PHYS 404
1st Midterm Exam-Spring 2019
Sunday 27th of October 2019

Instructor: Dr. V. Lempesis

Student Grade: ……./15

Please answer all questions

1. If $f(x) = \sum_{k=0}^{\infty} A_k P_k(x)$, then prove Parseval's identity:

$$\int_{-1}^{1} f^2(x)\,dx = \sum_{k=0}^{\infty} \frac{2A_k^2}{2k+1}$$

You are given that:

$$\int_{-1}^{1} P_k^2(x)\,dx = \frac{2}{2k+1}$$

(5 marks)

Solution:

$$f(x) = \sum_{k=0}^{\infty} A_k P_k(x), \quad f(x) = \sum_{m=0}^{\infty} A_m P_m(x)$$

$$f^2(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} A_k A_m P_k(x) P_m(x)$$

$$\int_{-1}^{1} f^2(x)\,dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} A_k A_m \int_{-1}^{1} P_k(x) P_m(x)\,dx$$

$$\int_{-1}^{1} f^2(x)\,dx = \sum_{k=0}^{\infty} A_k^2 \int_{-1}^{1} P_k^2(x)\,dx$$

$$\int_{-1}^{1} f^2(x)\,dx = \sum_{k=0}^{\infty} \frac{2A_k^2}{(2k+1)}$$

2. (a) Use the Rodrigues formula and find the Legendre polynomial $P_2(x)$.

(2 marks)

(b) Find the Legendre functions $P_2^2(x)$.

(2 marks)

(c) Calculate the spherical harmonic $Y_2^{-2}(\theta, \phi)$ (1 mark).

(1 mark)

You are given that:

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n, \qquad P_n^m(x) = \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x) \quad m>0$$

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \quad m>0$$

$$Y_l^m(\theta,\ \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \quad \text{for any } m$$

Solution:

a)

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

$$P_2(x) = \frac{1}{2^2 2!}\left(\frac{d}{dx}\right)^2 (x^2-1)^2$$

$$P_2(x) = \frac{1}{8}\frac{d}{dx}[4x(x^2-1)]$$

$$P_2(x) = \frac{1}{8}\frac{d}{dx}[4x^3-4x]$$

$$P_2(x) = \frac{1}{8}[12x^2-4]$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

b)

$$P_n^m(x) = \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x) \quad m>0$$

$$P_2^2(x) = (1-x^2)^{2/2}\frac{d^2}{dx^2}\frac{1}{2}(3x^2-1)$$

$$P_2^2(x) = \frac{1}{2}(1-x^2)\frac{d}{dx}6x$$

$$P_2^2(x) = 3(1-x^2) = 3sin^2\theta$$

c)

$$Y_l^m(\theta,\ \phi)=(-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi},\quad \text{for any } m$$

$$Y_2^{-2}(\theta,\phi)=(-1)^{-2}\sqrt{\frac{(2\cdot 2+1)}{4\pi}\frac{(2-(-2))!}{(2-2)!}}P_2^{-2}(cos\theta)e^{-i2\phi}$$

$$Y_2^{-2}(\theta,\phi)=\sqrt{\frac{30}{\pi}}P_2^{-2}(cos\theta)e^{-i2\phi}$$

But

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)\quad m>0$$

Thus

$$P_2^{-2}(x)=(-1)^{-2}\frac{(2-2)!}{(2+2)!}3(1-x^2)=\frac{3}{4!}(1-x^2)$$

$$P_2^{-2}(x)=\frac{1}{8}(1-x^2)=\frac{1}{8}sin^2\theta$$

So

$$Y_2^{-2}(\theta,\phi)=\sqrt{\frac{30}{\pi}}\left(\frac{1}{8}sin^2\theta\right)e^{-i2\phi}=\sqrt{\frac{30}{64\pi}}sin^2\theta e^{-2i\phi}=\sqrt{\frac{15}{32\pi}}sin^2\theta e^{-2i\phi}$$

3. We know that:

$$P_{n+1}^{'}(x)-P_{n-1}^{'}(x)=(2n+1)P_n(x).$$

Calculate the general expression for the integral $\int_0^1 P_n(x)dx$

(5 marks)

You are given that:

$$P_{2n}(0)=\frac{\left(-1\right)^n\left(2n\right)!}{2^{2n}\left(n!\right)^2} \text{ and } P_{2n+1}(0)=0\,, \quad P_n(1)=1$$

Solution:

$$P'_{n+1}(x)-P'_{n-1}(x)=\left(2n+1\right)P_n(x)\Rightarrow P_n(x)=\frac{1}{\left(2n+1\right)}\left[P'_{n+1}(x)-P'_{n-1}(x)\right]$$

$$\int_0^1 P_n(x)dx=\frac{1}{(2n+1)}\int_0^1[P'_{n+1}(x)-P'_{n-1}(x)]dx$$

$$\int_0^1 P_n(x)dx=\frac{1}{(2n+1)}\left\{P_{n+1}(x)\Big|_0^1-P_{n-1}(x)\Big|_0^1\right\}$$

$$\int_0^1 P_n(x)dx=\frac{1}{(2n+1)}\{P_{n+1}(1)-P_{n+1}(0)-P_{n-1}(1)+P_{n-1}(0)\}$$

$$\int_0^1 P_n(x)dx=\frac{1}{(2n+1)}\{-P_{n+1}(0)+P_{n-1}(0)\}$$

If n is even then n+1, n-1 are odd. In this case the integral is zero.