# HYS 404
# 1st Midterm Exam-Fall 2017
# Tuesday 31st October 2017

## Instructor: Dr. V. Lempesis

## Student Grade: ……./20

***Please answer all questions***

**1.** Expand the function $f(x)=\begin{cases} -1 & 0<x<1 \\ 1 & -1<x<0 \end{cases}$ in a series of the form $f(x)=\sum_{k=0}^{\infty} A_k P_k(x)$ . You are given that: $A_k=\frac{2k+1}{2}\int_{-1}^{1} P_k(x)f(x)dx$ , $P'_{k+1}(x)-P'_{k-1}(x)=(2k+1)P_k(x)$ , $P_{2k}(0)=\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}$ and $P_{2k+1}(0)=0$ .

Also you are given that $P_k(1)=1$ and that $P_k(-1)=(-1)^k$. (8 marks)

**Solution**

$$A_k=\frac{2k+1}{2}\int_{-1}^{+1} P_k(x)f(x)dx=\frac{2k+1}{2}\left\{\int_{-1}^{0} P_k(x)f(x)dx+\int_{0}^{1} P_k(x)f(x)dx\right\}=$$
$$\frac{2k+1}{2}\left\{\int_{-1}^{0} P_k(x)dx-\int_{0}^{1} P_k(x)dx\right\} \qquad (1)$$

But

$$P'_{k+1}(x)-P'_{k-1}(x)=(2k+1)P_k(x)\Rightarrow P_k(x)=\frac{1}{2k+1}\left(P'_{k+1}(x)-P'_{k-1}(x)\right) \qquad (2)$$

By inserting (2) into (1) we have:

$$A_k=\frac{2k+1}{2}\left\{\int_{-1}^{0} P_k(x)dx-\int_{0}^{1} P_k(x)dx\right\}=$$
$$\frac{2k+1}{2}\left\{\int_{-1}^{0}\frac{1}{2k+1}\left(P'_{k+1}(x)-P'_{k-1}(x)\right)dx-\int_{0}^{1}\frac{1}{2k+1}\left(P'_{k+1}(x)-P'_{k-1}(x)\right)dx\right\}=$$
$$\frac{1}{2}\left\{\int_{-1}^{0}\left(P'_{k+1}(x)-P'_{k-1}(x)\right)dx-\int_{0}^{1}\left(P'_{k+1}(x)-P'_{k-1}(x)\right)dx\right\}=$$
$$\frac{1}{2}\left\{\int_{-1}^{0} P'_{k+1}(x)dx-\int_{0}^{1} P'_{k+1}(x)dx-\int_{-1}^{0} P'_{k-1}(x)dx+\int_{0}^{1} P'_{k-1}(x)dx\right\}=$$

$$\frac{1}{2}\left\{P_{k+1}(0)-P_{k+1}(-1)-P_{k+1}(1)+P_{k+1}(0)-P_{k-1}(0)+P_{k-1}(-1)+P_{k-1}(1)-P_{k-1}(0)\right\}=$$

$$\frac{1}{2}\left\{2P_{k+1}(0)-P_{k+1}(-1)-2P_{k-1}(0)+P_{k-1}(-1)\right\}=$$

A) if $k$=*even* then

$$P_{k+1}(0)=P_{k-1}(0)=0$$

thus

$$A_k=\frac{1}{2}\left\{-P_{k+1}(-1)+P_{k-1}(-1)\right\}=\frac{1}{2}\left\{(-1)^{k-1}-(-1)^{k+1}\right\}=0$$

B) if $k$=odd then $k$+1, $k$-1 are even thus

$$P_{k+1}(-1)=\left(-1\right)^{k+1}=1,\ \ P_{k-1}(-1)=\left(-1\right)^{k-1}=1$$

Thus

$$A_k=P_{k+1}(0)-P_{k-1}(0)$$

if $k$ =2$n$+1 then

$$A_k=P_{2(n+1)}(0)-P_{2n}(0)=\frac{\left(-1\right)^{n+1}(2(n+1))!}{2^{2(n+1)}((n+1)!)^2}-\frac{\left(-1\right)^{n}(2n)!}{2^{2n}(n!)^2}.$$

**2.** A) Write the general solution of the differential equation:

$$x^2y''+xy'+\left(x^2-\frac{1}{9}\right)y=0$$

(2 marks)

**Solution:**

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0 \Rightarrow \frac{x^2}{x^2}y'' + \frac{x}{x^2}y' + \left(\frac{x^2}{x^2} - \frac{1}{9x^2}\right)y = 0 \Rightarrow$$

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{9x^2}\right)y = 0 \Rightarrow y'' + \frac{1}{x}y' + \left(1 - \frac{(1/3)^2}{x^2}\right)y = 0$$

This is a Bessel equation with $\nu$=1/3. Thus the general solution is:

$$y(x) = AJ_{1/3}(x) + BJ_{-1/3}$$

B) Verify the following: $1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$ . You are given the generating function $e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

(6 marks)

**Solution:**

$$e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \underset{t=1}{\Rightarrow} e^{\left(\frac{x}{2}\right)\left(1-\frac{1}{1}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) \Rightarrow e^{\left(\frac{x}{2}\right)0} = \sum_{n=-\infty}^{\infty} J_n(x) \Rightarrow$$

$$1 = \sum_{n=-\infty}^{\infty} J_n(x) \Rightarrow 1 = \sum_{n=-\infty}^{-1} J_n(x) + J_0(x) + \sum_{n=1}^{\infty} J_n(x) \Rightarrow$$

$$1 = \sum_{n=1}^{\infty} J_{-n}(x) + J_0(x) + \sum_{n=1}^{\infty} J_n(x) \Rightarrow 1 = J_0(x) + \sum_{n=1}^{\infty}\left[J_n(x) + J_{-n}(x)\right] \Rightarrow$$

$$1 = J_0(x) + \sum_{n=1}^{\infty}\left[J_n(x) + (-1)^n J_n(x)\right] \Rightarrow 1 = J_0(x) + \sum_{n=1}^{\infty}\left[1 + (-1)^n\right]J_n(x)$$

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$$

**3.** You are given that

$$P_3(x) = \frac{1}{2}x\left(5x^2 - 3\right).$$

A) Calculate the polynomial $P_3^{-2}(x)$ (2 marks).

B) Calculate the spherical harmonic $Y_3^{-1}(\theta, \phi)$ (2 marks).

$$P_n^m(x) = \left(1 - x^2\right)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} P_n(x)$$

$$Y_l^m\left(\theta,\ \phi\right)=\varepsilon\sqrt{\frac{\left(2l+1\right)}{4\pi}\frac{\left(l-\left|m\right|\right)!}{\left(l+\left|m\right|\right)!}}P_l^m\left(\cos\theta\right)e^{im\phi}$$

$$\varepsilon=\begin{cases}\left(-1\right)^m, & m\geq 0\\ 1 & m<0\end{cases}$$

**Solution:**

A)

$$P_3^{-2}(x)=\left(1-x^2\right)^{\left|-2\right|/2}\frac{d^{\left|-2\right|}}{dx^{\left|-2\right|}}P_3\ (x)=\left(1-x^2\right)\frac{d^2}{dx^2}P_3\ (x)=\left(1-x^2\right)\frac{d^2}{dx^2}P_3\ (x)=$$

$$\left(1-x^2\right)\frac{d^2}{dx^2}\left[\frac{1}{2}x\left(5x^2-3\right)\right]=\frac{1}{2}\left(1-x^2\right)\frac{d}{dx}\left[\frac{d}{dx}\left(x\left(5x^2-3\right)\right)\right]=\frac{1}{2}\left(1-x^2\right)\frac{d}{dx}\left[\left(5x^2-3\right)+x10x\right]=$$

$$\frac{1}{2}\left(1-x^2\right)\frac{d}{dx}\left(15x^2-3\right)=\frac{1}{2}\left(1-x^2\right)30x=\frac{30}{2}x\left(1-x^2\right)$$

B)

$$Y_3^{-1}\left(\theta,\ \phi\right)=\sqrt{\frac{\left(2\cdot 3+1\right)}{4\pi}\frac{\left(3-\left|-1\right|\right)!}{\left(3+\left|-1\right|\right)!}}P_3^{-1}\left(\cos\theta\right)e^{-i\phi}=\sqrt{\frac{7}{4\pi}\frac{2!}{4!}}P_3^{-1}\left(\cos\theta\right)e^{-i\phi}$$

But

$$P_3^{-1}(x)=\left(1-x^2\right)^{\left|-1\right|/2}\frac{d^{\left|-1\right|}}{dx^{\left|-1\right|}}P_3\ (x)=\left(1-x^2\right)^{1/2}\frac{d}{dx}P_3\ (x)=\frac{d}{dx}P_3\ (x)=$$

$$\left(1-x^2\right)^{1/2}\frac{d}{dx}\left[\frac{1}{2}x\left(5x^2-3\right)\right]=\frac{1}{2}\left(1-x^2\right)^{1/2}\frac{d}{dx}\left[x\left(5x^2-3\right)\right]=\frac{1}{2}\left(1-x^2\right)^{1/2}\frac{d}{dx}\left[\left(5x^3-3x\right)\right]=$$

$$\frac{1}{2}\left(1-x^2\right)^{1/2}\left(15x^2-3\right)=\frac{1}{2}\left(1-\cos^2\theta\right)^{1/2}\left(15\cos^2\theta-3\right)=\frac{1}{2}\sin\theta\left(15\cos^2\theta-3\right)=\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right)$$

Thus

$$Y_3^{-1}\left(\theta,\ \phi\right)=\sqrt{\frac{7}{4\pi}\frac{2!}{4!}}P_3^{-1}\left(\cos\theta\right)e^{-i\phi}=\sqrt{\frac{7}{4\pi}\frac{2!}{4!}}\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right)e^{-i\phi}=$$

$$\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta-1\right)e^{-i\phi}$$