Financial Mathematics

Prof. Mohamad Alghamdi

Department of Mathematics

April 30, 2025

Main Content

1. Macaulay duration
2. Volatility or Modified duration.
3. Convexity

Section 6.1: Macaulay duration

■ In the reality of companies with thousands of claims, thousands of accounts, thousands of bonds and other investments such as insurance companies or banks, the company may have to sell bonds to meet unexpected obligations at various times, and it also faces **interest rate risk**.

■ The concept of **duration** gives the investment manager a measure of the company's interest rate risk and a tool to manage that risk by matching the assets and liabilities of the entire portfolio.

■ There are two related types of duration: Macaulay duration and modified duration.

■ A simple way to describe the Macaulay duration of an investment is: *it is the weighted average time over which the investment payments will be made.*

Example of weighted average explanation. Let $x_1, x_2, \ldots, x_n$ be a set of n real numbers and $w_1, w_2, \ldots, w_n$ be a set of n positive real numbers such that $w_1 + w_2 + \cdots + w_n = 1$. The weighted average of $x_1, x_2, \ldots, x_n$ with the weights $w_1, w_2, \ldots, w_n$ is the sum $x_1 w_1 + x_2 w_2 + \cdots + x_n w_n$. For example, the weighted average of the numbers 1, 2, and 3 with weights 0.5, 0.3, 0.2, respectively, is $0.5(1) + 0.3(2) + 0.2(3) = 1.7$.

■ For an investment that has n cash flows

Time in years	t_1	t_2	t_3	...	t_n
Contributions	C_1	C_2	C_3	...	C_n

the Macaulay duration ($\bar{d}$) is a weighted average of the times of payment: $t_1, t_2, \ldots, t_n$.

■ The weights are based on the present values of the cash flows. The total present value (or price) of this investment is: $P = \sum_{k=1}^{n} C_{t_k} \nu^{t_k}$. The weight for the payment at time t_k is just the present value of that payment $C_{t_k} \nu^{t_k}$, divided by the total present value (P):

$$w_k = \frac{C_{t_k} \nu^{t_k}}{P} = \frac{C_{t_k} \nu^{t_k}}{\sum_{k=1}^{n} C_{t_k} \nu^{t_k}}$$

Note that $w_1 + w_2 + w_3 + \cdots + w_n = 1$.

- The Macaulay duration is defined as:

$$\begin{aligned}\bar{d} &= t_1 w_1 + t_2 w_2 + \cdots + t_n w_n \\ &= t_1 \frac{C_{t_1} \nu^{t_1}}{P} + t_2 \frac{C_{t_2} \nu^{t_2}}{P} + \cdots + t_n \frac{C_{t_n} \nu^{t_n}}{P} \\ &= \frac{t_1\ C_{t_1} \nu^{t_1} + t_2\ C_{t_2} \nu^{t_2} + \cdots + t_n\ C_{t_n} \nu^{t_n}}{P} = \frac{\sum_k^n t_k\ C_{t_k} \nu^{t_k}}{\sum_{k=1}^n C_{t_k} \nu^{t_k}}\end{aligned}$$

- For n equally–spaced cash flows,

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

the duration (or Macaulays duration) is

$$\bar{d} = \frac{\sum_{j=1}^n jC_j \nu^j}{\sum_{j=1}^n C_j \nu^j} = \sum_{j=1}^n j \frac{C_j \nu^j}{\sum_{k=1}^n C_k \nu^k} = \sum_{j=1}^n j\ w_j$$

where $w_j = \frac{C_j \nu^j}{\sum_{k=1}^n C_k \nu^k}$ satisfy $w_j > 0$ and $\sum_{j=1}^n w_j = 1$

Notes.

- The units of the duration are years.
- The Macaulay duration is a measure of the price sensitivity of a cashflow to interest rate changes.
- w_j is the fraction of the present value of contribution at time j over the present value of the whole cashflow.

Example. An investment pays 1000 at the end of year two and 1000 at the end of year 12. The annual effective rate of interest is 8%. Calculate the Macaulay duration for this investment.

Solution.

$$\bar{d} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{(2)(1000)(1.08)^{-2} + (12)(1000)(1.08)^{-12}}{(1000)(1.08)^{-2} + (1000)(1.08)^{-12}} = 5.165633881s \text{ years}$$

Theorem. Let $r > 0$. If the Macaulay duration of the cashflow

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

is $\bar{d}$, then the Macaulay duration of the cashflow

Time in years	1	2	3	...	n
Contributions	rC_1	rC_2	rC_3	...	rC_n

is $\bar{d}$.

Proof. The duration of the modified cashflow is

$$\frac{\sum_{j=1}^{n} jrC_j\nu^j}{\sum_{j=1}^{n} rC_j\nu^j} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \bar{d}$$

Example. The Macaulay duration of a 10-year annuity-immediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10year annuityimmediate with annual payments of \$50000.

Solution. Note that $50000 = 1000 \times 50$, so from the above theory where $r = 50$, we have that the duration of both cash flows is 5.6 years.

Theorem. If the Macaulay duration of the cashflow

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

is $\bar{d}$, then the Macaulay duration of the cashflow

Time in years	$t+1$	$t+2$	$t+3$	...	$t+n$
Contributions	C_1	C_2	C_3	...	C_n

is $t + \bar{d}$.

Proof. The duration of the modified cashflow is

$$\frac{\sum_{j=1}^{n}(t+j)C_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{\sum_{j=1}^{n} tC_j\nu^j + \sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{\sum_{j=1}^{n} tC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} + \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = t + \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = t + \bar{d}$$

Example. The Macaulay duration of a 10-year annuity-immediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10-year annuity-due with annual payments of \$5000.

Solution. Since the cashflow of an annuity-due is obtained from the cashflow of an annuity-immediate by transfaring payments 1 year, the answer is $5.6 - 1 = 4.6$ years.

Theorem. Suppose that two cashflows have durations $\bar{d}_1$ and $\bar{d}_2$, respectively, present values P_1 and P_2, respectively. Then, the duration of the combined cashflow is

$$\bar{d} = \frac{P_1\ \bar{d}_1 + P_2\ \bar{d}_2}{P_1 + P_2}$$

Proof. Suppose that the considered cashflows are

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

Time in years	1	2	3	...	n
Contributions	D_1	D_2	D_3	...	D_n

Then, the combined cashflow is

Time in years	1	2	3	...	n
Contributions	$C_1 + D_1$	$C_2 + D_2$	$C_3 + D_3$	...	$C_n + D_n$

We have that $P_1 = \sum_{j=1}^n C_j \nu^j$ and $P_2 = \sum_{j=1}^n D_j \nu^j$. By definition of duration,

$$\bar{d}_1 = \frac{\sum_{j=1}^n jC_j \nu^j}{\sum_{j=1}^n C_j \nu^j} = \frac{\sum_{j=1}^n jC_j \nu^j}{P_1} \text{ and } \bar{d}_2 = \frac{\sum_{j=1}^n jD_j \nu^j}{\sum_{j=1}^n D_j \nu^j} = \frac{\sum_{j=1}^n jD_j \nu^j}{P_2}$$

Hence,

$$\bar{d} = \frac{\sum_{j=1}^n j(C_j + D_j)\nu^j}{\sum_{j=1}^n (C_j + D_j)\nu^j} = \frac{\sum_{j=1}^n jC_j \nu^j + \sum_{j=1}^n jD_j \nu^j}{\sum_{j=1}^n C_j \nu^j + \sum_{j=1}^n D_j \nu^j} = \frac{\bar{d}_1 P_1 + \bar{d}_2 P_2}{P_1 + P_2}$$

Note that $\bar{d}_1 P_1 = \frac{\sum_{j=1}^n j\ C_j \nu^j}{P_1} P_1 = \sum_{j=1}^n jC_j \nu^j$ and $\bar{d}_2 P_2 = \frac{\sum_{j=1}^n j\ D_j \nu^j}{P_2} P_2 = \sum_{j=1}^n jD_j \nu^j$

Section 6.1: Macaulay duration

Note. By induction the previous formula holds for a combination of finitely many cashflows. Suppose that we have n cashflows. The j-the cashflow has present value P_j and duration . Then, the duration of the combined cashflow is

$$\frac{\sum_{j=1}^{n} P_j \bar{d}_j}{\sum_{j=1}^{n} P_j}$$

Example. An insurance has the following portfolio of investments:
(i) Bonds with a value of \$1,520,000 and duration 4.5 years.
(ii) Stock dividends payments with a value of \$1,600,000 and duration 14.5 years.
(iii) Certificate of deposits payments with a value of \$2,350,000 and duration 2 years.
Calculate the duration of the portfolio of investments.

Solution. The duration of the portfolio is

$$\frac{\sum_{j=1}^{n} P_j \bar{d}_j}{\sum_{j=1}^{n} P_j} = \frac{(4.5)(1,520,000) + (14.5)(1,600,000) + (2)(2,350,000)}{1,520,000 + 1,600,000 + 2,350,000} = 6.351005484 \text{ years}$$

Theorem. The Macaulay duration of a level payments annuityimmediate is $\bar{d} = \dfrac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}}$

Proof. We have that $\bar{d} = \dfrac{\sum_{j=1}^{n} jP\nu^j}{\sum_{j=1}^{n} P\nu^j} = \dfrac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}}$.

Example. Calculate Macaulay the duration of a 15-year annuity immediate with level payments if the current effective interest rate per annum is 5%.

Solution. The Macaulay the duration is

$$\bar{d} = \frac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}} = \frac{(Ia)_{\overline{15|}5\%}}{a_{\overline{15|}5\%}} = \frac{73.66768937}{10.37965804} = 7.097313716$$

Remember. $(Ia)_{\overline{n|}} = \dfrac{\ddot{a}_{\overline{n|}} - n\nu^n}{i}$

Theorem. The duration of a level payments perpetuity-immediate is

$$\bar{d} = \frac{1+i}{i}$$

Proof. We have that $\bar{d} = \dfrac{\sum_{j=1}^{\infty} jP\nu^j}{\sum_{j=1}^{\infty} P\nu^j} = \dfrac{(Ia)_{\overline{\infty|}i}}{a_{\overline{\infty|}i}} = \dfrac{\frac{1+i}{i^2}}{\frac{1}{i}} = \frac{1+i}{i}$

Example. Suppose that the Macaulay duration of a perpetuity immediate with level payments of 1000 at the end of each year is 21. Find the current effective rate of interest.

Solution. We have that $\bar{d} = \frac{1+i}{i} = 21 \Rightarrow 1 + i = 21i \Rightarrow 20i = 1 \Rightarrow i = \frac{1}{20} = 5\%$

Theorem. The duration of n year bond with $r\%$ annual coupons, face value F and redemption value C is

$$\bar{d} = \frac{Fr(Ia)_{\overline{n|}i} + Cn\nu^n}{Fra_{\overline{n|}i} + C\nu^n}$$

Proof. We have the cashflow

Time in years	1	2	...	$n-1$	n
Contributions	Fr	Fr	...	Fr	$Fr + C$

the duration is

$$\bar{d} = \frac{Fr\sum_{j=1}^{n} j\nu^j + Cn\nu^n}{Fr\sum_{j=1}^{n} \nu^j + C\nu^n} = \frac{Fr(Ia)_{\overline{n|}i} + Cn\nu^n}{Fra_{\overline{n|}i} + C\nu^n}$$

Example. Megan buys a 10-year 1000-face-value bond with a redemption value of 1200 which pay annual coupons at rate 7.5%. Calculate the Macaulay duration if the effective rate of interest per annum is 8%.

Solution. We have that $\bar{d} = \frac{Fr(Ia)_{\overline{n|}i}+Cn\nu^n}{Fra_{\overline{n|}i}+C\nu^n}$, so

$$\bar{d} = \frac{(1000)(0.075)(Ia)_{\overline{10|}8\%} + (1200)(10)(1.08)^{-10}}{(1000)(0.075)a_{\overline{10|}8\%} + (1200)(1.08)^{-10}} = \frac{(75)(32.68691288)+5558.321857}{(75)(6.710081399)+555.8321857} = 7.562958059$$

Section 6.2: Volatility (Modified duration)

- The modified duration (also referred to as the volatility) represents the rate of change in price as a percent of price.

- The modified duration ($\bar{\nu}$) is defined as the negative of the derivative $\frac{dP}{di}$, divided by the price P:

$$\bar{\nu} = -\frac{\frac{dP}{di}}{P} = -\frac{P'}{P} = -\frac{d\ln(P)}{di}$$

- We know that $P = \sum_{j=1}^{n} C_j \nu^j = \sum_{j=1}^{n} C_j (1+i)^{-j}$, then $P' = \sum_{j=1}^{n} C_j(-j)(1+i)^{-(j+1)}$. So the modified duration is

$$\begin{aligned} \bar{\nu} &= \frac{\sum_{j=1}^{n} jC_j \nu^{j+1}}{\sum_{j=1}^{n} C_j \nu^j} \\ &= \nu \frac{\sum_{j=1}^{n} jC_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j} \\ &= \nu \bar{d} \qquad \text{Remember. } \bar{d} = \frac{\sum_{j=1}^{n} jC_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j} \end{aligned}$$

- The modified duration measures the loss of present value of the cash flow as i increases relative to the present value of the cash flow.

- The modified duration is measured in years.

- Since $\bar{\nu} = \nu \bar{d}$, we have that the modified duration satisfies some of the properties of the duration. Suppose that we have n cash flows. The j-the cashflow has present value P_j and duration $\bar{\nu}_j$. Then, the duration of the combined cashflow is

$$\bar{\nu} = \frac{\sum_{j=1}^{n} P_j \bar{\nu}_j}{\sum_{j=1}^{n} P_j}$$

Example. A portfolio consists of four bonds. The prices and modified durations of the four bonds are given by the table:

Bond	Present value	Modified duration in years
Bond A	\$ 15050	4.3
Bond B	\$ 10350	10.4
Bond C	\$ 67080	7.6
Bond D	\$ 16750	6.5

Find the modified duration (or the volatility) of the whole portfolio.

Solution. $\bar{\nu} = \frac{\sum_{j=1}^{n} P_j \bar{\nu}_j}{\sum_{j=1}^{n} P_j} = \frac{(15050)(4.3)+(10350)(10.4)+(67080)(7.6)+(16750)(6.5)}{15050+10350+67080+16750} = 7.241948183$ years

■ Since P, $i \geq 0$, is a decreasing function of i, then $\ln(P)$ is a decreasing function of i. Hence,

$$0 < -\frac{d\ln(P)}{di} = -\frac{P'}{P} = -\frac{\sum_{j=1}^{n} C_j(-j)\nu^{(j+1)}}{\sum_{j=1}^{n} C_j \nu^j} = \bar{\nu}$$

This implies that if $C_j \geq 0$ for $1 \leq j \leq n$, $\bar{\nu} > 0$

■ Note that
$P' = \sum_{j=1}^{n} C_j(-j)(1+i)^{-(j+1)} \Rightarrow P'' = \sum_{j=1}^{n} C_j\ (-j)(-(j+1))(1+i)^{-(j+2)} = \sum_{j=1}^{n} C_j\ j(j+1)\nu^{(j+2)} > 0$ for each $i \geq 0$. This implies that P is a decreasing convex function (concave upward) on i.

■ Let h be change rate in interest rates (h close to zero), so we say interest rates change from i into $i+h$. Let $P(i)$ be the present value of a portfolio, when i is the effective rate of interest. By a Taylor expansion,

$$P(i+h) \approx P(i) + P'(i)h = P(i) + (-\bar{\nu}P(i)h) = P(i) - P(i)\bar{\nu}h = P(i)(1-\bar{\nu}h) = P(i)(1-\nu\bar{d}h)$$

Example. A portfolio of bonds is worth 535000 at the current rate of interest of 4.75%. Its Macaulay duration is 6.375. Estimate the value of the portfolio if interest rates decrease by 0.10%.

Solution. We have $P = 535000$, $i = 4.75\%$, $\bar{d} = 6.375$ and $h = -0.10\%$. From $P(i+h) \approx P(i)(1-\nu\bar{d}h)$, we have

$$P(0.0475 - 0.0010) \approx 535000(1 - \frac{1}{(1.0475)}(6.375)(-0.001)) = 538255.9666$$

■ If interest rates change from i into $i+h$, **the percentage of change in the present value** of the portfolio is

$$\frac{P(i+h) - P(i)}{P(i)} \approx \frac{P(i)(1-\nu\bar{d}h) - P(i)}{P(i)} = \frac{P(i)(1-\nu\bar{d}h-1)}{P(i)} = 1 - \nu\bar{d}h - 1 = -\nu\bar{d}h = -\bar{v}h$$

Example. A bond has a volatility of 4.5 years, at the current annual interest rate of 5%. Calculate the percentage of loss of value of the bond if the annual effective interest rate increase 250 basis points.

Solution. **Note:** 1 basis point $\frac{1}{100}\% = 0.01\%$, so when you say "the annual effective interest rate increases 250 basis points," it means the rate rises by $\frac{250}{100}\% = 2.5\%$.

Now, we have $\bar{v} = 4.5$ and $h = 2.5\%$, so the percentage of change is $-\bar{v}h = -(4.5)(2.5\%) = -0.1125 = -11.25\%$. The bond loses 11.25% of its value.

Remember. The price of a bond decreases as the rate of interest increases.

■ Suppose that you believe that interest rates will drop soon. You want to make a benefit by buying a bond today and selling it later for a higher price. The profit you make is

$$P(i+h) - P(i)$$

where i is **the interest you buy the bond** and $i+h$ is **the interest rate when you sell the bond**.

Notice that you make a benefit if $h < 0$ (i.e. interest falls and therefore price rises). The rate of return in your investment is

$$\frac{P(i+h) - P(i)}{P(i)} \approx -\bar{\nu}h$$

■ Of all the possible bonds, you will make the most profit by investing in the bond with the highest possible volatility.

Example. Suppose that you are comparing two five-year bonds with a face value of 1000, and are expecting a drop in yields of 1% almost immediately. The current yield is 8%. Bond 1 has 6% annual coupons and bond 2 has annual 12% coupons. You would like to invest 100,000 in the bond giving you the biggest return.

(1) Which would provide you with the highest potential gain if your outlook for rates actually occurs?

(2) Find the duration of each bond.

Example. Suppose that you are comparing two fiveyear bonds with a face value of 1000, and are expecting a drop in yields of 1% almost immediately. The current yield is 8%. Bond 1 has 6% annual coupons and bond 2 has annual 12% coupons. You would like to invest 100,000 in the bond giving you the biggest return.

(1) Which would provide you with the highest potential gain if your outlook for rates actually occurs?

(2) Find the duration of each bond.

Solution. We have $F = C = 1000$, $n = 5$, $i = 8\%$ and $h = -1\%$
(1) For **Bond 1:** $r = 6\%$ and $Fr = 60$
The price of the bond 1: $P = (60)\ a_{\overline{5}|8\%} + (1000)(1.08)^{-5} = 920.15$
The price of the bond 1 after the change of interest rates: $P = (60)\ a_{\overline{5}|7\%} + (1000)(1.07)^{-5} = 959.00$
The gain is $959.00 - 920.15 = 38.85$. The percentage of change is $\frac{959.00-920.15}{920.15} = 4.22\%$

For **Bond 2:** $r = 12\%$ and $Fr = 120$
The price of the bond 2: $P = (120)\ a_{\overline{5}|8\%} + (1000)(1.08)^{-5} = 1159.71$
The price of the bond 2 after the change of interest rates: $P = (120)\ a_{\overline{5}|7\%} + (1000)(1.07)^{-5} = 1205.01$
The gain is $1205.01 - 1159.71 = 45.30$. The percentage of change is $\frac{1205.01-1159.71}{1159.71} = 3.91\%$
Bond 1 is a better investment than bond 2

(2) For **Bond 1:** the price $P = 920.15$ and its Macaulay duration is

$$\bar{d} = \frac{Fr(Ia)_{\overline{n}|i} + Cn\nu^n}{Fra_{\overline{n}|i} + C\nu^n} = \frac{60(Ia)_{\overline{5}|8\%} + (1000)(5)(1.08)^{-5}}{60a_{\overline{5}|8\%} + (1000)(1.08)^{-5}} = 4.4393$$

For **Bond 2:** the price $P = 1159.71$ and its Macaulay duration is

$$\bar{d} = \frac{120(Ia)_{\overline{5}|8\%} + (1000)(5)(1.08)^{-5}}{120a_{\overline{5}|8\%} + (1000)(1.08)^{-5}} = 4.1103$$

The convexity of the cashflow

Time in years	1	2	...	n
Contributions	C_1	C_2	...	C_n

is the second derivative of price with respect to interest rate, expressed as a percent of price:

$$\bar{c} = \frac{P''}{P} = \frac{\sum_{j=1}^{n} C_j \, j \, (j+1)\nu^{(j+2)}}{\sum_{j=1}^{n} C_j \nu^j}$$

Note. As was mentioned in connection with modified duration, this formula is based on equally spaced cash flows.

Notes.

- Convexity is measured in years ^{2}.
- Convexity is typically measured in percentage change in price per unit change in yield squared.
- Convexity measures the rate of change of the volatility (Modified duration):

$$\frac{d}{di}\bar{\nu} = -\frac{d}{di}\frac{P'}{P} = -\frac{P''P - P'P'}{P^2} = -\frac{P''P}{P^2} + \frac{(P')^2}{P^2} = -\frac{P''}{P} + \left(\frac{P'}{P}\right)^2 = \bar{\nu}^2 - \bar{c}$$

- The second order Taylor expansion of the present value with respect to the yield is:

$$P(i+h) \approx P(i) + P'(i)h + \frac{h^2}{2}P''(i) = P(i) - \bar{\nu}P(i)h + \frac{h^2}{2}\bar{c}P(i) = P(i)(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c})$$

$\bar{\nu} = -\frac{P'}{P} \Rightarrow P' = -P\,\bar{\nu}$ and $\bar{c} = \frac{P''}{P} \Rightarrow P'' = P\,\bar{c}$

■ Using duration and convexity, we measure of how sensitive the present value of a cash flow is to interest rate changes and we have the following Taylor expansion:

$$P(i+h) \approx P(i)(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c})$$

The percentage change in the PV of a cashflow is

$$\frac{P(i+h) - P(i)}{P(i)} \approx \frac{P(i)(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c}) - P(i)}{P(i)} = \frac{P(i)(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c} - 1)}{P(i)} = -\bar{\nu}h + \frac{h^2}{2}\bar{c}$$

■ Convexity can be used to compare bonds. If two bonds offer the same duration and yield but one exhibits greater convexity, the bond with greater convexity is more affected by interest rates.

Example. A portfolio of bonds is worth 350000 at the current rate of interest of 5.2%. Its modified duration is 7.22. Its convexity is 370. Estimate the value of the portfolio if interest rates increase by 0.2%.

Solution. We have that $P = 350000$, $i = 5.2\%$, $h = 0.2\%$, $\bar{\nu} = 7.22$ and $\bar{c} = 370$
From $P(i+h) \approx P(i)(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c})$, we have $P(5.2\% + 0.2\%) \approx (350000)(1 - (7.22)(0.2\%) + \frac{(0.2\%)^2}{2}(370)) = 345205$

Example. Calculate the duration, the modified duration and the convexity of a \$5000 face value 15-year zero-coupon bond if the current effective annual rate of interest is 7.5%.

Solution. *Remember.* For zero-coupon bond, the price is $P = F\ \nu^n$. Now, we have $n = 15$, $F = 5000$ and $i = 7.5\%$, so

$$P = 5000(1+i)^{-15} \Rightarrow P' = 5000(-15)(1+i)^{-16} \Rightarrow P'' = 5000(-15)(-16)(1+i)^{-17}$$

For $i = 7.5\%$, we have

(1) The modified duration: $\bar{\nu} = -\frac{P'}{P} = \frac{5000(-15)(1+i)^{-16}}{5000(1+i)^{-15}} = 15(1+7.5\%)^{-1} = 13.95348837$ years

(2) The duration: $\bar{d} = (1+i)\bar{\nu} = (1.075)(13.95348837) = 15$ years

(3) The convexity: $\bar{c} = \frac{P''}{P} = \frac{5000(-15)(-16)(1+i)^{-17}}{5000(1+i)^{-15}} = (-15)(-16)(1+0.75)^{-2} = 78.36734694$ years ^{2}.

Example. Calculate the duration, the modified duration and the convexity of a level payments perpetuity-immediate with payments at the end of the year if the current effective annual rate of interest is 5%.

Solution. For perpetuity-immediate , we have $P = \frac{k}{i} \Rightarrow P' = \frac{-k}{i^2} \Rightarrow P'' = \frac{2k}{i^3}$.

(1) The modified duration: $\bar{\nu} = -\frac{P'}{P} = -\frac{-\frac{k}{i^2}}{\frac{k}{i}} = \frac{1}{i} = \frac{1}{0.05} = 20$ years.

(2) The duration: $\bar{d} = (1+i)\bar{\nu} = (1.05)(20) = 21$ years.

(3) The convexity: $\bar{c} = \frac{P''}{P} = \frac{\frac{2k}{i^3}}{\frac{k}{i}} = \frac{2}{i^2} = \frac{2}{(0.05)^2} = 800$ years ^{2}

Example. A 100 par value 3 year bond pays annual coupons at a rate 7% coupon rate (with annual coupon payments). The current annual effective interest rate is 7%.
(1) Calculate the duration, the modified duration and the convexity of the bond.
(2) If the interest rate change from 7% to 8%, what is the percentage change in the price of the bond?
(3) Using the duration rule, including convexity, what is the percentage change in the bond price?

Solution. We have $F = C = 100$, $r = 7\%$, $n = 3$, $i = 7\%$ and $Fr = (100)(7\%) = 7$.
(1) The cashflow is

Time in years	1	2	3
Contributions	7	7	7+100

$$\text{The duration: } \bar{d} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{(1)(7)(1.07)^{-1} + (2)(7)(1.07)^{-2} + (3)(107)(1.07)^{-3}}{(7)(1.07)^{-1} + (7)(1.07)^{-2} + (107)(1.07)^{-3}} = 2.808018 \text{ years}$$

The modified duration $\bar{\nu} = \nu\bar{d} = \frac{\bar{d}}{(1+i)} = \frac{2.808018}{1.07} = 2.6243$ years.

The convexity: $\bar{c} = \frac{\sum_{j=1}^{n} C_j\ j\ (j+1)\nu^{(j+2)}}{\sum_{j=1}^{n} C_j\nu^j} = \frac{(7)(1)(2)(1.07)^{-3}+(7)(2)(3)(1.07)^{-4}+(107)(3)(4)(1.07)^{-5}}{(7)(1.07)^{-1}+(7)(1.07)^{-2}+(107)(1.07)^{-3}} = 9.58944$ years.

(2) If i = 7%, the price of the bond is $P = Fr\ a_{\overline{n}|i} + C(1+i)^{-n} = 7\ a_{\overline{3}|7\%} + (100)(1.07)^{-3} = 100$

If i = 8%, the price of the bond is $P = 7\ a_{\overline{3}|8\%} + (100)(1.08)^{-3} = 97.4229$

The change in percentage is $\frac{97.4229-100}{100} = -2.5771\%$.

(3) The estimation in the change in percentage is $-\bar{\nu}h + \frac{h^2}{2}\bar{c} = -(2.6243)(0.01) + \frac{(0.01)2}{2}(9.58944) = -2.576353\%$