n-1},$$

$$\left[p, x^2\right] = -2i\hbar x, \quad \left[p, x^3\right] = -3i\hbar x^2 \ldots \quad \left[p, x^n\right] = -in\hbar x^{n-1}.$$

Then show that: $\left[x, \mathrm{A}(x,p)\right] = i\hbar \dfrac{\partial A}{\partial p}$ and $\left[p, \mathrm{A}(x,p)\right] = -i\hbar \dfrac{\partial A}{\partial x}$

Where, $\mathrm{A}(x,p)$ is a function of x and p smooth enough to be developed in a Taylor series with respect either to x or p.

25. A) Prove the Enhrenfest's theorem in the case of the one dimensional motion of a particle subject to a force $F=F(x)$. B) Apply your result to the following cases i) when the force is constant $F=F_0$ and ii) when the force is that of a simple harmonic oscillator $F=-kx$.