Financial Mathematics

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Main Content

Annuity. An annuity is a sequence of cash flows (paid/received) made at equal intervals of time. For example, house rents, mortgage payments and interest payments on money invested.

Notes:

- The interval between annuity payments is called payments period.
- An annuity is said to have level payments if all payments C_j are equal.
- An annuity has nonlevel payments if some payments C_j are different from other ones.
- The payments can be made either at **the beginning** or at **the end** of intervals of time:

Annuity-immediate. For an annuity-immediate, payments are made at the end of the intervals of time.

Example: Consider the intervals: $[0, 1], [1, 2], [2, 3], \ldots[n-1, n]$, the annuity–immediate payments are made at the end of the periods: $[0, 1], [1, 2], [2, 3], \ldots[n-1, n]$.

The cashflow under the annuity–immediate is of the type:

Time	0	1	2	...	n
Contributions	0	C_1	C_2	...	C_n

Annuity-due. For an annuity–due, payments are made at the beginning of the intervals of time.

Example: Consider the intervals: $[0, 1], [1, 2], [2, 3], \ldots[n-1, n]$, the annuity–due payments are made at the end of the periods: $[0, 1], [1, 2], [2, 3], \ldots[n-1, n]$.

The cashflow under the annuity–due is of the type:

Time	0	1	...	$n-1$	n
Contributions	C_0	C_1	...	C_{n-1}	0

Section 2.2: Annuity-immediate calculation

Recall.

Present value of a cashflow: If deposits/withdrawals are made according with the table

Time (in periods)	t_1	t_2	...	t_n
Investments	C_1	C_2	...	C_n

Remember. Under the accumulation function $a(t)$,

- The present value of the considered cashflow at time zero is $PV(t) = \sum_{j=1}^{n} C_j \frac{1}{a(t_j)}$
- The present value at time t of the cashflow (the future value or the equation of value) is $FV(t) = \sum_{j=1}^{n} C_j \frac{a(t)}{a(t_j)}$

Now, consider an annuity-immediate with level payments of 1 (A **unit annuity** is one for which each regular payment is 1). The cashflow of the annuity is

Time	0	1	2	...	n
Contributions	0	1	1	...	1

- The present value of annuity–immediate:

$$a_{\overline{n|}} = \frac{1}{a(1)} + \frac{1}{a(2)} + \cdots + \frac{1}{a(n)} = \sum_{j=1}^{n} \frac{1}{a(j)}$$

- The future value of an annuity–immediate:

$$s_{\overline{n|}} = \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \cdots + \frac{a(n)}{a(n)} = \sum_{j=1}^{n} \frac{a(n)}{a(j)}$$

Example: You are given that $\delta_t = \frac{1}{8+t}$, $t > 0$. Find $a_{\overline{n|}}$ and $s_{\overline{n|}}$.

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Solution: *Remember.* $a(t) = e^{\int_0^t \delta_s \, ds}$.

$$\int_0^t \delta_s \, ds = \int_0^t \frac{1}{8+s} \, ds = \ln(8+s)\Big|_0^t = \ln(8+t) - \ln(8) = \ln\left(\frac{8+t}{8}\right) \Rightarrow a(t) = e^{\ln\left(\frac{8+t}{8}\right)} = \frac{8+t}{8}$$

$$a_{\overline{n|}} = \sum_{j=1}^{n} \frac{1}{a(j)} = \sum_{j=1}^{n} \frac{8}{8+j}$$

$$s_{\overline{n|}} = \sum_{j=1}^{n} \frac{a(n)}{a(j)} = \sum_{j=1}^{n} \frac{\frac{8+n}{8}}{\frac{8+j}{8}} = \sum_{j=1}^{n} \frac{8+n}{8+j}$$

Example: You are given that $\delta_t = \frac{1}{8+t}$, $t > 0$. Find $a_{\overline{n|}}$ and $s_{\overline{n|}}$.

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- *Remember in Chapter 1.* Under compound interest $a(t) = (1+i)^t$:
- The present value of the considered cashflow at time zero is

$$PV(t) = \sum_{j=1}^{n} C_j \frac{1}{(1+i)^{t_j}} = \sum_{j=1}^{n} C_j (1+i)^{-t_j} = \sum_{j=1}^{n} C_j \nu^{t_j}$$

Remember in Chapter 1. $\nu = \frac{1}{1+i}$

- The present value at time t of the cashflow (the future value or the equation of value) is

$$FV(t) = \sum_{j=1}^{n} C_j \frac{(1+i)^t}{(1+i)^{t_j}} = \sum_{j=1}^{n} C_j (1+i)^{t-t_j}$$

Section 2.2: Annuity-immediate calculation

- Consider unit annuity under which payments of 1 are made at the end of each period for n periods:

Time	0	1	2	...	n
Contributions	0	1	1	...	1

- The **present value** of annuity–immediate is the sum of the individual present values of the payments of 1:

$$a_{\overline{n|}} = \sum_{j=1}^{n} \nu^j$$

$$= \nu + \nu^2 + \nu^3 + \cdots + \nu^{n-1} + \nu^n$$

$$= \nu \frac{1-\nu^n}{1-\nu} \quad \text{geometric series:} 1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}, r \neq 1$$

$$= \nu \frac{1-\nu^n}{i\nu} \quad \text{Remember: } 1 - \nu = d \text{ and } d = i\nu \Rightarrow 1 - \nu = i\nu$$

$$= \frac{1-\nu^n}{i}$$

- The **future value** of annuity–immediate is the sum of the individual future values of the payments of 1:

$$s_{\overline{n|}} = \sum_{j=1}^{n} (1+i)^{n-j}$$

$$= (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1$$

$$= \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}$$

Section 2.2: Annuity-immediate calculation

Theorem. If $i \neq 0$, then for an annuity-immediate with level payments of one, we have

- $a_{\overline{n|}i} = a_{\overline{n|}} = \frac{1-\nu^n}{i} = \frac{1-(1+i)^{-n}}{i}$
- $s_{\overline{n|}i} = s_{\overline{n|}} = \frac{(1+i)^n-1}{i}$

- *Notes.*
- **Annuities with level payments other than 1.** If an immediate annuity has payment P, then

- The present value: $PV = P\, a_{\overline{n|}}$
- The future value: $FV = P\, s_{\overline{n|}}$

- **Relationships between present and future values.**

- $s_{\overline{n|}} = (1+i)^n a_{\overline{n|}} = \frac{(1+i)^n-1}{i}$

Proof. $(1+i)^n a_{\overline{n|}} = (1+i)^n\ \frac{1-\nu^n}{i} = \frac{(1+i)^n-(1+i)^n\nu^n}{i} = \frac{(1+i)^n-1}{i} = s_{\overline{n|}}$

- $a_{\overline{n|}} = \nu^n s_{\overline{n|}}$

Proof. $\nu^n s_{\overline{n|}} = \nu^n\ \frac{(1+i)^n-1}{i} = \frac{\nu^n\ (1+i)^n-\nu^n}{i} = \frac{1-\nu^n}{i}$

Example: Calculate the present and future values of $5000 paid at the end of each year for 15 years using an annual effective interest rate of 7.5%.

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Proof. $(1+i)^n a_{\overline{n}|} = (1+i)^n\, \frac{1-\nu^n}{i} = \frac{(1+i)^n-(1+i)^n\nu^n}{i} = \frac{(1+i)^n-1}{i} = s_{\overline{n}|}$

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Proof. $\nu^n s_{\overline{n}|} = \nu^n\, \frac{(1+i)^n-1}{i} = \frac{\nu^n\,(1+i)^n-\nu^n}{i} = \frac{1-\nu^n}{i}$

Example: Calculate the present and future values of \$5000 paid at the end of each year for 15 years using an annual effective interest rate of 7.5%.

Solution: $PV = 5000 a_{\overline{15}|0.075} = (5000)\frac{1-(1+0.075)^{-15}}{0.075} = 44135.59873$

$FV = 5000 s_{\overline{15}|0.075} = (5000)\frac{(1+0.075)^{15}-1}{0.075} = 130591.824$

Example: If $i = 5\%$ and $n = 10$, find $s_{\overline{n|}}$.

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Solution:

$$s_{\overline{n|}i} = s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}$$

$$\Rightarrow s_{\overline{10|}0.05} = \frac{(1+0.05)^{10} - 1}{0.05} = 12.5779$$

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Example: If $n = 15$ and $i = 6\%$, find
(1) $a_{\overline{n|}}$
(2) $s_{\overline{n|}}$.

Example: If $i = 5\%$ and $n = 10$, find $s_{\overline{n|}}$.

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Example: If $n = 15$ and $i = 6\%$, find
(1) $a_{\overline{n|}}$
(2) $s_{\overline{n|}}$.

Solution:
(1)

$$a_{\overline{15|}} = \frac{1 - (1+0.06)^{-15}}{0.06} = 9.712$$

(2)

$$s_{\overline{15|}} = \frac{(1+0.06)^{15} - 1}{0.06} = 23.276$$

- Consider unit annuity under which payments of 1 are made at the beginning of each period for n periods:

Time	0	1	...	$n-1$	n
Contributions	1	1	...	1	0

- The present value of annuity–due:

$$\ddot{a}_{\overline{n|}} = 1 + \frac{1}{a(1)} + \frac{1}{a(2)} + \cdots + \frac{1}{a(n-1)} = \sum_{j=0}^{n-1} \frac{1}{a(j)}$$

- The future value of an annuity–due:

$$\ddot{s}_{\overline{n|}} = a(n) + \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \cdots + \frac{a(n)}{a(n-1)} = \sum_{j=0}^{n-1} \frac{a(n)}{a(j)}$$

- Now, we want to drive an expression for $\ddot{a}_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$ under compound interest $a(t) = (1+i)^t$:
- The present value of annuity–due is the sum of the individual present values of the payments of 1:

$$\begin{aligned}\ddot{a}_{\overline{n|}} &= \sum_{j=0}^{n-1} \nu^j = 1 + \nu + \nu^2 + \nu^3 + \cdots + \nu^{n-1} \\ &= \frac{1-\nu^n}{1-\nu} \qquad \text{geometric series:} 1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}, r \neq 1 \\ &= \frac{1-\nu^n}{d} \qquad d = 1 - \nu\end{aligned}$$

- The future value of annuity–due is the sum of the individual future values of the payments of 1:

$$\begin{aligned}
\ddot{s}_{\overline{n|}} &= \sum_{j=0}^{n-1}(1+i)^{n-j} \\
&= (1+i)+(1+i)^2+(1+i)^3+\cdots+(1+i)^{n-1}+(1+i)^n \\
&= (1+i)\frac{(1+i)^n-1}{(1+i)-1} \\
&= \frac{(1+i)^n-1}{d} \qquad \text{Remember: } d=\frac{i}{1+i}
\end{aligned}$$

Theorem. If $i \neq 0$, then for an annuity-due with level payments of one, we have

- $\ddot{a}_{\overline{n|}} = \frac{1-\nu^n}{d}$
- $\ddot{s}_{\overline{n|}} = \frac{(1+i)^n-1}{d}$

- *Notes.*

- **Annuities with level payments other than 1.** If due annuity has payment P, then

- $PV = P\ \ddot{a}_{\overline{n|}}$
- $FV = P\ \ddot{s}_{\overline{n|}}$

- **Relationships between annuity-immediate and annuity-due**
- $\ddot{a}_{\overline{n|}} = \frac{i}{d}\ a_{\overline{n|}} = (1+i)\ a_{\overline{n|}}$

Proof.

$$\begin{aligned}
\ddot{a}_{\overline{n|}} &= \frac{1-\nu^n}{d} = i\,\frac{1-\nu^n}{i\ d} = \frac{i}{d}\ \frac{1-\nu^n}{i} \\
&= \frac{i}{d}\ a_{\overline{n|}} = (1+i)\ a_{\overline{n|}} \qquad \text{Remember: } d=\frac{i}{1+i} \Rightarrow 1+i=\frac{i}{d} \text{ and } a_{\overline{n|}}=\frac{1-\nu^n}{i}
\end{aligned}$$

■ $\ddot{s}_{\overline{n|}} = \frac{i}{d}\ s_{\overline{n|}} = (1+i)\ s_{\overline{n|}}$

Proof.

$$\begin{aligned}\ddot{s}_{\overline{n|}} &= \frac{(1+i)^n - 1}{d}\\ &= i\,\frac{(1+i)^n - 1}{i\ d}\\ &= \frac{i}{d}\ \frac{(1+i)^n - 1}{i} = \frac{i}{d}\ s_{\overline{n|}}\\ &= (1+i)\ s_{\overline{n|}}\end{aligned}$$

Remember: $d = \frac{i}{1+i} \Rightarrow 1+i = \frac{i}{d}$ and $s_{\overline{n|}} = \frac{(1+i)^n - 1}{d}$

■ $\ddot{s}_{\overline{n|}} = (1+i)^{n+1} a_{\overline{n|}}$

Proof.

$$\begin{aligned}(1+i)^{n+1} a_{\overline{n|}} &= (1+i)^{n+1}\,\frac{1-\nu^n}{i}\\ &= (1+i)\ \frac{(1+i)^n.1 - (1+i)^n \nu^n}{i}\\ &= \frac{(1+i)^n - 1}{d} = \ddot{s}_{\overline{n|}}\end{aligned}$$

$(1+i)^{n+1} = (1+i).(1+i)^n$

Example: If $i = 5\%$ and $n = 10$, find $\ddot{a}_{\overline{n|}}$.

Section 2.3: Annuity-due calculation

Example: If $i = 5\%$ and $n = 10$, find $\ddot{a}_{\overline{n|}}$.

Solution: We know that $\ddot{a}_{\overline{n|}} = \frac{1-\nu^n}{d}$ where $\nu = \frac{1}{1+i}$ and $d = \frac{i}{1+i}$

So,

$$\ddot{a}_{\overline{10|}} = \frac{1 - \left(\frac{1}{1+0.05}\right)^{10}}{\frac{0.05}{1+0.05}} = 8.1078$$

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Example: If $i = 6\%$, find $\ddot{a}_{\overline{15|}}$ and $\ddot{s}_{\overline{15|}}$.

Solution:

$$\ddot{a}_{\overline{15|}} = \frac{1 - (\frac{1}{1+0.06})^{15}}{\frac{0.06}{1+0.06}} = 10.295$$

$$\ddot{s}_{\overline{15|}} = \frac{(1+0.06)^{15} - 1}{\frac{0.06}{1+0.06}} = 24.673$$

Example: If $i = 5\%$ and $n = 10$, find $\ddot{a}_{\overline{n|}}$.

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Example: You want to accumulate $12,000 in a 5% account by making a level deposit at the beginning of each of the next 9 years. Find the required level payment.

Example: If $i = 5\%$ and $n = 10$, find $\ddot{a}_{\overline{n|}}$.

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So,

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Example: You want to accumulate \$12,000 in a 5% account by making a level deposit at the beginning of each of the next 9 years. Find the required level payment.

Solution: $FV = P\,\ddot{s}_{\overline{n|}} = P\,\frac{(1+i)^n - 1}{d}$ where

$$d = \frac{i}{1+i} = \frac{0.05}{1+0.05}$$

This implies

$$12000 = P\,\frac{(1.05)^9 - 1}{\frac{0.05}{1+0.05}} \Rightarrow P = \frac{12000}{\frac{(1.05)^9 - 1}{\frac{0.05}{1+0.05}}} = 1036.46$$

Section 2.3: Annuity-due calculation

Example: A loan for 8,000 must be repaid with 6 year end payments at an annual rate of 11%. What is the annual payment?

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Solution:

$$PV = P\, a_{\overline{n|}} \Rightarrow P = \frac{PV}{a_{\overline{n|}}}$$

$$\text{where } a_{\overline{n|}} = \frac{1-\nu^n}{i} \Rightarrow a_{\overline{n|}} = \frac{1-(\frac{1}{1+i})^6}{0.11} = 4.2305 \Rightarrow P = \frac{8000}{4.2305} = 1,891.01$$

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Example: You wish to make a deposit of now in an account earning 6% annually so that you can get a payment of 250 at the end of each of the next 8 years. How much should you deposit today?

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Example: You wish to make a deposit of now in an account earning 6% annually so that you can get a payment of 250 at the end of each of the next 8 years. How much should you deposit today?

Solution:

$$FV = P\, s_{\overline{n|}} = 250\,\frac{(1+0.06)^8-1}{0.06} = 2474.367$$

Now, $s_{\overline{n|}} = (1+i)^n\, a_{\overline{n|}} \Rightarrow a_{\overline{n|}} = \frac{s_{\overline{n|}}}{(1+i)^8} = \frac{2474.367}{(1+0.06)^8} = 1552.45$

OR

$$PV = Pa_{\overline{n|}i} \Rightarrow PV = 250a_{\overline{8|}6\%} = (250)(6.20979) = 1552.45$$

Section 2.4: Perpetuity

Perpetuity. A perpetuity is an annuity whose payments continue forever.

An example of a perpetuity is when someone purchases a property and then rents it out. The owner is entitled to an infinite stream of cash flow from the renter as long as the property continues to exist (assuming the renter continues to rent).

Perpetuity–immediate

The present value of a perpetuity–immediate that pays 1 per period is denoted by $a_{\overline{\infty|}}$

$$\begin{aligned} a_{\overline{\infty|}} &= \nu + \nu^2 + \nu^3 + \ldots \\ &= \frac{\nu}{1-\nu} \quad \text{Remember: } \sum_{j=1}^{n} \nu^{t_j} = \nu \frac{1-\nu^n}{1-\nu} \quad \text{Remember: } d = i\nu \text{ and } d = 1-\nu \\ &= \frac{\nu}{i\nu} \\ &= \frac{1}{i} \end{aligned}$$

OR

$$a_{\overline{\infty|}} = \lim_{n\to\infty} a_{\overline{n|}} = \lim_{n\to\infty} \frac{1-\nu^n}{i} = \frac{1}{i} \text{ where } \lim_{n\to\infty} \nu^n = \lim_{n\to\infty} \frac{1}{(1+i)^n} = 0$$

Perpetuity-due

The present value of a perpetuity–due that pays 1 per period is denoted by $\ddot{a}_{\overline{\infty|}}$

$$\ddot{a}_{\overline{\infty|}} = \lim_{n\to\infty} \ddot{a}_{\overline{n|}} = \lim_{n\to\infty} \frac{1-\nu^n}{d} = \frac{1}{d} = \frac{1+i}{i} \text{ where } \lim_{n\to\infty} \nu^n = 0$$

Section 2.4: Perpetuity

Theorem. Let $i \neq 0$.

- For an perpetuity-immediate,
  - With level payments of **one**: $a_{\overline{\infty|}} = \frac{1}{i}$
  - With level payments of **P**: $PV = P\, a_{\overline{\infty|}} = \frac{P}{i}$
- For an perpetuity-due,
  - With level payments of **one**: $\ddot{a}_{\overline{\infty|}} = \frac{1}{d} = \frac{1+i}{i}$
  - With level payments of **P**: $PV = P\, \ddot{a}_{\overline{\infty|}} = \frac{P}{d}$

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Example: If $i = 5\%$, find (1) $a_{\overline{\infty|}}$ (2) $\ddot{a}_{\overline{\infty|}}$.

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- For an perpetuity-due,
- With level payments of **one**: $\ddot{a}_{\overline{\infty|}} = \frac{1}{d} = \frac{1+i}{i}$
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Example: If $i = 5\%$, find (1) $a_{\overline{\infty|}}$ (2) $\ddot{a}_{\overline{\infty|}}$.

Solution:

(1) $a_{\overline{\infty|}} = \frac{1}{i} = \frac{1}{0.05} = 20$

(2) $\ddot{a}_{\overline{\infty|}} = \frac{1}{d} = \frac{1+i}{i} = \frac{1.05}{0.05} = 21$

Section 2.4: Perpetuity

Theorem. Let $i \neq 0$.

- For an perpetuity-immediate,
  - With level payments of **one**: $a_{\overline{\infty|}} = \frac{1}{i}$
  - With level payments of **P**: $PV = P\, a_{\overline{\infty|}} = \frac{P}{i}$
- For an perpetuity-due,
  - With level payments of **one**: $\ddot{a}_{\overline{\infty|}} = \frac{1}{d} = \frac{1+i}{i}$
  - With level payments of **P**: $PV = P\, \ddot{a}_{\overline{\infty|}} = \frac{P}{d}$

Example: If $i = 5\%$, find (1) $a_{\overline{\infty|}}$ (2) $\ddot{a}_{\overline{\infty|}}$.

Solution:

(1) $a_{\overline{\infty|}} = \frac{1}{i} = \frac{1}{0.05} = 20$

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Example: Find the present value of a perpetuity immediate with $i = 8\%$.

Solution: $a_{\overline{\infty|}} = \frac{1}{i} = \frac{1}{0.08} = 12.5$

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Example: John uses his retirement fund to buy a perpetuity-due of 20,000 per year based on an annual nominal interest 8% compounded monthly. Find Johns retirement fund.

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Example: If $i = 5\%$, find (1) $a_{\overline{\infty|}}$ (2) $\ddot{a}_{\overline{\infty|}}$.

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Example: Find the present value of a perpetuity immediate with $i = 8\%$.

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Example: John uses his retirement fund to buy a perpetuity-due of 20,000 per year based on an annual nominal interest 8% compounded monthly. Find Johns retirement fund.

Solution: We use

$$PV = \frac{P}{d} \text{ where } d = \frac{i}{1+i}$$

First, we have to find i.

Remember.

$$1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow 1 + i = (1 + \frac{0.08}{12})^{12} \Rightarrow i = (1 + \frac{0.08}{12})^{12} - 1 = 0.08299950681 = 8.299950681\%$$

$d = \frac{0.08299950681}{1.08299950681} = 0.076638545$, then $PV = \frac{20000}{0.076638545} = 260963.2872$

Continuous annuity. Continuous annuity is a special case of annuities payable more frequently than convertible interest and it is a state in which the frequency of payment becomes infinite, i.e. payments are made continuously.

The present value of an annuity payable continuously for n interest conversion periods, such that the total amount paid during each interest conversion period is 1, by the symbol $\bar{a}_{\overline{n}|}$ and is defined as follows:

$$\bar{a}_{\overline{n}|} = \int_0^n \nu^t \, dt = \int_0^n e^{t \ln \nu} \, dt \quad \text{Remember: } \nu^t = e^{\ln \nu^t} = e^{t \ln \nu}$$

$$= \int_0^n e^{-t \ \ln(1+i)} \, dt \quad \text{Remember: } \nu = \frac{1}{1+i} = (1+i)^{-1} \Rightarrow e^{\ln \nu} = e^{\ln(1+i)^{-1}} = e^{-\ln(1+i)}$$

$$= \int_0^n e^{-\delta t} \, dt \quad \text{Remember: } \delta = \ln(1+i)$$

$$= \frac{-1}{\delta} e^{-\delta t} \Big|_0^n \quad \text{Remember: } \int e^{at} \, dt = \frac{1}{a} \int a \, e^{at} \, dt = \frac{1}{a} e^{at} + c$$

$$= \frac{-1}{\delta} \left[e^{-\delta n} - 1 \right] = \frac{-1}{\delta} \left[(e^{-\delta})^n - 1 \right] = \frac{-1}{\delta} \left[(e^{\ln(1+i)^{-1}})^n - 1 \right]$$

$$= \frac{-1}{\delta} \left[(e^{\ln \nu})^n - 1 \right] = \frac{-1}{\delta} (\nu^n - 1) = \frac{1}{\delta} \, (1 - \nu^n)$$

OR

$$\bar{a}_{\overline{n}|} = \int_0^n \nu^t \, dt = \frac{1}{\ln \nu} \int_0^n \nu^t \, \ln \nu \, dt$$

$$= \frac{1}{\ln \nu} \left[\nu^t \right]_0^n \quad \text{Remember: } \int a^{u(t)} \, u'(t) \, \ln a \, dt = a^{u(t)} + c$$

$$= \frac{1}{-\ln(1+i)} (\nu^n - 1) = \frac{1 - \nu^n}{\delta} \quad \text{Remember: } \delta = \ln(1+i)$$

Section 2.5: Continuous Annuities

Similarly, the future value is

$$\begin{aligned}
\bar{s}_{\overline{n}|} &= \int_0^n (1+i)^t \, dt \\
&= \frac{1}{\ln(1+i)} \int_0^n (1+i)^t \, \ln(1+i) \, dt \\
&= \frac{1}{\delta} \left[(1+i)^t\right]_0^n \qquad \text{Remember: } \delta = \ln(1+i) \\
&= \frac{1}{\delta} \left((1+i)^n - 1\right) \\
&= \frac{(1+i)^n - 1}{\delta}
\end{aligned}$$

Theorem.

- The present value of a continuous annuity with rate $C(t) = 1$, $0 \le t \le n$, is

$$\bar{a}_{\overline{n}|} = \frac{1 - \nu^n}{\delta}$$

- The future value at time n of a continuous annuity with rate of one is

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

Theorem.

- $\bar{a}_{\overline{n|}} = \frac{i}{\delta}\ a_{\overline{n|}}$

Proof.

$$\begin{aligned}\bar{a}_{\overline{n|}} &= \frac{1-\nu^n}{\delta} \\ &= i\ \frac{1-\nu^n}{i\ \delta} \\ &= \frac{i}{\delta}\ \frac{1-\nu^n}{i} = \frac{i}{\delta}\ a_{\overline{n|}}\end{aligned}$$

- $\bar{s}_{\overline{n|}} = \frac{i}{\delta}\ s_{\overline{n|}}$

Proof.

$$\begin{aligned}\bar{s}_{\overline{n|}} &= \frac{(1+i)^n-1}{\delta} \\ &= i\ \frac{(1+i)^n-1}{i\ \delta} \\ &= \frac{i}{\delta}\ \frac{(1+i)^n-1}{i} = \frac{i}{\delta}\ s_{\overline{n|}}\end{aligned}$$

Example: If $i = 5\%$ and $n = 10$, find $\bar{a}_{\overline{n|}}$ and $\bar{s}_{\overline{n|}}$.

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Solution: First, find $\delta = \ln(1+i)$, then

$$\bar{a}_{\overline{10|}} = \frac{1 - (\frac{1}{1+0.05})^{10}}{\ln(1+0.05)} = 7.9132$$

$$\bar{s}_{\overline{10|}} = \frac{(1+0.05)^{10} - 1}{\ln(1+0.05)} = 12.8898$$

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Example: If $i = 6\%$ and $n = 15$, find $\bar{a}_{\overline{n|}}$ and $\bar{s}_{\overline{n|}}$.

Solution: First, find $\delta = \ln(1+i)$, then

$$\bar{a}_{\overline{15|}} = \frac{1 - (\frac{1}{1+0.06})^{15}}{\ln(1+0.06)} = 10.0008$$

$$\bar{s}_{\overline{15|}} = \frac{(1+0.06)^{15} - 1}{\ln(1+0.06)} = 23.9675$$

Section 2.6: Basic Annuity Problems

Example: A loan for 20,000 must be repaid by 5 year-end payments with interest at an annual effective rate of 12%. What is the annual payment?

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Solution: Apply the formula: $PV = P\ a_{\overline{n|}}$ where $a_{\overline{n|}} = \frac{1-\nu^n}{i}$.

For $n = 5$ and $i = 12\%$, we have $a_{\overline{5|}} = \frac{1-(\frac{1}{1+0.12})^5}{0.12} = 3.60478$

The annual payment:

$$20000 = P\ a_{\overline{5|}} \Rightarrow P = \frac{20000}{a_{\overline{5|}}} = \frac{20000}{3.60478} = 5548.1946$$

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Example: You have a 5,000 balance in an account earning a 4.5% annual effective rate. You want to increase your balance to 20,000 at the end of 12 years by making a level deposit at the beginning of each of the next 12 years. Find the required level payment.

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Solution: We have that

$$FV = ka(t) + P\ddot{s}_{\overline{n|}}$$

where $ka(t) = k(1+i)^t$ and $\ddot{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{d}$, and $d = \frac{i}{1+i}$.
This implies

$$20000 = 5000(1+0.045)^{12} + P\ \frac{(1+0.045)^{12} - 1}{\frac{0.045}{1+0.045}}$$

$$20000 = 5000(1+0.045)^{12} + 16.1599\ P$$

$$\Rightarrow P = \frac{20000 - 5000(1+0.045)^{12}}{16.1599}$$

$$\Rightarrow P = 712.912$$

Example: An account earning a 5% annual effective rate has a current balance of 6,000. If a deposit of 1,500 is made at the end of each year for 20 years, what will be the balance in the account at the end of 20 years?

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Solution: $FV = ka(t) + Ps_{\overline{n|}}$ where $ka(t) = k(1+i)^t$ and $s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}$. So,

$$FV = 6000(1+0.05)^{20} + 1500\frac{(1+0.05)^{20} - 1}{0.05} = 65,518.72$$

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Example: You want to accumulate at least 20,000 in an account earning a 5% annual effective rate. You will make a level deposit of 1,000 at the beginning of each year for n years. What is the value of n?

Section 2.6: Basic Annuity Problems

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Example: You want to accumulate at least 20,000 in an account earning a 5% annual effective rate. You will make a level deposit of 1,000 at the beginning of each year for n years. What is the value of n?

Solution:

$$FV = P\ddot{s}_{\overline{n|}}$$

$$20000 = 1000\,\frac{(1+0.05)^n}{\frac{0.05}{1.05}}$$

$$\Rightarrow (1+0.05)^n = \frac{20000}{1000}\cdot\frac{0.05}{1.05}$$

$$\Rightarrow \ln(1+0.05)^n = \ln(1.9524)$$

$$\Rightarrow n\ \ln(1.05) = \ln(1.9524) \Rightarrow n = \frac{\ln(1.9524)}{\ln(1.05)} = 13.71$$

For $n = 13$, we have $FV = 1000\ddot{s}_{\overline{13|}} = 18,598.63$ but this amount is less than 20,000.
For $n = 14$, we have $FV = 1000\ddot{s}_{\overline{14|}} = 20,578.56$
Thus, 14th payment is required to reach at least 20,000.

Section 2.7: Deferred Annuities

Deferred annuity is an annuity whose first payment takes place at future period.

- Consider an annuity of under which payments of 1 (**unit annuity**) are made at the end of each period for n periods:

Time	0	1	2	3	4	5	6	7	8
Contributions	0	1	1	1	1	1	1	1	1

- The present value of an annuity-immediate with level payments of one is $a_{\overline{8|}}$
- Assume that the payment starts from the end the third period.

Time	0	1	2	3	4	5	6	7	8
Contributions	0	0	0	1	1	1	1	1	1

- The present value of the annuity would be $\nu^2 a_{\overline{6|}}$ (the present value at the end of the third period discounted for two periods).
- To see this imagine the payment starts at the end of the first period:

Time	0	1	2	3	4	5	6	7	8
Contributions	0	**1**	**1**	1	1	1	1	1	1

The present value at $t = 0$ is $a_{\overline{8|}}$. However, we must remove the present value of the imaginary payments, which is $a_{\overline{2|}}$. Thus, the present value at the end of the third period:

$$a_{\overline{8|}} - a_{\overline{2|}} = \frac{1-\nu^8}{i} - \frac{1-\nu^2}{i} = \frac{-\nu^8+\nu^2}{i} = \nu^2\,\frac{1-\nu^6}{i} = \nu^2 a_{\overline{6|}}$$

An annuity like this is called a deferred annuity.

- **The present value of an n–year unit annuity–immediate deferred for k years is** $\nu^k a_{\overline{n|}}$.
- **Notation** for the present value of an n–year unit annuity–immediate deferred for k years is ${}_{k|}a_{\overline{n|}}$, so we have:

$$ {}_{k|}a_{\overline{n|}} = \nu^k a_{\overline{n|}} $$

Notes.

- From above explanation, we have

$$ \nu^k a_{\overline{n|}} = a_{\overline{k+n|}} - a_{\overline{k|}} $$

  Also,

$$ a_{\overline{k+n|}} = a_{\overline{k|}} + \nu^k a_{\overline{n|}} $$

  The present value of an annuity–immediate for $n+k$ periods is the sum of the present value of a k–period annuity–immediate and an n–period annuity deferred for k periods.

  Take this example:

$$ a_{\overline{5|}} = \nu + \nu^2 + \nu^3 + \nu^4 + \nu^5 = (\nu + \nu^2 + \nu^3) + (\nu^4 + \nu^5) = (\nu + \nu^2 + \nu^3) + \nu^3(\nu + \nu^2) = a_{\overline{3|}} + \nu^3 a_{\overline{2|}} $$

  The present value of a five-period annuity–immediate can be expressed as the sum of a 3-year annuity–immediate and a 2-year annuity–immediate deferred for 3 years:

- The present value of an n-year annuity–immediate deferred for k years with level of payment P is

$$ P\ {}_{k|}a_{\overline{n|}} = P\ \nu^k a_{\overline{n|}} $$

Section 2.7: Deferred Annuities

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Solution: From

$$a_{\overline{k+n|}} = a_{\overline{k|}} + \nu^k a_{\overline{n|}}$$

we have

$$a_{\overline{8|}} = a_{\overline{4|}} + \nu^4 a_{\overline{4|}} = 3.5460 + (0.8227)(3.5460) = 6.463$$

Example: If $a_{\overline{4|}} = 3.5460$ and $\nu^4 = 0.8227$. Find $a_{\overline{8|}}$

Solution: From

$$a_{\overline{k+n|}} = a_{\overline{k|}} + \nu^k a_{\overline{n|}}$$

we have

$$a_{\overline{8|}} = a_{\overline{4|}} + \nu^4 a_{\overline{4|}} = 3.5460 + (0.8227)(3.5460) = 6.463$$

Example: Based on a 5% annual interest rate, find the present value of a 10-year annuity with level payments of 100 each, with the first payment occurring 4 years from now.

Example: If $a_{\overline{4|}} = 3.5460$ and $\nu^4 = 0.8227$. Find $a_{\overline{8|}}$

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we have

$$a_{\overline{8|}} = a_{\overline{4|}} + \nu^4 a_{\overline{4|}} = 3.5460 + (0.8227)(3.5460) = 6.463$$

Example: Based on a 5% annual interest rate, find the present value of a 10-year annuity with level payments of 100 each, with the first payment occurring 4 years from now.

Solution:
The first payment at the end of the 4th year i.e, no payments during the first 3 years, so this is a 3–year–deferred annuity immediate.

$$P\ {}_{3|}a_{\overline{10|}} = 100\ \nu^3 a_{\overline{10|}} = 100\ \left(\frac{1}{1+0.05}\right)^3\ \frac{1-\left(\frac{1}{1+0.05}\right)^{10}}{0.05} = 667.03$$

OR

Use the relation:

$$\nu^k a_{\overline{n|}} = a_{\overline{k+n|}} - a_{\overline{k|}}$$

$$100\nu^3 a_{\overline{10|}} = 100(a_{\overline{13|}} - a_{\overline{3|}}) = 100\left(\frac{1-\left(\frac{1}{1+0.05}\right)^{13}}{0.05} - \frac{1-\left(\frac{1}{1+0.05}\right)^{3}}{0.05}\right) = 100(9.3936 - 2.7232) = 667.03$$

Now again, assume that the payment starts from the end the third period:

Time	0	1	2	3	4	5	6	7	8
Contributions	0	0	0	1	1	1	1	1	1

The future value of the annuity would be at $t = 8$: $(1+i)^2 s_{\overline{6|}}$.
To see this imagine the payment starts at the end of the first period:

Time	0	1	2	3	4	5	6	7	8
Contributions	0	**1**	**1**	1	1	1	1	1	1

The future value at $t = 8$ is $s_{\overline{8|}}$. However, we must remove the future value of the imaginary payments, which is $s_{\overline{2|}}$. Thus, the future value at the end of the last period:

$$s_{\overline{8|}} - s_{\overline{2|}} = \frac{(1+i)^8 - 1}{i} - \frac{(1+i)^2 - 1}{i} = \frac{(1+i)^8 - (1+i)^2}{i} = (1+i)^2 \, \frac{(1+i)^6 - 1}{i} = (1+i)^2 s_{\overline{6|}}$$

In general, **the future value of an n–year unit annuity–immediate deferred for k years is** $(1+i)^k s_{\overline{n|}}$.

Note. ${}_{k|}s_{\overline{n|}} = (1+i)^k s_{\overline{n|}}$.

Remember.

- An annuity is said to have level payments if all payments C_j are equal.
- An annuity has nonlevel payments if some payments C_j are different from other ones.

For any type of varying annuity, the present value (the accumulated value) can be evaluated by taking the present value (the accumulated value) of each payment separately and summing the results:

Time (in periods)	t_1	t_2	...	t_n
Investments	C_1	C_2	...	C_n

Under the accumulation function $a(t)$,

- The present value of the considered cashflow at time zero: $PV(t) = \sum_{j=1}^{n} C_j \frac{1}{a(t_j)}$
- The future value at time t of the cashflow: $FV(t) = \sum_{j=1}^{n} C_j \frac{a(t)}{a(t_j)}$

In this section, we consider special annuities with varying payments. In particular, we focus on varying annuities for which simple expressions can be developed:

- (1) Annuities with Terms in Arithmetic Progression.
- (2) Annuities with Terms in Geometric Progression.

(1) Annuities with Terms in Arithmetic Progression.

(A) Increasing Annuities with Terms in Arithmetic Progression

- An annuity whose n payments are $1, 2, 3, \ldots, n$ is called a unit increasing annuity.
- If payments are made at the end of each period, it is an increasing annuity–immediate.
- The present value of the increasing annuity–immediate for n payments is denoted by $(Ia)_{\overline{n|}}$

Time	1	2	3	...	n	Present Value
Payments	1	1	1	...	1	$\frac{1-\nu^n}{i}$
		1	1	...	1	$\nu.a_{\overline{n-1\|}} = \nu.\frac{1-\nu^{n-1}}{i} = \frac{\nu-\nu^n}{i}$
			1	...	1	$\nu^2.a_{\overline{n-2\|}} = \nu^2.\frac{1-\nu^{n-2}}{i} = \frac{\nu^2-\nu^n}{i}$
						...
					1	$\nu^{n-1}.a_{\overline{1\|}} = \nu^{n-1}.\frac{1-\nu}{i} = \frac{\nu^{n-1}-\nu^n}{i}$
Total	1	2	3	...	n	$\frac{\sum_{t=0}^{n-1}\nu^t - n\nu^n}{i} = \frac{\ddot{a}_{\overline{n\|}} - n\nu^n}{i}$

Now, we have

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i}$$

Example: Let i $= 5\%$ and $n = 4$. If the annuity–immediate payments are 1, 2, 3, 4, then find the present value.

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Solution: Apply $(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i}$ for i = 5% and $n = 4$:

$$(Ia)_{\overline{4|}} = \frac{\ddot{a}_{\overline{4|}} - 4\nu^4}{0.05} = 8.6488$$

Example: Let i = 5% and $n = 4$. If the annuity–immediate payments are 1, 2, 3, 4, then find the present value.

Solution: Apply $(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i}$ for i = 5% and $n = 4$:

$$(Ia)_{\overline{4|}} = \frac{\ddot{a}_{\overline{4|}} - 4\nu^4}{0.05} = 8.6488$$

- If payments are made at the beginning of each period, it is an increasing annuity–due.

Remember: $\ddot{a}_{\overline{n|}} = \frac{i}{d} a_{\overline{n|}} = (1+i)a_{\overline{n|}}$

From this, we have $(I\ddot{a})_{\overline{n|}} = \frac{i}{d}(Ia)_{\overline{n|}} = \frac{i}{d} . \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{d}$

$$(I\ddot{a})_{\overline{n|}} = \frac{i}{d}(Ia)_{\overline{n|}} = (1+i)(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{d}$$

- For increasing continuously payable annuity, remember that $\bar{a}_{\overline{n|}} = \frac{i}{\delta} a_{\overline{n|}}$ so we have

$$(I\bar{a})_{\overline{n|}} = \frac{i}{\delta}(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{\delta}$$

- In the case of an increasing perpetuity, the present value is the limit of the appropriate formula as n approaches infinity:
- Increasing unit perpetuity–immediate: (Remember: $\ddot{a}_{\overline{n|}} = \frac{1-\nu^n}{d}$)

$$(Ia)_{\overline{\infty|}} = \lim_{n\to\infty} (Ia)_{\overline{n|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i} = \frac{\frac{1}{d} - 0}{i} = \frac{1}{id}$$

- Increasing unit perpetuity–due:

$$(I\ddot{a})_{\overline{\infty|}} = \lim_{n\to\infty} (I\ddot{a})_{\overline{n|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{d} = \frac{\frac{1}{d} - 0}{d} = \frac{1}{d^2}$$

- Increasing continuously payable perpetuity:

$$(I\bar{a})_{\overline{\infty|}} = \lim_{n\to\infty} (I\bar{a})_{\overline{n|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{\delta} = \frac{\frac{1}{d} - 0}{\delta} = \frac{1}{\delta d}$$

Example: Let $i = 5\%$ and $n = 4$, find $(I\ddot{a})_{\overline{4|}}$.

Example: Let $i = 5\%$ and $n = 4$, find $(I\ddot{a})_{\overline{4|}}$.

Solution: Apply $(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{d}$

we have $(I\ddot{a})_{\overline{4|}} = \frac{\ddot{a}_{\overline{4|}} - 4\nu^4}{d} = 9.0812$

Example: Let $i = 5\%$ and $n = 4$, find $(I\ddot{a})_{\overline{4|}}$.

Solution: Apply $(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{d}$

we have $(I\ddot{a})_{\overline{4|}} = \frac{\ddot{a}_{\overline{4|}} - 4\nu^4}{d} = 9.0812$

■ The future value of an increasing unit annuity–immediate is denoted by $(Is)_{\overline{n|}}$. **Remember.** $s_{\overline{n|}} = (1+i)^n a_{\overline{n|}}$

Notes. We use the previous relation to find expressions to calculate $(Is)_{\overline{n|}}$, and for $(I\ddot{a})_{\overline{n|}}$, $(I\bar{a})_{\overline{n|}}$ students can follow same procedure:

$$(Is)_{\overline{n|}} = (1+i)^n (Ia)_{\overline{n|}} = (1+i)^n \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i} = \frac{(1+i)^n\ \ddot{a}_{\overline{n|}} - (1+i)^n\ n\nu^n}{i} = \frac{\ddot{s}_{\overline{n|}} - \ n}{i}$$

$$(Is)_{\overline{n|}} = (1+i)^n (Ia)_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{i}$$

$$(I\ddot{s})_{\overline{n|}} = (1+i)^n (I\ddot{a})_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{d} = \frac{i}{d}(Is)_{\overline{n|}}$$

$$(I\bar{s})_{\overline{n|}} = (1+i)^n (I\bar{a})_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{\delta} = \frac{i}{\delta}(Is)_{\overline{n|}}$$

(B) Decreasing Annuities with Terms in Arithmetic Progression

- An annuity whose n payments are $n, n-1, n-2, \ldots, 1$ is called a unit decreasing annuity.
- If payments are made at the end of each period, it is an decreasing annuity–immediate.
- The present value of the decreasing annuity–immediate for n payments is denoted by $(Da)_{\overline{n|}}$

Time	1	2	3	...	...	$n-1$	n
Payments	n	$n-1$	$n-2$	...	...	2	1

Now,

Time	1	2	3	...	$n-2$	$n-1$	n
decreasing annuity-immediate	n	$n-1$	$n-2$	...	3	2	1
increasing annuity-immediate	1	2	3	...	$n-2$	$n-1$	n
Total	$n+1$	$n+1$	$n+1$	...	$n+1$	$n+1$	$n+1$

This implies that $PV = (n+1)a_{\overline{n|}}$ i.e. $(Da)_{\overline{n|}} + (Ia)_{\overline{n|}} = (n+1)a_{\overline{n|}}$.

We can rearrange that formula and solve for $(Da)_{\overline{n|}}$:

$$\begin{aligned}
(Da)_{\overline{n|}} &= (n+1)a_{\overline{n|}} - (Ia)_{\overline{n|}} \\
&= (n+1)\frac{1-\nu^n}{i} - \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i} \\
&= \frac{(n+1) - (n+1)\nu^n - \ddot{a}_{\overline{n|}} + n\nu^n}{i} \\
&= \frac{(n+1) - \ddot{a}_{\overline{n|}} - \nu^n}{i} = \frac{n - (\ddot{a}_{\overline{n|}} - 1 + \nu^n)}{i} \\
&= \frac{n - a_{\overline{n|}}}{i}
\end{aligned}$$

Note:

$$\ddot{a}_{\overline{n|}} - 1 + \nu^n = \frac{1-\nu^n}{d} - 1 + \nu^n = \frac{1-\nu^n-d+d\nu^n}{d} = \frac{(1-d)-\nu^n(1-d)}{d} = \frac{(1-d)(1-\nu^n)}{d} = \frac{1-\nu^n}{i} = a_{\overline{n|}}$$

where $\frac{1-d}{d} = \frac{1}{i}$. So ,we have

$$(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i}$$

■ For decreasing unit annuity–due, remember that $\ddot{a}_{\overline{n|}} = \frac{i}{d} a_{\overline{n|}} = (1+i)a_{\overline{n|}}$.

So we have $(D\ddot{a})_{\overline{n|}} = \frac{i}{d}(Da)_{\overline{n|}} = \frac{i}{d} \cdot \frac{n - a_{\overline{n|}}}{i} = \frac{n - a_{\overline{n|}}}{d}$

$$(D\ddot{a})_{\overline{n|}} = \frac{i}{d}(Da)_{\overline{n|}} = (1+i)(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{d}$$

■ For increasing continuously payable annuity, remember that $\bar{a}_{\overline{n|}} = \frac{i}{\delta} a_{\overline{n|}}$.

So we have

$$(D\bar{a})_{\overline{n|}} = \frac{i}{\delta}(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{\delta}$$

Section 2.8: Annuities with Varying Payments

- The future value of an decreasing unit annuity–immediate is denoted by $(Ds)_{\overline{n|}}$. Remember: $s_{\overline{n|}} = (1+i)^n a_{\overline{n|}}$.

Note: We use the previous relation to find expressions to calculate $(Ds)_{\overline{n|}}$, and for $(D\ddot{a})_{\overline{n|}}$, $(D\bar{a})_{\overline{n|}}$ students can follow same procedure:

$$(Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}} = (1+i)^n \frac{n - a_{\overline{n|}}}{i} = \frac{(1+i)^n\ n - (1+i)^n\ a_{\overline{n|}}}{i} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

$$(Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

$$(D\ddot{s})_{\overline{n|}} = (1+i)^n (D\ddot{a})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{d}$$

$$(D\bar{s})_{\overline{n|}} = (1+i)^n (D\bar{a})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{\delta}$$

- The future value of an decreasing unit annuity–immediate is denoted by $(Ds)_{\overline{n|}}$. Remember: $s_{\overline{n|}} = (1+i)^n a_{\overline{n|}}$.

Note: We use the previous relation to find expressions to calculate $(Ds)_{\overline{n|}}$, and for $(D\ddot{a})_{\overline{n|}}$, $(D\bar{a})_{\overline{n|}}$ students can follow same procedure:

$$(Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}} = (1+i)^n \frac{n - a_{\overline{n|}}}{i} = \frac{(1+i)^n \; n - (1+i)^n \; a_{\overline{n|}}}{i} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

$$(Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

$$(D\ddot{s})_{\overline{n|}} = (1+i)^n (D\ddot{a})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{d}$$

$$(D\bar{s})_{\overline{n|}} = (1+i)^n (D\bar{a})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{\delta}$$

Example: Given $i = 5\%$ and $n = 4$, find $(Da)_{\overline{4|}}$

Section 2.8: Annuities with Varying Payments

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Note: We use the previous relation to find expressions to calculate $(Ds)_{\overline{n|}}$, and for $(D\ddot{a})_{\overline{n|}}$, $(D\bar{a})_{\overline{n|}}$ students can follow same procedure:

$$(Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}} = (1+i)^n \frac{n - a_{\overline{n|}}}{i} = \frac{(1+i)^n\ n - (1+i)^n\ a_{\overline{n|}}}{i} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

$$(Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

$$(D\ddot{s})_{\overline{n|}} = (1+i)^n (D\ddot{a})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{d}$$

$$(D\bar{s})_{\overline{n|}} = (1+i)^n (D\bar{a})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{\delta}$$

Example: Given $i = 5\%$ and $n = 4$, find $(Da)_{\overline{4|}}$

Solution: $(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i}$, so for $n = 4$ and $i = 0.05$,we have

$$(Da)_{\overline{4|}} = \frac{4 - a_{\overline{4|}}}{0.05} = \frac{4 - 3.546}{0.05} = 9.08$$

Exercises

Exercise 1: An annuity pays 1 at the end of each of the next four years and 2 at the end of each of the four following years. Based on a 5% annual effective rate, what is the present value of this annuity?

Exercises

Exercise 1: An annuity pays 1 at the end of each of the next four years and 2 at the end of each of the four following years. Based on a 5% annual effective rate, what is the present value of this annuity?

Solution: This sequence of payments can be broken down into an 8-year annuity–immediate with payments of 1, plus a 4–year–deferred annuity–immediate with 4 payments of 1 each:

Time	1	2	3	4	5	6	7	8	Present Value
Payments	1	1	1	1	1	1	1	1	$a_{\overline{8}\vert} = \frac{1-\nu^8}{0.05}$
					1	1	1	1	$\nu^4.a_{\overline{4}\vert} = \nu^4.\frac{1-\nu^4}{0.05}$
Total	1	1	1	1	2	2	2	2	$a_{\overline{8}\vert} + \nu^4.a_{\overline{4}\vert}$

$$a_{\overline{8}|} + \nu^4.a_{\overline{4}|} = \frac{1-\nu^8}{0.05} + \nu^4.\frac{1-\nu^4}{0.05} = 9.38$$

Another approach is to analyze this series of payments:

a 4–year annuity–immediate with payments of 1, plus a 4–year–deferred annuity–immediate with 4 payments of 2 each:

$$a_{\overline{4}|} + \nu^4.2a_{\overline{4}|} = \frac{1-\nu^4}{0.05} + 2\nu^4.\frac{1-\nu^4}{0.05} = 9.38$$

Exercises

Exercise 2: An annuity pays 100 at the end of each of the next 10 years and 200 at the end of each of the five subsequent years. If $i = 0.08$, find the present value of the annuity.

Exercises

Exercise 2: An annuity pays 100 at the end of each of the next 10 years and 200 at the end of each of the five subsequent years. If $i = 0.08$, find the present value of the annuity.

Solution: This sequence of payments can be broken down into an 10-year annuity–immediate with payments of 100, plus a 5–year–deferred annuity–immediate with 5 payments of 100 each:

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Present Value
Payments	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	$100a_{\overline{15}\mid}$
											100	100	100	100	100	$100\nu^{10}.a_{\overline{5}\mid}$
Total	100	100	100	100	100	100	100	100	100	100	200	200	200	200	200	$100(a_{\overline{15}\mid} + \nu^{10}.a_{\overline{5}\mid})$

$$100(a_{\overline{15}\mid} + \nu^{10}.a_{\overline{5}\mid}) = 100\frac{1-\nu^{15}}{0.08} + 100\nu^{10}.\frac{1-\nu^{5}}{0.08} = 1,040.89$$

Another approach is to analyze this series of payments:

a 10–year annuity–immediate with payments of 100, plus a 10–year–deferred annuity–immediate with 5 payments of 200 each:

$$100a_{\overline{10}\mid} + \nu^{10}.200a_{\overline{5}\mid} = 100\frac{1-\nu^{10}}{0.08} + 200\nu^{10}.\frac{1-\nu^{5}}{0.08} = 1,040.89$$

Exercises

Exercise 3: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. There are 10 further payments of 500. Find the present value of this annuity at 6.5%.

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Solution:

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Payments	100	200	300	400	500	500	500	500	500	500	500	500	500	500	500
	$100(Ia)_{\overline{5}\rvert}$					$\nu^5.500.a_{\overline{10}\rvert}$									

$$PV = 100(Ia)_{\overline{5}\rvert} + \nu^5.500.a_{\overline{10}\rvert} = 100(11.9445) + (0.7299)(500)(7.1888) = 3,817.95$$

Exercises

Exercise 3: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. There are 10 further payments of 500. Find the present value of this annuity at 6.5%.

Solution:

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Payments	100	200	300	400	500	500	500	500	500	500	500	500	500	500	500
	$100(Ia)_{\overline{5}\rvert}$					$\nu^5.500.a_{\overline{10}\rvert}$									

$$PV = 100(Ia)_{\overline{5}\rvert} + \nu^5.500.a_{\overline{10}\rvert} = 100(11.9445) + (0.7299)(500)(7.1888) = 3,817.95$$

Exercise 4: An annuity–immediate has 5 annual payments of 100, followed by a perpetuity of 200 starting in the 6th year. Find the present value at 8%.

Exercises

Exercise 3: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. There are 10 further payments of 500. Find the present value of this annuity at 6.5%.

Solution:

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Payments	100	200	300	400	500	500	500	500	500	500	500	500	500	500	500
	$100(Ia)_{\overline{5}\vert}$					$\nu^5.500.a_{\overline{10}\vert}$									

$$PV = 100(Ia)_{\overline{5}\vert} + \nu^5.500.a_{\overline{10}\vert} = 100(11.9445) + (0.7299)(500)(7.1888) = 3,817.95$$

Exercise 4: An annuity–immediate has 5 annual payments of 100, followed by a perpetuity of 200 starting in the 6th year. Find the present value at 8%.

Solution:

Time	1	2	3	4	5	6	7	8	...
Payments	100	100	100	100	100	200	200	...	...
	$100a_{\overline{5}\vert}$					$\nu^5.200.a_{\overline{\infty}\vert} = \nu^5.200.\frac{1}{i}$			

$$PV = 100a_{\overline{5}\vert} + \nu^5.200.a_{\overline{\infty}\vert} = 399.271 + 1701.458 = 2100.73$$

Exercises

Exercise 5: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. Find the present value of this annuity at 6.5%.

Exercises

Exercise 5: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. Find the present value of this annuity at 6.5%.

Solution:

Time	1	2	3	4	5
Payments	100	200	300	400	500

The present value of this cashflow is

$$PV = 100(Ia)_{\overline{5|}} = 100(11.9445) = 1194.45$$

Exercises

Exercise 5: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. Find the present value of this annuity at 6.5%.

Solution:

Time	1	2	3	4	5
Payments	100	200	300	400	500

The present value of this cashflow is

$$PV = 100(Ia)_{\overline{5|}} = 100(11.9445) = 1194.45$$

Exercise 6: Find the present value of a 15year decreasing annuity-immediate paying 150000 the first year and decreasing by 10000 each year thereafter. The effective annual interest rate of 4.5%.

Exercises

Exercise 5: An annuity–immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. Find the present value of this annuity at 6.5%.

Solution:

Time	1	2	3	4	5
Payments	100	200	300	400	500

The present value of this cashflow is

$$PV = 100(Ia)_{\overline{5|}} = 100(11.9445) = 1194.45$$

Exercise 6: Find the present value of a 15year decreasing annuity-immediate paying 150000 the first year and decreasing by 10000 each year thereafter. The effective annual interest rate of 4.5%.

Solution: The cashflow of payments is

Time	1	2	3	...	15
Payments	(15) (10000)	(14) (10000)	(13) (10000)	...	(1) (10000)

The present value of this cashflow is

$$PV = 10000(Da)_{\overline{15|}} = (10000)(94.6767616) = 946767.616$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

Suppose the first payment in an annuity–immediate is P and the subsequent payments change by Q per period, where Q can be either positive or negative. If the annuity has n payments, the sequence of payments is:

$$P, P+Q, P+2Q, \ldots, P+(n-1)Q$$

We find the present value for this sequence of payments:

Time	1	2	3	...	n	Present Value
Payments	P	P	P	...	P	$Pa_{\overline{n\vert}}$
		Q	Q	...	Q	$\nu.Q.a_{\overline{n-1\vert}} = \nu.Q.\frac{1-\nu^{n-1}}{i} = Q.\frac{\nu-\nu^n}{i}$
			Q	...	Q	$\nu^2.Q.a_{\overline{n-2\vert}} = \nu^2.Q.\frac{1-\nu^{n-2}}{i} = Q.\frac{\nu^2-\nu^n}{i}$
						...
					Q	$\nu^{n-1}.Q.a_{\overline{1\vert}} = \nu^{n-1}.Q.\frac{1-\nu}{i} = Q.\frac{\nu^{n-1}-\nu^n}{i}$
Total	P	$P+Q$	$P+2Q$	...	$P+(n-1)Q$	$Pa_{\overline{n\vert}} + Q\frac{\sum_{t=1}^{n}\nu^t - n\nu^n}{i} = Pa_{\overline{n\vert}} + Q\left(\frac{a_{\overline{n\vert}} - n\nu^n}{i}\right)$ add $Q\frac{\nu^n-\nu^n}{i}$

The present value of this annuity is:

$$PV = Pa_{\overline{n\vert}} + Q\left(\frac{a_{\overline{n\vert}} - n\nu^n}{i}\right)$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

- Another approach to find the present value for the previous sequence of payments is as follows:

$$P, P+Q, P+2Q, \ldots, P+(n-1)Q$$

Time	1	2	3	4	...	n	Present Value
Payments	P	P	P	P	...	P	$Pa_{\overline{n}\vert}$
		Q	$2Q$	$3Q$	...	$(n-1)Q$	$Q.\nu.(Ia)_{\overline{n-1}\vert}$

$$\begin{aligned} PV &= Pa_{\overline{n}|} + Q.\nu.(Ia)_{\overline{n-1}|} \\ &= Pa_{\overline{n}|} + Q.\nu.\frac{\ddot{a}_{\overline{n-1}|} - (n-1)\nu^{n-1}}{i} \\ &= Pa_{\overline{n}|} + Q.\frac{a_{\overline{n-1}|} - (n-1)\nu^{n}}{i} \qquad \nu.\ddot{a}_{\overline{n-1}|} = \nu\frac{1-\nu^{n-1}}{d} = \frac{\nu}{d}(1-\nu^{n-1}) = \frac{1-\nu^{n-1}}{i} = a_{\overline{n-1}|} \\ &= Pa_{\overline{n}|} + Q.\frac{a_{\overline{n-1}|} - n\nu^{n} + \nu^{n}}{i} \\ &= Pa_{\overline{n}|} + Q.\left(\frac{a_{\overline{n}|} - n\nu^{n}}{i}\right) \qquad a_{\overline{n-1}|} + \nu^{n} = \frac{1-\nu^{n-1}}{i} + \nu^{n} = \frac{1-\nu^{n}(\nu^{-1}-i)}{i} = \frac{1-\nu^{n}(1+i-i)}{i} = a_{\overline{n}|} \end{aligned}$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

Notes. For the formula: $PV = Pa_{\overline{n|}} + Q\left(\frac{a_{\overline{n|}} - n\nu^n}{i}\right)$,

- In the previous formula, if $P = 1$ and $Q = 1$, then we have **the present value of the unit increasing annuity–immediate**

$$PV = a_{\overline{n|}} + \frac{a_{\overline{n|}} - n\nu^n}{i} = \frac{i\ a_{\overline{n|}} + a_{\overline{n|}} - n\nu^n}{i}$$

$$= \frac{(1+i)a_{\overline{n|}} - n\nu^n}{i}$$

$$= \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i} \qquad \text{Remember: } \ddot{a}_{\overline{n|}} = (1+i)a_{\overline{n|}}$$

$$= (Ia)_{\overline{n|}}$$

- In the previous formula, if $P = n$ and $Q = -1$, then we have **the present value of the unit decreasing annuity–immediate**

$$PV = na_{\overline{n|}} - \frac{a_{\overline{n|}} - n\nu^n}{i} = \frac{i\ n\ a_{\overline{n|}} - a_{\overline{n|}} + n\nu^n}{i}$$

$$= \frac{n(ia_{\overline{n|}} + \nu^n) - a_{\overline{n|}}}{i}$$

$$= \frac{n(1 - \nu^n + \nu^n) - a_{\overline{n|}}}{i} \qquad \text{Note: } ia_{\overline{n|}} = i\frac{1-\nu^n}{i} = 1 - \nu^n$$

$$= \frac{n - a_{\overline{n|}}}{i} = (Da)_{\overline{n|}}$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

- The present value of an increasing perpetuity–immediate of the form $P, P+Q, P+2Q, \ldots$:

$$\begin{aligned}
\lim_{n\to\infty} PV &= \lim_{n\to\infty} Pa_{\overline{n|}} + Q\Big(\frac{a_{\overline{n|}} - n\nu^n}{i}\Big) \\
&= Pa_{\overline{\infty|}} + Q\Big(\frac{a_{\overline{\infty|}} - 0}{i}\Big) \\
&= \frac{P}{i} + Q\Big(\frac{\frac{1}{i}}{i}\Big) = \frac{P}{i} + \frac{Q}{i^2}
\end{aligned}$$

- To develop a formula for the future value of the annuity at time n, we can multiply the present value by $(1+i)^n$ as follows:

$$\begin{aligned}
(1+i)^n\ PV &= P\ (1+i)^n\ a_{\overline{n|}} + Q\Big(\frac{(1+i)^n\ a_{\overline{n|}} - n(1+i)^n\ \nu^n}{i}\Big) \\
FV &= P\ s_{\overline{n|}} + Q\Big(\frac{s_{\overline{n|}} - n}{i}\Big)
\end{aligned}$$

The future value of this annuity is:

$$FV = P\ s_{\overline{n|}} + Q\Big(\frac{s_{\overline{n|}} - n}{i}\Big)$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

For annuity–due, suppose the first payment is P and the subsequent payments change by Q per period, where Q can be either positive or negative. If the annuity has n payments, the sequence of payments is: $P, P+Q, P+2Q, \ldots, P+(n-1)Q$:

Time	0	1	2	3	...	$n-1$	Present Value
Payments	P	P	P	P	...	P	$P\ddot{a}_{\overline{n}\vert}$
		Q	$2Q$	$3Q$	...	$(n-1)Q$	$Q.\nu.(I\ddot{a})_{\overline{n-1}\vert}$

$$
\begin{aligned}
PV &= P\ddot{a}_{\overline{n}\vert} + Q.\nu.(I\ddot{a})_{\overline{n-1}\vert} \\
&= P\ddot{a}_{\overline{n}\vert} + Q.\nu.\frac{\ddot{a}_{\overline{n-1}\vert} - (n-1)\nu^{n-1}}{d} \\
&= P\ddot{a}_{\overline{n}\vert} + Q.\frac{a_{\overline{n-1}\vert} - (n-1)\nu^{n}}{d} \qquad \nu.\ddot{a}_{\overline{n-1}\vert} = \nu\frac{1-\nu^{n-1}}{d} = \frac{\nu}{d}(1-\nu^{n-1}) = \frac{1-\nu^{n-1}}{i} = a_{\overline{n-1}\vert} \\
&= P\ddot{a}_{\overline{n}\vert} + Q.\frac{a_{\overline{n-1}\vert} - n\nu^{n} + \nu^{n}}{d} \\
&= P\ddot{a}_{\overline{n}\vert} + Q.\Big(\frac{a_{\overline{n}\vert} - n\nu^{n}}{d}\Big) \qquad a_{\overline{n-1}\vert} + \nu^{n} = \frac{1-\nu^{n-1}}{i} + \nu^{n} = \frac{1-\nu^{n}(\nu^{-1}-i)}{i} = \frac{1-\nu^{n}(1+i-i)}{i} = a_{\overline{n}\vert}
\end{aligned}
$$

The present value of this annuity is:

$$PV = P\ddot{a}_{\overline{n}\vert} + Q.\Big(\frac{a_{\overline{n}\vert} - n\nu^{n}}{d}\Big)$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

- Another approach to find the present value for the previous sequence of payments is as follows:

We know that

$$\ddot{a}_{\overline{n|}} = (1+i)a_{\overline{n|}}$$

and the present value for the annuity–immediate:

$$PV = Pa_{\overline{n|}} + Q\left(\frac{a_{\overline{n|}} - n\nu^n}{i}\right)$$

This implies that the present value for annuity–due

$$PV = P(1+i)a_{\overline{n|}} + Q(1+i)\frac{a_{\overline{n|}} - n\nu^n}{i}$$

Hence,

$$PV = P\ddot{a}_{\overline{n|}} + Q\frac{a_{\overline{n|}} - n\nu^n}{d} \qquad d = \frac{i}{1+i}$$

Notes:

■ In the previous formula, if $P = 1$ and $Q = 1$, then we have present value of the unit increasing annuity–due

$$\begin{aligned} PV = \ddot{a}_{\overline{n}|} + \frac{a_{\overline{n}|} - n\nu^n}{d} &= \frac{d\ \ddot{a}_{\overline{n}|} + a_{\overline{n}|} - n\nu^n}{d} \\ &= \frac{d\ddot{a}_{\overline{n}|} + \nu.\ddot{a}_{\overline{n}|} - n\nu^n}{d} \qquad a_{\overline{n}|} = \nu.\ddot{a}_{\overline{n}|} \\ &= \frac{\ddot{a}_{\overline{n}|}(d+\nu) - n\nu^n}{d} \\ &= \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{d} \qquad \text{Note: } d + \nu = \frac{i}{1+i} + \frac{1}{1+i} = \frac{1+i}{1+i} = 1 \\ &= (I\ddot{a})_{\overline{n}|} \end{aligned}$$

■ In the previous formula, if $P = n$ and $Q = -1$, then we have present value of the unit decreasing annuity–due

$$\begin{aligned} PV = n\ddot{a}_{\overline{n}|} - \frac{a_{\overline{n}|} - n\nu^n}{d} &= \frac{d\ n\ a_{\overline{n}|} - a_{\overline{n}|} + n\nu^n}{d} \\ &= \frac{n(d\ddot{a}_{\overline{n}|} + \nu^n) - a_{\overline{n}|}}{d} \\ &= \frac{n - a_{\overline{n}|}}{d} = (D\ddot{a})_{\overline{n}|} \qquad \text{Note: } d\ddot{a}_{\overline{n}|} + \nu^n = d\,\frac{1-\nu^n}{d} + \nu^n = 1 \end{aligned}$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

- The present value of an increasing perpetuity–due of the form $P, P+Q, P+2Q, \ldots$:

$$\begin{aligned}\lim_{n\to\infty} PV &= \lim_{n\to\infty} P\ddot{a}_{\overline{n}|} + Q\Big(\frac{a_{\overline{n}|} - n\nu^n}{d}\Big)\\ &= P\ddot{a}_{\overline{\infty}|} + Q\Big(\frac{a_{\overline{\infty}|} - 0}{d}\Big)\\ &= \frac{P}{d} + Q\Big(\frac{\frac{1}{i}}{d}\Big) = \frac{P}{d} + \frac{Q}{id}\end{aligned}$$

- To develop a formula for the future value of the annuity at time n, we can multiply the present value by $(1+i)^n$ as follows:

$$(1+i)^n\ PV = P\ (1+i)^n\ \ddot{a}_{\overline{n}|} + Q\Big(\frac{(1+i)^n\ a_{\overline{n}|} - n(1+i)^n\ \nu^n}{d}\Big)$$

$$FV = P\ \ddot{s}_{\overline{n}|} + Q\Big(\frac{s_{\overline{n}|} - n}{d}\Big)$$

The future value of this annuity is:

$$FV = P\ \ddot{s}_{\overline{n}|} + Q\Big(\frac{s_{\overline{n}|} - n}{d}\Big)$$

Section 2.9: A Single Formula for Annuities with Terms in Arithmetic Progression

Exercise 3: A 10-year annuity-immediate has a first-year payment of 500. The subsequent payments increase by 100 each year. Find the present value of this annuity based on an annual effective rate of 5%.

Exercise 3: A 10-year annuity-immediate has a first-year payment of 500. The subsequent payments increase by 100 each year. Find the present value of this annuity based on an annual effective rate of 5%.

Solution:

$$PV = Pa_{\overline{n|}} + Q\left(\frac{a_{\overline{n|}} - n\nu^n}{i}\right)$$

where $n = 10$, $i = 0.05$, $P = 500$, $Q = 100$.

$$PV = 500\frac{1-(1.05)^{-10}}{0.05} + 100\left(\frac{\frac{1-(1.05)^{-10}}{0.05} - \frac{10}{1.05}}{0.05}\right) = 7,026.07$$

Section 2.10: Annuities with Terms in Geometric Progression

■ A **geometric annuity** is an annuity where the payments increase geometrically with a common ratio.

For example, consider the sequence of three payments 1, 1.05, $1.1025 = (1.05)^2$ made at the end of the years 1, 2, 3:

Time	0	1	2	3
Payments		1	1.05	$(1.05)^2$

■ The payments increase geometrically with a common ratio of 1.05: $\frac{1.05}{1} = \frac{(1.05)^2}{1.05}$

■ The payments increase geometrically with a growth rate $g = 0.05$: $1, (1+0.05), (1+0.05)^2$

■ More generally, we can consider an n-year geometric annuity-immediate with growth rate g. Its payments are $1, (1+g), (1+g)^2, (1+g)^3, \ldots, (1+g)^{n-1}$.

■ The present value is represented by the symbol $a^{g}_{\overline{n}|i}$ or $a^{g}_{\overline{n}|}$. Sometimes present value is represented by the symbol $(Ga)_{\overline{n}|i,r}$

■ The first payment in the geometric series is 1.

■ The annual rate of change, g, can be either positive or negative, reflecting payments that increase or decrease geometrically.

Section 2.10: Annuities with Terms in Geometric Progression

Now, we have the following cashflow:

Time	1	2	3	...	n
Payments	1	$(1+g)$	$(1+g)^2$	...	$(1+g)^{n-1}$

■ We can express the present value of a **geometric annuity–immediate** as follows:

$$\begin{aligned} a^{g}_{\overline{n}|} &= \frac{1}{1+i} + \frac{(1+g)}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \cdots + \frac{(1+g)^{n-1}}{(1+i)^n} \\ &= \frac{1}{1+i}\left[1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \cdots + \left(\frac{1+g}{1+i}\right)^{n-1}\right] \\ &= \frac{1}{1+i}\,\frac{1-\left(\frac{1+g}{1+i}\right)^n}{1-\left(\frac{1+g}{1+i}\right)} \quad \text{we apply the formula: } \frac{1-r^n}{1-r} \text{ for } r = \frac{(1+g)}{(1+i)} \\ &= \frac{1}{1+i}\,\frac{1-\left(\frac{1+g}{1+i}\right)^n}{\frac{(1+i)-(1+g)}{1+i}} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \end{aligned}$$

$$a^{g}_{\overline{n}|} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}$$

Notes:

■ If $g = 0$, then $a^{g}_{\overline{n|}} = \frac{1-\nu^n}{i} = a_{\overline{n|}}$

■ If $g = i$,

$$a^{g}_{\overline{n|}} = \frac{1}{1+i}\left[1 + (\frac{1+g}{1+i}) + (\frac{1+g}{1+i})^2 + \cdots + (\frac{1+g}{1+i})^{n-1}\right]$$
$$= \frac{1}{1+i}\left[1+1+1+\cdots+1\right] = \frac{n}{1+i}$$

■ For **geometric annuities–due**:

$$\ddot{a}^{g}_{\overline{n|}} = (1+i)a^{g}_{\overline{n|}} = (1+i)\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}$$
$$= \frac{1}{\nu}.\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \qquad \nu = \frac{1}{1+i} \Rightarrow \frac{1}{\nu} = 1+i$$
$$= \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i\nu - g.\nu}$$
$$= \frac{1-\left(\frac{1+g}{1+i}\right)^n}{d - g.\nu} \qquad i\nu = d$$

Section 2.10: Annuities with Terms in Geometric Progression

■ For the future values of geometric annuities:

$$s^{g}_{\overline{n}|} = (1+i)^n . a^{g}_{\overline{n}|} = (1+i)^n . \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g}$$
$$= \frac{(1+i)^n - (1+g)^n}{i - g}$$

and

$$\ddot{s}^{g}_{\overline{n}|} = (1+i)^n . \ddot{a}^{g}_{\overline{n}|} = (1+i)^n . \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{d - g.\nu}$$
$$= \frac{(1+i)^n - (1+g)^n}{d - g.\nu}$$

■ For the perpetuity, (1) annuity–immediate

$$\lim_{n\to\infty} a^{g}_{\overline{n}|} = \lim_{n\to\infty} \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g}$$
$$a^{g}_{\overline{\infty}|} = \frac{1-0}{i-g} = \frac{1}{i-g} \text{ for } i > g$$

If $g \geq i$, the present value of the perpetuity is infinite.

(2) annuity–due

$$\lim_{n\to\infty} \ddot{a}^{g}_{\overline{n|}} = \lim_{n\to\infty} \frac{1-\left(\frac{1+g}{1+i}\right)^n}{d-g.\nu} \Rightarrow \ddot{a}^{g}_{\overline{\infty|}} = \frac{1-0}{d-g.\nu} = \frac{1}{d-g.\nu} \text{ for } i > g$$

(1) geometric annuity–immediate

$$a^{g}_{\overline{n|}} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \text{ and } s^{g}_{\overline{n|}} = \frac{(1+i)^n-(1+g)^n}{i-g}$$

$$\text{If } g=0,\ a^{g}_{\overline{n|}} = \frac{1-\nu^n}{i} = a_{\overline{n|}} \text{ and if } g=i,\ a^{g}_{\overline{n|}} = \frac{n}{1+i}$$

(2) geometric annuity–due

$$\ddot{a}^{g}_{\overline{n|}} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{d-g.\nu} \text{ and } \ddot{s}^{g}_{\overline{n|}} = \frac{(1+i)^n-(1+g)^n}{d-g.\nu}$$

$$\text{For } i > g,\ a^{g}_{\overline{\infty|}} = \frac{1}{i-g} \text{ and } \ddot{a}^{g}_{\overline{\infty|}} = \frac{1}{d-g.\nu}$$

Section 2.10: Annuities with Terms in Geometric Progression

For an annuity–immediate or for future values, you must first calculate the present value of the annuity–due, and then adjust it by the appropriate interest factor, using $(1+i)$:

(1)

$$\frac{\ddot{a}^{g}_{\overline{n|}}}{1+i} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{(1+i)d-(1+i)g.\nu} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{(1+i)\frac{i}{1+i}-g} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{i-g} = a^{g}_{\overline{n|}}$$

(2)

$$(1+i)^{n-1}.\ddot{a}^{g}_{\overline{n|}} = \frac{(1+i)^{n-1}-\frac{(1+g)^{n}}{1+i}}{d-g.\nu} = \frac{\frac{(1+i)^{n}-(1+g)^{n}}{(1+i)}}{d-g.\nu} = \frac{(1+i)^{n}-(1+g)^{n}}{(1+i).d-(1+i)g.\nu} = \frac{(1+i)^{n}-(1+g)^{n}}{i-g} = s^{g}_{\overline{n|}}$$

(3)

$$(1+i)^n.\ddot{a}^{g}_{\overline{n|}} = \frac{(1+i)^n - (1+i)^n.(\frac{1+g}{1+i})^n}{d - g.\nu}$$
$$= \frac{(1+i)^n - (1+g)^n}{d - g.\nu} = \ddot{s}^{g}_{\overline{n|}}$$

$$a^{g}_{\overline{n|}} = \frac{\ddot{a}^{g}_{\overline{n|}}}{1+i}$$

$$s^{g}_{\overline{n|}} = (1+i)^{n-1}.\ddot{a}^{g}_{\overline{n|}}$$

$$\ddot{s}^{g}_{\overline{n|}} = (1+i)^n.\ddot{a}^{g}_{\overline{n|}}$$

Example: Given $i = 10\%$, find the present value of the sequence of payments: $1.05, (1.05)^2, \ldots, (1.05)^{10}$. Payments are made at the beginning of each year.

Section 2.10: Annuities with Terms in Geometric Progression

(3)

$$(1+i)^n.\ddot{a}^{g}_{\overline{n|}} = \frac{(1+i)^n - (1+i)^n.\left(\frac{1+g}{1+i}\right)^n}{d - g.\nu}$$
$$= \frac{(1+i)^n - (1+g)^n}{d - g.\nu} = \ddot{s}^{g}_{\overline{n|}}$$

$$a^{g}_{\overline{n|}} = \frac{\ddot{a}^{g}_{\overline{n|}}}{1+i}$$

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Example: Given $i = 10\%$, find the present value of the sequence of payments: $1.05, (1.05)^2, \ldots, (1.05)^{10}$. Payments are made at the beginning of each year.

Solution: Note that this series starts with 1.05, not 1, and that payments are made at the beginning of each period with $g = 0.05$.

$$(1.05).\ddot{a}^{0.05}_{\overline{10|}0.10} = (1.05).\frac{1 - \left(\frac{1.05}{1.10}\right)^{10}}{\frac{0.10}{1.10} - \frac{0.05}{1.10}} = 8.59$$

Section 2.10: Annuities with Terms in Geometric Progression

Example: You want to save 1,000,000 for retirement. You plan to make annual deposits at the beginning of each year into an account that earns an annual effective rate of 7.5%. You will increase the amount of your deposit each year by 4%. If you plan to retire in 40 years, what should be the amount of your first deposit?

Section 2.10: Annuities with Terms in Geometric Progression

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Solution: Let P be the amount of the first deposit.

$$FV = P.\ddot{s}^{g}_{\overline{n|}} = P.\frac{(1+i)^n - (1+g)^n}{d - g.\nu}$$

$$1,000,000 = P\ \ddot{s}^{0.04}_{\overline{40|}0.075} = P\ \frac{(1.075)^{40} - (1.04)^{10}}{\frac{0.075}{1.075} - \frac{0.04}{1.075}}$$

$$1,000,000 = P(406.756)$$

$$\Rightarrow P = \frac{1,000,000}{406.756} = 2,458.48.$$

Section 2.10: Annuities with Terms in Geometric Progression

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Example: An annuity provides for 10 annuals payments, the first payment a year hence being \$2600. The payments increase in such a way that each payment is 3% greater than the previous one. The annual effective rate of interest is 4%. Find the present value of this annuity.

Section 2.10: Annuities with Terms in Geometric Progression

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Example: An annuity provides for 10 annuals payments, the first payment a year hence being \$2600. The payments increase in such a way that each payment is 3% greater than the previous one. The annual effective rate of interest is 4%. Find the present value of this annuity.

Solution: Let P be the first payment.

$$PV = P.a^{g}_{\overline{n}|} = P.\frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g}$$

$$PV = (2600)a^{0.03}_{\overline{10}|0.04} = (2600)\frac{1 - \left(\frac{1.03}{1.04}\right)^{10}}{0.04 - 0.03}$$

$$= 23945.54454$$

Section 2.11: Annuities With More Complex Payment Patterns

Example: The present value of a series of payments of 3 at the end of every eight years, forever, is equal to 9.5. Calculate the effective annual rate of interest.

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Example: The present value of a series of payments of 3 at the end of every eight years, forever, is equal to 9.5. Calculate the effective annual rate of interest.

Solution: **Remember:** The effective rate of interest over a certain period of time depends only on the length of this period, i.e. for $0 \leq s \leq t$,

$$\frac{A(t)-A(s)}{A(s)} = \frac{A(0)(1+i)^t - A(0)(1+i)^s}{A(0)(1+i)^s} = (1+i)^{t-s} - 1 = \frac{A(t-s)-A(0)}{A(0)}$$

where i is the effective annual rate of interest.

So, the 8-year effective interest rate is $(1+i)^8 - 1$. We have that

$$PV = Pa_{\overline{\infty}|(1+i)^8-1} \Rightarrow 9.5 = (3)a_{\overline{\infty}|(1+i)^8-1}$$

Remember: $a_{\overline{\infty}|i} = \dfrac{1}{i}$

$$\Rightarrow 9.5 = (3)\frac{1}{(1+i)^8 - 1}$$

$$\Rightarrow (1+i)^8 - 1 = \frac{3}{9.5}$$

$$\Rightarrow (1+i)^8 = \frac{3}{9.5} + 1$$

$$\Rightarrow 1 + i = \sqrt[8]{\frac{3}{9.5} + 1}$$

$$\Rightarrow i = \sqrt[8]{\frac{3}{9.5} + 1} - 1 = 0.03489979511$$

$$\Rightarrow i = 3.489979511\%$$

Section 2.11: Annuities With More Complex Payment Patterns

Example: A perpetuity pays \$1 at the end of every year plus an additional \$1 at the end of every second year. The effective rate of interest is $i = 5\%$. Find the present value of the perpetuity at time 0.

Section 2.11: Annuities With More Complex Payment Patterns

Example: A perpetuity pays \$1 at the end of every year plus an additional \$1 at the end of every second year. The effective rate of interest is $i = 5\%$. Find the present value of the perpetuity at time 0.

Solution:

Time	0	1	2	3	4	5	6	...	Present value
Payments		1	1	1	1	1	1	...	$a_{\overline{\infty}\rvert} = \frac{1}{i}$
			1		1		1	...	$a_{\overline{\infty}\rvert (1+i)^2-1} = \frac{1}{(1+i)^2-1}$ The 2-year interest factor is $(1+i)^2$, so the 2-year effective interest rate is $(1+i)^2 - 1$.
Total		1	2	1	2	1	2	...	$\frac{1}{i} + \frac{1}{(1+i)^2-1}$

$$PV = \frac{1}{i} + \frac{1}{(1+i)^2 - 1} = \frac{(1+i)^2 - 1 + i}{i\left[(1+i)^2 - 1\right]} = 29.7561$$

Section 2.11: Annuities With More Complex Payment Patterns

Example: An annuity-immediate has a first payment of 100, and its payments increase by 100 each year until they reach 500. The remaining payments are a perpetuity-immediate of 500 beginning in year 6. Find the present value at 6.5%.

Section 2.11: Annuities With More Complex Payment Patterns

Example: An annuity-immediate has a first payment of 100, and its payments increase by 100 each year until they reach 500. The remaining payments are a perpetuity-immediate of 500 beginning in year 6. Find the present value at 6.5%.

Solution:

Time	1	2	3	4	5	6	7	...
Payments	100	200	300	400	500	500	...	...
	$100(Ia)_{\overline{5}\vert 0.065}$					$\nu^5.500.a_{\overline{\infty}\vert} = \nu^5.500.\frac{1}{i}$		

$$PV = 100\frac{\ddot{a}_{\overline{5}|0.065} - 5(\frac{1}{1.065})^5}{0.065} + (\frac{1}{1.065})^5.500.\frac{1}{0.065} = 1194.45 + 5614.47 = 6808.92$$

Section 2.12: Annuities with Payments More Frequent than Annual

Example: John invest \$500 into an account at the end of each month for 5 years. The annual effective interest rate is 4.5%. Calculate the balance of this account at the end of 5 years.

Section 2.12: Annuities with Payments More Frequent than Annual

Example: John invest \$500 into an account at the end of each month for 5 years. The annual effective interest rate is 4.5%. Calculate the balance of this account at the end of 5 years.

Solution: The number of payments made is $(5)(12) = 60$.

Remember: $FV = P.s_{\overline{n}|i} = P.\frac{(1+i)^n - 1}{i}$

However, we use

$$FV = P.s_{\overline{n\times m}|\,\frac{i^{(m)}}{m}}$$

where m is number of payments each year.

Now, we want to find $\frac{i^{(m)}}{m}$:

$$(1+i) = (1+\frac{i^{(m)}}{m})^m \Rightarrow \frac{i^{(12)}}{12} = (1+0.045)^{\frac{1}{12}} - 1$$

The one-month effective interest rate is $(1.045)^{1/12} - 1$.

Hence, the balance of this account at the end of 5 years is

$$FV = 500.s_{\overline{60}|(1.045)^{1/12}-1} = (500)\frac{((1.045)^{1/12} - 1 + 1)^{60} - 1}{(1.045)^{1/12} - 1} = (500)\frac{(1.045)^5 - 1}{(1.045)^{1/12} - 1} = 33495.8784$$

Section 2.12: Annuities with Payments More Frequent than Annual

Example: Find the level monthly payment for a 30-year mortgage loan of 300,000 at an interest rate of 6% convertible monthly.

Section 2.12: Annuities with Payments More Frequent than Annual

Example: Find the level monthly payment for a 30-year mortgage loan of 300,000 at an interest rate of 6% convertible monthly.

Solution: The number of payments made is (30)(12) = 360.

Remember: $PV = P.a_{\overline{n}|i}$

However, we use

$$PV = P.a_{\overline{n\times m}|\frac{i^{(m)}}{m}}$$

where m is number of payments each year.

A nominal rate of interest 6% convertible monthly means an interest rate of $\frac{i^{(m)}}{m} = \frac{6\%}{12} = 0.5\% = 0.005$ per month.

We can simply use the usual annuity functions, recognizing that the period is 1 month instead of 1 year:

$$\begin{aligned} PV &= P.a_{\overline{n\times m}|\frac{i^{(m)}}{m}} \\ 300,000 &= P.a_{\overline{360}|0.005} \\ \Rightarrow P &= \frac{300,000}{a_{\overline{360}|0.005}} \\ P &= \frac{300,000}{166.7916} \qquad a_{\overline{360}|0.005} = \frac{1-(\frac{1}{1.005})^{360}}{0.005} \\ P &= 1798.65 \end{aligned}$$

Section 2.12: Annuities with Payments More Frequent than Annual

Example: An annuity-immediate has 20 initial quarterly payments of 25 each, followed by a perpetuity of quarterly payments of 50 starting in the sixth year. Find the present value at 8% convertible quarterly.

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Example: An annuity-immediate has 20 initial quarterly payments of 25 each, followed by a perpetuity of quarterly payments of 50 starting in the sixth year. Find the present value at 8% convertible quarterly.

Solution: A nominal rate of interest 8% convertible quarterly means an interest rate of $\frac{8\%}{4} = 2\% = 0.02$ per quarter.

$$
\begin{aligned}
PV &= P.a_{\overline{n}|i} + \nu^n.P.a_{\overline{\infty}|i} \\
&= (25)a_{\overline{20}|2\%} + (50)\nu^{20}a_{\overline{\infty}|2\%} \\
&= (25)(16.3514) + (50)(0.6729)(50) \qquad \text{Remember: } a_{\overline{\infty}|i} = \frac{1}{i} \\
&= 2091.2142
\end{aligned}
$$

Section 2.12: Annuities with Payments More Frequent than Annual

■ Many annuities, such as loan payments or pensions, have more frequent payments (e.g., monthly or quarterly). We will develop methods for annuities whose payments are more frequent than annual.

■ The symbol $a^{(m)}_{\overline{n}|i}$ indicates the present value of an annuity-immediate that pays $\frac{1}{m}$ at the end of each $\frac{1}{m}$ of a year. For example, consider an annuity that makes 12 payments per year. The symbol $a^{(12)}_{\overline{n}|i}$ is the present value of an annuity that pays $\frac{1}{12}$ at the end of each month for n years.

■ The cashflow value of the annuity-immediate with level payments of one per year and m payments per year is

Time	$\frac{1}{m}$	$\frac{2}{m}$	$\frac{3}{m}$	...	...	$n \times m$
Payments	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	...	...	$\frac{1}{m}$

$$a^{(m)}_{\overline{n}|} = \frac{1}{m} a_{\overline{m \times n}|\frac{i^{(m)}}{m}} = \frac{1}{m} \frac{1-\left(1+\frac{i^{(m)}}{m}\right)^{-m \times n}}{\frac{i^{(m)}}{m}} = \frac{1-\left[\left(1+\frac{i^{(m)}}{m}\right)^{m}\right]^{-n}}{i^{(m)}} = \frac{1-\left[1+i\right]^{-n}}{i^{(m)}} = \frac{1-\nu^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|}$$

$$a^{(m)}_{\overline{n}|} = \frac{1-\nu^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|}, \qquad s^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|}$$

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{(1-\nu^n}{d^{(m)}} = \frac{i}{d^{(m)}} a_{\overline{n}|} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|}, \qquad \ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} s_{\overline{n}|} = \frac{d}{d^{(m)}} \ddot{s}_{\overline{n}|}$$

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{m(\nu^{-\frac{1}{m}} - 1)} = \frac{1}{i^{(m)}}, \qquad \ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{m} \frac{1}{1-\nu^{\frac{1}{m}}} = \frac{1}{d^{(m)}}$$

Section 2.12: Annuities with Payments More Frequent than Annual

Example: John invest $500 into an account at the end of each month for 5 years. The annual effective interest rate is 4.5%. Calculate the balance of this account at the end of 5 years.

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Solution: Remember:

$$s_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} s_{\overline{n}|}$$

$$FV = P.s_{\overline{n}|}^{(m)} = P.\frac{i}{i^{(m)}} s_{\overline{n}|}$$

This implies

$$FV = (500)\frac{0.045}{(1.045)^{1/12} - 1}\frac{(1+0.045)^5 - 1}{0.045} = 33495.8784$$

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Example: A saver deposits 100 into a bank account at the end of every month for 10 years. If the account earns interest at an annual effective rate of 6%, what is the savers balance at the end of 10 years?

Section 2.12: Annuities with Payments More Frequent than Annual

Example: John invest $500 into an account at the end of each month for 5 years. The annual effective interest rate is 4.5%. Calculate the balance of this account at the end of 5 years.

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$$FV = P.s_{\overline{n}|}^{(m)} = P.\frac{i}{i^{(m)}} s_{\overline{n}|}$$

This implies

$$FV = (100)\frac{0.06}{(1.06)^{1/12} - 1}\frac{(1.045)^{10} - 1}{0.06} = 16247.344$$

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

We will address situations where a continuously payable annuitys rate of payment changes continuously.

■ **(A) Arithmetic Annuities**

■ (1) The continuously increasing annuity.

■ The symbol for the present value of this annuity is $(\bar{I}\bar{a})_{\overline{n}|}$.

■ The bar over the a indicates that payments are continuous.

■ The bar over the I indicates that the increases in the rate of payment also occur continuously.

At the beginning of the first year (at time 0), the payment rate is 0 and the payment rate at the end of the first year is 1 per year. It builds up from 0 at time 0 to 1 at time 1, and continues to increase, reaching a rate of n per year at time n.

At any time t, the payment rate is t per year, so the amount paid during a small time interval dt is $(t.dt)$, and the present value of that amount at time 0 is $(t.\nu^t.dt)$.

To find the present value of all the payments from time 0 to time n, we integrate this value from 0 to n:

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t.\nu^t \; dt$$

Using integration by part:

$$u = t \Rightarrow du = dt \; ,$$

$$dw = \nu^t \; dt \Rightarrow w = \int \nu^t \; dt = -\frac{1}{\delta} e^{-\delta t}$$ Revise the continuous annuity (slide 36)

$$\begin{aligned}
\int t.\nu^t \; dt &= -\frac{t}{\delta}e^{-\delta t} + \frac{1}{\delta}\int e^{-\delta t}\; dt \\
&= -\frac{t}{\delta}e^{-\delta t} - \frac{1}{\delta^2}e^{-\delta t} \\
&= -\frac{t}{\delta}\nu^t - \frac{1}{\delta^2}\nu^t \qquad e^{-\delta t} = (e^{\ln(1+i)^{-1}})^t = \nu^t \\
&= \frac{-\nu^t}{\delta}\Big(t + \frac{1}{\delta}\Big)
\end{aligned}$$

This implies

$$\begin{aligned}
(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t.\nu^t \; dt &= \frac{-\nu^t}{\delta}\Big(t + \frac{1}{\delta}\Big)\Big|_0^n \\
&= \frac{-\nu^n}{\delta}(n + \frac{1}{\delta}) + \frac{1}{\delta}(\frac{1}{\delta}) \\
&= \frac{1}{\delta}\Big(- n\nu^n - \frac{\nu^n}{\delta} + \frac{1}{\delta}\Big) \\
&= \frac{1}{\delta}\Big(- n\nu^n + \frac{1-\nu^n}{\delta}\Big) \\
&= \frac{1}{\delta}\Big(- n\nu^n + \bar{a}_{\overline{n}|}\Big) \\
&= \frac{\bar{a}_{\overline{n}|} - n\nu^n}{\delta}
\end{aligned}$$

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

This gives us the formula for the present value of a continuously increasing arithmetic annuity:

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - n\nu^n}{\delta}$$

In the case of a continuously increasing perpetuity, we have:

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\bar{a}_{\overline{n}|} - n\nu^n}{\delta}$$

This implies

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \frac{1}{\delta^2}$$

For the future values of these annuities:

$$(\bar{I}\bar{s})_{\overline{n}|} = (1+i)^n(\bar{I}\bar{a})_{\overline{n}|} = (1+i)^n \frac{\bar{a}_{\overline{n}|} - n\nu^n}{\delta} = \frac{\bar{s}_{\overline{n}|} - n}{\delta}$$

- (2) The continuously decreasing annuity.

A continuously decreasing annuity $(\bar{D}\bar{a})_{\overline{n}|}$ makes continuous payments at a rate that decreases linearly from n per year at time 0 to a rate of 0 at time n. We can develop a formula for $(\bar{D}\bar{a})_{\overline{n}|}$ as follows:

$$(\bar{I}\bar{a})_{\overline{n}|} + (\bar{D}\bar{a})_{\overline{n}|} = n\bar{a}_{\overline{n}|} \Rightarrow (\bar{D}\bar{a})_{\overline{n}|} = n\bar{a}_{\overline{n}|} - (\bar{I}\bar{a})_{\overline{n}|} \Rightarrow n.\frac{1-\nu^n}{\delta} - \frac{\bar{a}_{\overline{n}|} - n\nu^n}{\delta} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}$$

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

$$(\bar{D}\bar{a})_{\overline{n}|} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}$$

For the future value:

$$(\bar{D}\bar{s})_{\overline{n}|} = (1+i)^n (\bar{D}\bar{a})_{\overline{n}|} = (1+i)^n \frac{n - \bar{a}_{\overline{n}|}}{\delta} = \frac{n(1+i)^n - \bar{s}_{\overline{n}|}}{\delta}$$

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

$$(\bar{D}\bar{a})_{\overline{n}|} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}$$

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$$(\bar{D}\bar{s})_{\overline{n}|} = (1+i)^n (\bar{D}\bar{a})_{\overline{n}|} = (1+i)^n \frac{n - \bar{a}_{\overline{n}|}}{\delta} = \frac{n(1+i)^n - \bar{s}_{\overline{n}|}}{\delta}$$

Example: An annuity provides continuous payments at the rate of 500 per year at time 0, and its payment rate increases continuously at a rate of 100 per year for 10 years. Calculate the accumulated value of this annuity at the end of 10 years, assuming an annual effective interest rate of 10%.

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

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Example: An annuity provides continuous payments at the rate of 500 per year at time 0, and its payment rate increases continuously at a rate of 100 per year for 10 years. Calculate the accumulated value of this annuity at the end of 10 years, assuming an annual effective interest rate of 10%.

Solution: We have two parts:

Part (1): The original payment rate of 500 per year is a level continuously payable annuity, $500\bar{s}_{\overline{10}|}$

Part (2): The increasing portion is a separate continuously increasing annuity, $100(\bar{I}\bar{s})_{\overline{10}|}$

The accumulated value is:

$$500\bar{s}_{\overline{10}|} + 100(\bar{I}\bar{s})_{\overline{10}|} = 500\frac{(1.10)^{10} - 1}{\ln(1.10)} + 100\frac{\frac{(1.10)^{10}-1}{\ln(1.10)} - 10}{\ln(1.10)} = 15,413.20$$

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

(B) Geometric Annuities

A continuously payable geometric annuity can also have a continuously changing rate of payment. The present value of such an annuity can be represented by the symbol $\bar{a}^{\bar{g}}_{\overline{n}|}$, where $\bar{g}$ is the continuously compounded rate of change in the payment rate. This means that the payment rate at time t is $e^{\bar{g}t}$. We can develop a formula for $\bar{a}^{\bar{g}}_{\overline{n}|}$ by integration:

$$\bar{a}^{\bar{g}}_{\overline{n}|} = \int_0^n e^{\bar{g}t} e^{-\delta t}\, dt = \int_0^n e^{-(\delta-\bar{g})t}\, dt = \frac{1-e^{-n(\delta-\bar{g})}}{\delta-\bar{g}} \qquad \bar{a}_{\overline{n}|} = \int_0^n \nu^t\, dt = \int_0^n e^{-\delta t}\, dt$$

In the case of a continuously perpetuity, we have:

$$\bar{a}^{\bar{g}}_{\overline{\infty}|} = \lim_{n\to\infty} \frac{1-e^{-n(\delta-\bar{g})}}{\delta-\bar{g}} = \frac{1}{\delta-\bar{g}} \quad \text{such that} \quad \delta > \bar{g}$$

The future value of a continuously varying geometric annuity is found by accumulating the present value to time n at the force of interest, δ:

$$\bar{s}^{\bar{g}}_{\overline{n}|} = \bar{a}^{\bar{g}}_{\overline{n}|} e^{n\delta} = \frac{1-e^{-n(\delta-\bar{g})}}{\delta-\bar{g}} e^{n\delta} = \frac{e^{n\delta}-e^{n.\bar{g}}}{\delta-\bar{g}} \qquad (1+i)^n = e^{n\ln(1+i)} = e^{n\delta}$$

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

$$\bar{a}_{\overline{n|}}^{\bar{g}} = \frac{1 - e^{-n(\delta - \bar{g})}}{\delta - \bar{g}}$$

$$\bar{a}_{\overline{\infty|}}^{\bar{g}} = \frac{1}{\delta - \bar{g}} \quad \text{such that} \quad \delta > \bar{g}$$

$$\bar{s}_{\overline{n|}}^{\bar{g}} = \frac{e^{n\delta} - e^{n.\bar{g}}}{\delta - \bar{g}}$$

Note: In some cases, the pattern of continuously varying payments is not arithmetic or geometric, and the interest rate also varies continuously. In this situation, the present value of the annuity must be expressed as an integral:

$$PV = \int_0^n \rho(t)\, e^{-\int_0^t \delta_s \, ds} \, dt$$

where $\rho(t)$ is a rate of payment per year that varies as a function of t and $e^{-\int_0^t \delta_s \, ds}$ is the present value factor.

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

Example: A 5–year annuity makes continuous payments at a rate of $(1000 + 50t)$ per year. Calculate the present value of this annuity at a continuously varying force of interest $\delta(t) = \frac{1}{20+t}$.

Section 2.13: Continuously Payable Annuities With Continuously Varying Payments

Example: A 5–year annuity makes continuous payments at a rate of $(1000 + 50t)$ per year. Calculate the present value of this annuity at a continuously varying force of interest $\delta(t) = \frac{1}{20+t}$.

Solution:

$$PV = \int_0^n \rho(t)\, e^{-\int_0^t \delta_s\, ds}\, dt$$

By substitution, we have

$$PV = \int_0^5 (1000 + 50t)\, e^{-\int_0^t \frac{1}{20+s}\, ds}\, dt$$

The present value factor in this expression is $e^{-\int_0^t \frac{1}{20+s}\, ds}$.

The integral can be evaluated as follows:

$$-\int_0^t \frac{1}{20+s}\, ds = -\ln(20+s)\Big|_0^t = -\Big[\ln(20+t) - \ln(20)\Big] = -\ln\left(\frac{20+t}{20}\right)$$

The present value factor is

$$e^{-\int_0^t \frac{1}{20+s}\, ds} = e^{\ln\left(\frac{20+t}{20}\right)^{-1}} = (\frac{20+t}{20})^{-1} = (1 + \frac{1}{20}t)^{-1} = (1+0.05t)^{-1} = \frac{1}{1+0.05t}$$

$$\Rightarrow PV = \int_0^5 (1000{+}50t)\, \frac{1}{1+0.05t}\, dt = 1000\int_0^5 (1{+}0.05t)\, \frac{1}{1+0.05t}\, dt = 1000\int_0^5 1\, dt = 1000\, t\Big|_0^5 = 1000(5{-}0) = 5000$$

Section : Exercises

Section 2.22: Basic review problems: all exercises.

Section 2.24 Sample exam problems: 1,3,4,7,8,11,14,18,19,20,22,23,27,28,29.