Question III:

(1) Prove that if $5n+6$ is odd, then n is odd.

Domain $\mathbb{Z}^+$

$5n+6$ is odd $\rightarrow$ n is odd

$5n+6=2K+1$

$\frac{5n}{5}=\frac{2K-5}{5}$

we cant know if this is odd or even then we cant use direct proof.

by contra position $(\neg q \rightarrow \neg p)$

n is even $\longrightarrow$ $5n+6$ is even

$n=2K$

$k \in ??$

$5n=5(2K)$

$5n+6=10K+6$

$5n+6=2(\underbrace{5K+3}_{s})$

$5n+6=2s$ which $s \in \mathbb{Z}^+$

1.5/2

then by contraposition we prove that if $5n+6$ is odd then n is odd

(2) Show that if x is a <u>rational</u> number, then $3x - 1$ is also <u>rational.</u>

$x=\frac{a}{b}$

x rational $\longrightarrow 3x-1$ rational

$3x=\frac{3a}{b}$

$3x-1=\frac{3a}{b}-1$

$3x=\frac{3a}{b}\cancel{-1}\cancel{+1}$

$\frac{3x}{3}=\frac{3a}{3b}$

$x=\frac{a}{b}$

1.5

then we prove that when x is rational then $3x-1$ is also rational

(3) Prove that the statement "$\forall n\,(n \le n^3)$ is false in the domain of integer number $\mathbb{Z}$.

by counter ex.

$n \le n^3$.

$n=-2$

try positiv num

$-2 \le -8$ so we prove that this statement is is false in the domain $\mathbb{Z}$.

1

$1 \le 1$

$2 \le 8$

$3 \le 27$

try Zero

$0 \le 0$

try negtive

$-1 \le -1$

$-2 \le -8$

$$p \to ((q \to r) \vee q) \equiv T$$

by L.H.S $p \to ((\neg q \vee r) \vee q)$

$\equiv p \to ((\neg q \vee q) \vee r)$

$\equiv p \to (T \vee r)$

$\equiv \neg p \vee (T \vee r)$

$\equiv (\neg p \vee T) \vee r$

$\equiv T \vee r$

$\equiv T$

2/2

So L.H.S $\equiv$ R.H.S

Question IV:

A. Prove <u>by cases</u> that $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for every real numbers $x, y \in \mathbb{R}$ and $y \neq 0$.

Case#1 $x=0, y>0$

$x=0 \quad y-y \qquad \frac{|0|}{|y|}=0$

then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \Leftarrow \left|\frac{0}{y}\right| = |0| = 0$

?

Case#2 $x>0, y>0$

$x>0, y<0$

$x=x, \quad y=y$

$\left|\frac{x}{y}\right| = \frac{x}{y} \qquad \frac{|x|}{|y|} = \frac{x}{y}$

then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

2.75

Case#3 $x<0, y<0$

$x=-x, \quad y=-y$

$\left|\frac{x}{-y}\right| = \frac{x}{y}, \quad \frac{|-x|}{|-y|} = \frac{x}{y}$

then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Case#4 $x<0, y>0$

$x=-x, \quad y=-y$

$\frac{|-x|}{|y|} = \frac{x}{y}, \quad \left|\frac{-x}{y}\right| = \frac{x}{y}$

then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Question Number	1	2	3	4	5	6	Total
Answer	~~d~~	b	c	a	~~d~~	a	2

Question I:

A. **Choose the correct answer, then fill in the table above:**

(1) The compound proposition $p \wedge \neg(p \vee T)$ is:

(a) a tautology (b) a contradiction (c) a contingency (d) None of the previous.

(2) The proposition "n is odd is a necessary and sufficient condition for n^2 to be odd" is logically equivalent to the proposition:

(a) If n is odd, then n^2 is odd. (b) n is odd if and only if n^2 is odd.

(c) If n^2 is odd, then n is odd (d) None of the previous

(3) Let $A = \{\phi\}$. Then $\mathcal{P}(A) =$

(a) $\{\emptyset\}$ (b) $\{\{\emptyset\}\}$ (c) $\{\emptyset, \{\emptyset\}\}$ (d) None of the previous

(4) $\forall x\,(x > 0 \vee x < 0)$ is false if the domain is:

(a) $\mathbb{Z}$ (b) $\mathbb{Z}^+$ (c) $\mathbb{Z}^-$ (d) None of the previous

(5) The converse of the proposition "We go camping whenever it is not raining ":

(a) If it is not raining, then we go camping. (b) If it is raining, then we cannot go campin

(c) If we go camping, then it is not raining (d) None of the previous

(6) Let $Q(x)$ be the statement" $x + 1 < 2x$" where the domain is the set of all intege then the truth value of $\exists x \neg Q(x)$ is:

$x+1 \geq 2x$
$0+1 \geq 2(0)$
$1 \geq 0$

(a) True (b) False (c) None of the prev

Q4: Use a proof by contraposition to show that if (xy) is an even number where $x, y \in \mathbb{Z}$, then x is even or y is even. (2 marks)

Sol

Let x is odd $\Rightarrow x = 2k+1$

and y is odd $\Rightarrow y = 2m+1$

ضرب

$xy = (2k+1)(2m+1)$

$xy = 4km + 2k + 2m + 1$

$xy = 2(2km + k + m) + 1$

xy is odd

xy is odd

$xy = 2 \cdot \square + 1$

الهدف

Q5: Let $x, y, z \in \mathbb{R}$ such $x + y + z = 21$, use a proof by contradiction to show that $x \geq 8$ or $y \geq 7$ or $z \geq 6$. (2 marks)

Sol

Let $x < 8$

and $y < 7$

and $z < 6$

جمع

$x + y + z < 21$

$\Longrightarrow x + y + z \neq 21$

Contradiction

$x \geq 8$ or $y \geq 7$ or $z \geq 6$

Q3: Determine the truth value of each of these statements and justify your answer. (2 marks)

(i) $\forall x \in \mathbb{R}, \quad \boxed{x^2 - 5x + 6 \geq 0.}$

$X = 2.5$

$(2.5)^2 - 5(2.5) + 6 = -0.25 \ngeq 0$

false

(ii) $\exists x \in \mathbb{R}, \quad x^4 < x^2.$

$x = \frac{1}{2}$

$\left(\frac{1}{2}\right)^4 < \left(\frac{1}{2}\right)^2$

$\frac{1}{16} < \frac{1}{4}$ ✓

True

Q2. (a) Let a and b be real numbers. Prove by contraposition that if $a + 2b > 10$, then $a > 4$ or $b > 3$. (2 pts)

Sol

Let $a \leq 4$

and $b \leq 3$. 2

$$a \leq 4$$

$$2b \leq 6$$

$$a + 2b \leq 10$$

$a+2b \leq 10$

Q3. (a) Prove that $3^{n-1} \geq 2^n + 1$ for all integers $n \geq 3$. (4 pts)

Sol

$$\boxed{P(n) : 3^{n-1} \geq 2^n + 1 \quad : n \geq 3}$$

B.S Prove: $P(n=3) : 3^{3-1} \geq 2^3 + 1$

$9 \geq 9$

Basis Step

I.S Hyp: $P(n=k) : 3^{k-1} \geq 2^k + 1$

Prove: $P(n=k+1) : 3^{k+1-1} \geq 2^{k+1} + 1$???

proof

$$3^{k+1-1} = 3^{k-1+1}$$

$$= 3^{k-1} \cdot 3^1 \qquad \text{from Hyp}$$

$$\geq (2^k + 1) \cdot 3$$

$$\geq \boxed{2^k \cdot 3 + 3} \longrightarrow \text{مهم}$$

$$\geq 2^k \cdot 2 + 1$$

$$\geq 2^{k+1} + 1$$

مع الحلاوة

(b) Assume that $\sqrt{7}$ is irrational. Give a proof by contradiction to show that $\frac{2+\sqrt{7}}{3}$ is irrational. (2 pts)

$\frac{a}{b}$

$\sqrt{7}$ $\sqrt{7}$

Sol

Let $\frac{2+\sqrt{7}}{3}$ is rational

$$\rightarrow \frac{2+\sqrt{7}}{3} = \frac{a}{b} : b \neq 0$$

$$\xrightarrow{3} 2+\sqrt{7} = \frac{3a}{b}$$

$$\sqrt{7} = \frac{3a}{b} - \frac{2}{1}$$

$$\sqrt{7} = \frac{3a-2b}{b} : b \neq 0$$

$\boxed{\sqrt{7} \text{ is rational}}$

Contradiction ⊅

$\frac{2+\sqrt{7}}{3}$ is irrational ⊅

(b) Without using truth tables, show that $p \to [(p \wedge q) \vee \neg q]$ is a tautology. (3 pts)

الحل

$$p \to [(p \wedge q) \vee \neg q]$$

$$\equiv \neg p \vee [(p \wedge q) \vee \neg q]$$ الشرط

$$\equiv \neg p \vee [\neg q \vee (p \wedge q)]$$ ابدال

$$\equiv (\neg p \vee \neg q) \vee (p \wedge q)$$ تجميع

$$\equiv \neg(p \wedge q) \vee (p \wedge q)$$ دي مورجان

$$\equiv T$$ النفي

Tautology

مع السلامة

I) Choose the correct answer:

$1+1 > 2(1)$
$2 > 2$ is false

1) Let $Q(x)$ be the statement $x+1 > 2x$, with domain the set of integers. Which of the following statements is **false**?

(A) $\exists x Q(x)$	(B) $\forall x Q(x)$	(C) $Q(0)$ True	(D) None

$F \to T$ is true

2) The conditional statement "*If* $1+5=3$, *then* $2+2=4$ is

(A) True	(B) False ×	(C) Undetermined	(D) None

$\neg(p \vee \neg p) \vee p$
$(\neg p \wedge p) \vee p$
$F \vee p \to p$

p	$\neg p$	$p \vee \neg p$	$(p \vee \neg p) \to p$
T	F	T	T
F	T	T	F

3) The proposition $(p \vee \neg p) \to p$ is a

(A) Tautology	(B) Contradiction	(C) Contingency	(D) None

5) $\neg(\exists x((x<0) \wedge (x^2>0)))$ is equivalent to $\forall_x (x \geq 0) \vee (x^2 \leq 0)$

(A) $\forall x((x \geq 0) \vee (x^2 \leq 0))$	(B) $\forall x((x \leq 0) \wedge (x^2 \leq 0))$	(C) $\exists x((x \geq 0) \vee (x^2 \leq 0))$	(D) None

$p \to q \Rightarrow \neg p \to \neg q$

6) The inverse of the conditional statement *"If n is odd, then n^2 is odd"* is

(A) If n^2 is even, then n is even	(B) If n^2 is odd, then n is odd	(C) If n is even, then n^2 is even	(D) None

II) A) **Without using truth tables**, prove that $p \to (q \land p) \equiv p \to q$.

$$p \to (q \land p) \equiv p \to q$$

$p \to (q \land p)$

$p \to (q \land p)$	$\equiv \neg p \lor (q \land p)$	As $p \to q \equiv \neg p \lor q$
	$\equiv (\neg p \lor q) \land (\neg p \lor p)$	Distributive law
	$\equiv (\neg p \lor q) \land T$	Negation law
	$\equiv \neg p \lor q$	Identity law
	$\equiv p \to q$	As $\neg p \lor q \equiv p \to q$

L.H.S $\equiv$ R.H.S

1.5/5

$p \quad \longleftrightarrow \quad q$

III) A) Prove that, if n is an integer number, then $n+2$ is even if and only if n^2-1 is odd.

① if $n+2$ is even then n^2-1 is odd

Suppose $n+2$ is even, $n+2=2k$; $k \in \mathbb{Z}$ then, $n = 2k-2$
$= 2(k-1)$ let $k-1 = m$, $m \in \mathbb{Z}$
$= 2m$ meaning n is even

by solving for $n = 2m$ in n^2-1 we have, $(2m)^2-1 = 4m^2-1$
$= 2(2m^2)-1$, let $2m^2 = b$, $b \in \mathbb{Z}$
$= 2b-1$, which is odd because $2b$ is even, and $2b+1$ is odd consecutively $2b-1$ is odd

$\therefore$ by direct proof if $n+2$ is even, n^2-1 is ~~odd~~ odd

② if n^2-1 is odd, then $n+2$ is even

using contraposition, if $n+2$ is odd, then n^2-1 is even

Suppose $n+2 = 2k+1$; $k \in \mathbb{Z}$
$n = 2k-1$, meaning n is odd

by solving for $n = 2k-1$ in n^2-1, we have, $(2k-1)^2-1 = 4k^2-4k+1-1$
$= 2(2k^2+2k)$, let $2k^2+2k = m$, $m \in \mathbb{Z}$
$= 2m$
meaning n^2-1 is even

$\therefore$ by contraposition if $n+2$ is odd then n^2-1 is even
thus, if n^2-1 is odd, then $n+2$ is even

4

Since $p \to q \wedge q \to p$ is proven true the statement $n+2$ is even if and only if n^2-1 is odd is true

B) Prove, by cases, that $|x| - x \geq 0$, for every real number x.

$|x| \geq x$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

case 1: $x = 0$
$|0| = 0 \to$ therefor $|x| - x \geq 0$ is true
$0 - 0 \geq 0$

case 2: $x > 0$
$|x| = x \to$ therefore $|x| - x \geq 0$ is true
$x - x \geq 0$
$0 \geq 0$

0.75

case 3: $x < 0$

~~$|x|$~~ ~~$x + x \geq 0$~~ therefore $|x| - x \geq 0$ is true
~~$|-x|$~~ $x - -x \geq 0$
$|x| = +x$ $x + x \geq 0$
~~$x + x \geq 0, 0 \geq 0$~~

Since case 1, 2, 3 are all true, $|x| - x \geq 0 \; \forall x \in \mathbb{R}$

IV) A) What is the truth value of $\forall n(4n^2 - 3n^3 \geq 0)$, if the domain is the set of integer numbers?

Suppose $n = 2$, then $4(2)^2 - 3(2)^3 \geq 0$

$4(4) - 3(8) \geq 0$

$16 - 24 \geq 0$

$-8 \geq 0 \rightarrow$ false

by counterexample, the truth value is false

1

I) Choose the correct answer (write it on the table above):

1) The truth value of the proposition "$10 - 1 = 8$ *if and only if* $3 + 2 = 1$" is

(A) True	(B) False	(C) Undetermined	(D) None

2) Let $P(x, y)$ be the statement "$y + x$ *is a perfect square*". Then

(A) $P(3, 2)$ is true	(B) $P(2, 2)$ is true	(C) $P(1, 1)$ is true	(D) None

3) The converse of the statement "*If* $x > 2$, *then* $x + 2 > x^2$ is

(A) If $x + 2 > x^2$, then $x > 2$	(B) If $x \leq 2$, then $x + 2 \leq x^2$	(C) If $x + 2 \leq x^2$, then $x \leq 2$	(D) None

6) The negation of $\exists x \in \mathbb{R}(x > 2 \wedge x \leq 3)$ is

(A) $\exists x \in \mathbb{R}$ $(x \leq 2 \vee x > 3)$	(B) $\forall x \in \mathbb{R}$ $(x \leq 2 \wedge x > 3)$	(C) $\forall x \in \mathbb{R}$ $(x \leq 2 \vee x > 3)$	(D) None

II) A) Without using truth tables, prove that $p \to (q \to r) \vee q$ is a tautology.

III) Prove the theorem:

"If n is an integer number, then n is odd if and only if $5n + 3$ is even".

IV) A) Prove, by cases, that, for every real number x, the inequality $x \geq -|x|$ holds.

B) Prove that the statement "$n^2 - 7n + 10 \geq 0$, *for all integer numbers* n" is false.

I) Choose the correct answer (write it on the table above): F

1) The truth value of the statement "For all positive integers x, $\underbrace{\text{if } \underbrace{x^2 < 0}, \text{ then } x + 1 > 2}_{T}$" is

✓ (A) True	(B) False	(C) Undetermined	(D) None

2) Let $P(x, y)$ be the statement "$x = 3y + 6$". Then

(A) $P(1, 6)$ is true	(B) $P(1, 9)$ is true	(C) $P(9, 1)$ is true ✓ $9 = 3(1) + 6$	(D) None

تكافؤ

3) The conditional statement $p \to q$ is logically equivalent to

(A) Its converse $q \to p$	النقيض العكسي (B) Its contrapositive $\neg q \to \neg p$ ✓	(C) Its inverse $\neg p \to \neg q$	(D) None

نفي

4) The negation of a contradiction is

(A) A tautology ✓	(B) A contradiction	(C) A contingency	(D) None

5) $\neg[\forall x(1 + 2x \geq 2 - x \vee x > 0)]$ is equivalent to

(A) $\exists x(1 + 2x \geq 2 - x \vee x > 0)$	(B) $\forall x(1 + 2x \wedge 2 - x \vee x > 0))$	(C) $\exists x(1 + 2x < 2 - x \wedge x \leq 0)$ ✓	(D) None

II) A) **Without using truth tables**, prove that $[\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology.

$$[\neg p \wedge (p \vee q)] \rightarrow q \overset{?}{\equiv} T$$

$$\text{L.H.S} = [\neg p \wedge (p \vee q)] \rightarrow q$$

$$\equiv \neg [\neg p \wedge (p \vee q)] \vee q$$

$$\equiv [p \vee \neg (p \vee q)] \vee q$$

$$\equiv [p \vee (\neg p \wedge \neg q)] \vee q$$

$$\equiv [\underline{\overset{T}{(p \vee \neg p)}} \wedge (p \vee \neg q)] \vee q$$

$$\equiv [T \wedge (p \vee \neg q)] \vee q \equiv (p \vee \neg q) \vee q \equiv p \vee (\neg q \vee q)$$

$$\equiv p \vee T \equiv T$$

III) A) Prove that, for all integers n, the following three statements are equivalent:

(a) n is odd

(b) $3n + 3$ is even

(c) $n - 5$ is even

we must prove that $a \to b$, $b \to c$ and $c \to a$.

1) n is odd $\to 3n+3$ is even.

by directed proof

suppose n is odd show that $3n+3$ is even

n is odd $\to \exists k \in \mathbb{Z}$ s.t $n = 2k+1$

$\to 3n+3 = 3(2k+1)+3 = 6k+3+3$

$= 6k+6$

$= 2(3k+2)$

$= 2s \quad s = 3k+2 \in \mathbb{Z}$

$\therefore 3n+3$ is even.

2) $3n+3$ is even $\to n-5$ is even.

by Contrapostive $\Rightarrow$ $n-5$ is odd $\to 3n+3$ is odd.

$n-5 = 2k+1 \to n = 2k+6$

$\to 3n+3 = 3(2k+6)+3$

$= 6k+18+3 = 6k+21$

$= 6k+20+1$

$= 2(3k+10)+1$ odd.

3) $n-5$ is even $\to n$ is odd.

$n-5 = 2k \to n = 2k+5$

$\to n = 2k+4+1$

$\to n = 2(k+2)+1$ which is odd.

B) Prove that the statement "*For all positive integers m and n, $mn \geq m+n$*" is false.

by Counter example.

let $m = 1 \quad n = 2$

$m.n = (1)(2) = 2$

$m+n = 1+2 = 3$

$2 \geq 3$ False.

$m.n \geq m+n$ is false by Counter examplo.

IV) A) Prove that, if x is any real number, then $x \geq -|x|$.

by cases

1) Case 1 if $x = 0$

$0 \geq -|0|$ ✓ $\rightarrow x \geq |x|$ ✓

2) Case 2 if $x < 0 \xrightarrow{?} x \geq -|x|$

$|x| = -x > 0$

$-|x| = x < 0$ يعني سالب

$\therefore x = -|x| \rightarrow x \leq -|x|$ true .

3) Case 3 if $x > 0 \xrightarrow{?} x \geq -|x|$

$|x| = x > 0$

$-|x| = -x < 0 \rightarrow x \geq -|x|$

from three cases

$x \geq -|x| \quad \forall x \in R.$

III) A) **Prove the theorem:**

"If the product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10".

$$xy > 100 \rightarrow x > 10 \text{ or } y > 10 \qquad x, y \in R^+$$

by contrapostive.

$$x \leq 10 \text{ and } y \leq 10 \overset{?}{\longrightarrow} xy \leq 100$$

suppose $x \leq 10$ and $y \leq 10$. and showthat $xy \leq 100$

$$x \leq 10$$
$$y \leq 10$$

$$xy \leq 100$$

Since the contrapositive is true.

$xy > 100 \rightarrow x > 10$ or $y > 10$ is true.

1.5 X

B) Prove that the statement "*For all positive integers m and n, $mn \geq m + n$*" is false.

$mn \geq m+n \quad \forall\, m, n \in Z^+$ is false.

by Counter example.

$m = 1 \qquad n = 5.$

$m.n = 5$

$m+n = 6$

$m.n \not\geq m+n.$

The statment is false.

Question Number	1	2	3	4	5	6	Total
Answer	a	a	c	a	b	b	

Question I:

A. **Choose the correct answer, then fill in the table above:**

$(p\wedge q)\vee(\neg p\vee\neg q)$
$\equiv(p\wedge q)\vee\neg(p\wedge q)$
$\equiv T$

$p\vee\neg p\equiv T$

(1) The compound proposition $(p \wedge q) \vee (\neg p \vee \neg q)$ is:

✓(a) a tautology (b) a contradiction (c) a contingency (d) None of the previous.

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(2) $\neg(\exists x: (x^2 > 2 \vee y \neq 2))$ is equivalent to

✓(a) $\forall x: (x^2 \leq 2 \wedge y = 2)$

(b) $\exists x: (x^2 \leq 2 \wedge y = 2)$

(c) $\forall x: (x^2 \leq 2 \vee y = 2)$

(d) None of the previous

---

(4) $\forall x\ (x > 0 \ \vee \ x < 0)$ is false if the domain is:

in $\mathbb{Z}$ $(0>0 \vee 0<0) \to$ False

$\neg\forall x(x>0 \vee x<0)$ False

✓(a) $\mathbb{Z}$ (b) $\mathbb{Z}^+$ (c) $\mathbb{Z}^-$ (d) None of the previous

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(5) The converse of the proposition "If today is Monday, then we have an exam" :

(a) "We have an exam whenever today is Monday" ✓(b) "If we have an exam, then today is Monday"

(c) "If today is not Monday, then we have not an exam" (d) None of the previous

---

(6) Let $Q(x)$ be the statement" $x = x + 1$" where the domain is the set of all integers, then the truth value of $\exists x \neg Q(x)$ is:

(a) True ✓(b) False (c) None of the previous

$Q(x):\ x=x+1$
$\neg Q(x):\ x\neq x+1$
Domain $\mathbb{Z}$

$\exists x\ (x\neq x+1)$ ✓

Since $\forall x\in\mathbb{Z}$ $x\neq x+1$ is false

the statment false

Question II:

A. **Without using truth tables** prove the following:

$$p \wedge (p \to q) \wedge (p \to \neg q) \equiv F$$

$L.H.S = p \wedge (p \to q) \wedge (p \to \neg q)$

$\equiv p \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

$\equiv (p \wedge \neg p) \vee \underline{(p \wedge q) \wedge \neg (p \wedge q)}$

$\equiv (p \wedge \neg p) \vee F$

$\equiv F \vee F$

$\equiv F$

$p \wedge \neg p \equiv F$

Question III:

A. If n is an integer, prove that "$7n + 4$ *is even if and only if* n *is even*".

$7n+4$ is even $\leftrightarrow$ n is even

we must show that

① $7n+4$ is even $\rightarrow$ n is even

by contrapostive

n is odd $\rightarrow$ $7n+4$ is odd

let n is odd

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k+1$

$\rightarrow 7n = 14k + 7$

$\rightarrow 7n+4 = 14k+11$

$= 14k+10+1$

$= 2(7k+5)+1$

$= 2t+1 \qquad t = 7k+5 \in \mathbb{Z}$

which is odd.

② if n is even $\rightarrow$ $7n+4$ is even

by direct proof.

suppose n is even show that $7n+4$ is even

let n is even

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k$

$\rightarrow 7n = 14k$

$\rightarrow 7n+4 = 14k+4$

$= 2(7k+2)$

$= 2s \qquad s = 7k+2 \in \mathbb{Z}$

$\therefore$ $7n+4$ is even.

B. Prove that if x is irrational, then $1/x$ is irrational.

by contrapostive

suppose $\frac{1}{x}$ is rational show that x is rational.

let $\frac{1}{x}$ be rational

$\rightarrow \frac{1}{x} = \frac{a}{b} \qquad a, b \in \mathbb{Z},\ b \neq 0 \qquad$ suppose $a \neq 0$

$\rightarrow x = \frac{b}{a} \qquad b, a \in \mathbb{Z},\ a \neq 0$

$\rightarrow x$ is rational

??

C. Prove that the statement "$\forall x \in \mathbb{R}, |x| > 0$" **is false** in the domain of integer number $\mathbb{Z}$.

since $|0| > 0$ is false

$\therefore \forall x, |x| > 0$

is false.

Question IV:

A. Prove **by cases** that "$n^2 + 3n$ *is even*", for every integer number n.

we have two cases

Case 1

if n is even, then n^2+3n is even

suppose n is even

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k$

$\rightarrow n^2 = 4k^2$

$\rightarrow n^2+3n = 4k^2+3(2k)$

$= 4k^2+6k$

$= 2(2k^2+3k) = 2s \qquad s = 2k^2+3k \in \mathbb{Z}$

which is even.

Case 2

if n is odd, then n^2+3n is even.

suppose n is odd

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k+1 \quad k \in \mathbb{Z}$

$\rightarrow n^2 = (2k+1)^2 = 4k^2+4k+1$

$\rightarrow n^2+3n = 4k^2+4k+1+3(2k+1)$

$= 4k^2+4k+1+6k+1$

$= 4k^2+10k+2$

$= 2(2k^2+5k+1)$

$= 2t \qquad t = 2k^2+5k+1 \in \mathbb{Z}$

which is even

Question I:

A. **Choose the correct answer, then fill in the table above:**

$p \wedge \neg(p \vee T)$
$\equiv p \wedge \neg T$
$\equiv p \wedge F$
$\equiv F$

(1) The compound proposition $p \wedge \neg(p \vee T)$ is:

(a) a tautology (b) a contradiction (c) a contingency (d) None of the previous.

(2) The proposition "n is odd is a necessary and sufficient condition for n^2 to be odd" is logically equivalent to the proposition:

$\leftrightarrow$

(a) If n is odd, then n^2 is odd. (b) n is odd if and only if n^2 is odd.

(c) If n^2 is odd, then n is odd (d) None of the previous

(4) $\forall x\,(x > 0 \vee x < 0)$ is false if the domain is:

since $0>0 \vee 0<0$ false
$\therefore \forall x\,(x>0 \vee x<0)$ is false in $\mathbb{Z}$

(a) $\mathbb{Z}$ (b) $\mathbb{Z}^+$ (c) $\mathbb{Z}^-$ (d) None of the previous

(5) The converse of the proposition "We go camping whenever it is not raining ":

statment: if it is not raning, then we go camping.

(a) If it is not raining, then we go camping. (b) If it is raining, then we cannot go camping.

(c) If we go camping, then it is not raining (d) None of the previous

(6) Let $Q(x)$ be the statement" $x + 1 < 2x$" where the domain is the set of all integers, then the truth value of $\exists x \neg Q(x)$ is:

(a) True (b) False (c) None of the previous

$\exists x \in \mathbb{Z}\,(x+1 \geq 2x)$
$1+1 \geq 2.1$ is true
The statment is true.

A. **Without using truth tables** prove the following:

$$p \to ((q \to r) \vee q) \equiv T$$

$$
\begin{aligned}
L.H.S = \; & p \to ((q \to r) \vee q) \\
\equiv \; & p \to ((\neg q \vee r) \vee q) \\
\equiv \; & p \to (r \vee (q \vee \neg q)) \\
\equiv \; & p \to (r \vee T) \\
\equiv \; & p \to T \\
\equiv \; & \neg p \vee T \\
\equiv \; & T \quad = R.H.S
\end{aligned}
$$

٦. **Question III:**

(1) Prove that if $5n + 6$ is odd, then n is odd.

by contraposition.

Suppose n is even show that $5n+6$ is even.

let n is even.

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k$

$\rightarrow 5n = 10k$

$\rightarrow 5n + 6 = 10k + 6$

$= 2(5k+3)$

$= 2m \qquad m = 5k+3 \in \mathbb{Z}$

which is even

(2) Show that if x is a rational number, then $3x - 1$ is also rational.

suppose that x is rational and show that $3x - 1$ is rational

let x be rational number

$\rightarrow x = \frac{a}{b} \qquad a, b \in \mathbb{Z}, \; b \neq 0$

$\rightarrow 3x = \frac{3a}{b}$

$\rightarrow 3x - 1 = \frac{3a}{b} - 1 = \frac{3a - b}{b} = \frac{m}{t} \qquad m = 3a - b \in \mathbb{Z}$

$t = b \in \mathbb{Z}$

$\therefore 3x - 1$ is also rational $\quad t \neq 0$

(3) Prove that the statement "$\forall n \, (n \leq n^3)$ is **false** in the domain of integer number $\mathbb{Z}$.

let $n = -2$

$n^3 = -8$

$-2 \leq -8$ false

$\therefore \forall n \, (n \leq n3)$ is false.

٦.

Question IV:

A. Prove by cases that $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for every real numbers $x, y \in \mathbb{R}$ and $y \neq 0$.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

There are 4 cases

Case 1.

let $x \geq 0, y > 0 \rightarrow \frac{x}{y} \geq 0$

$|x| = x$

$|y| = y$

$\frac{|x|}{|y|} = \frac{x}{y}$

$\left|\frac{x}{y}\right| = \frac{x}{y}$ = which is true

Case 2

$x \geq 0, y < 0 \rightarrow \frac{x}{y} \leq 0$

$|x| = x$

$|y| = -y$

$\frac{|x|}{|y|} = \frac{x}{-y}$

$\left|\frac{x}{y}\right| = -\frac{x}{y}$ = which is true.

Case 3

$x < 0, y > 0 \rightarrow \frac{x}{y} < 0$

$|x| = -x$

$|y| = y$

$\frac{|x|}{|y|} = \frac{-x}{y}$

$\left|\frac{x}{y}\right| = -\frac{x}{y}$ = which is true.

Case 4

$x < 0, y < 0 \rightarrow \frac{x}{y} > 0$

$|x| = -x$

$|y| = -y$

$\frac{|x|}{|y|} = \frac{-x}{-y} = \frac{x}{y}$ which is true

$\left|\frac{x}{y}\right| = \frac{x}{y}$

from all cases the statment is true.

GZ

Question I:

∃! (1/1 = 1)
-1 ∈

A. **Choose the correct answer, then fill in the table above:**

$N = \{1, 2, 3, 4, ---\}$

$a = 1 \rightarrow \frac{1}{a} = \frac{1}{1} = 1 \in N$
$a = 2 \rightarrow \frac{1}{a} = \frac{1}{2} \notin N$

(1) $\exists!\, a \in \mathbb{N}$ *such that* $\frac{1}{a} \in \mathbb{N}$ is

(a) True (b) False (c) None of the previous

يقبل القسمة على 1 او نفسه فقط

(2) Let $P(x, y)$ be the statement" $y - x$ *is prime* " then أصغر عدد أولي 2 $\{2, 3, 5, 7, 11, ---\}$

(a) $P(1,5)$ is true (b) $P(2,5)$ is true (c) $P(3,3)$ is true (d) None of the previous

$5 - 1 = 4$ not prime

$5 - 3 = 3$ prime

$3 - 3 = 0$

---

(4) $\neg(\forall x: (x > 1 \lor x < -1))$ is equivalent to

(a) $\forall x\, (-1 \leq x < 1)$

(b) $\exists x\, (x \leq 1 \land x \geq -1)$

(c) $\forall x\, (x \leq 1 \land x \geq -1)$

(d) None of the previous

---

(5) If $\forall x\, (x \in A \leftrightarrow x \in B)$ then

(a) $A \subseteq B$ (b) $A = B$ (c) $B \subseteq A$ (d) None of the previous

---

Question II:

A. **Without using truth tables** prove the following:

$$\neg p \vee (r \to \neg q) \equiv (p \to \neg r) \vee \neg q$$

$$\begin{aligned}
L.H.S &= \neg p \vee (r \to \neg q) \\
&= \neg p \vee (\neg r \vee \neg q) \\
&\equiv (\neg p \vee \neg r) \vee \neg q \\
&\equiv (p \to \neg r) \vee \neg q \\
&\equiv R.H.S
\end{aligned}$$

Question III:

(1) Prove that if n is odd if and only if $5n^2 + 3$ is even.

n is odd $\leftrightarrow$ $5n^2+3$ is even

we must proof

1) n is odd $\rightarrow$ $5n^2+3$ is even.

by using direct proof

Suppose n is odd and Show that $5n^2+3$ is even

suppose n is odd

$\rightarrow \exists k \in Z$ s.t $n = 2k+1$

$n^2 = (2k+1)^2 = 4k^2+4k+1$

$5n^2 = 20k^2+20k+5$

$5n^2+3 = 20k^2+20k+8$

$= 2(\underbrace{10k^2+10k+4}_{s})$

$= 2s$ $\quad s = 10k^2+10k+4 \in Z$

which is even

2) $5n^2+3$ is even $\rightarrow$ n is odd.

by using Contrapostive

n is even $\rightarrow$ $5n^2+3$ is odd

suppose n is even

$\rightarrow \exists k \in Z$ s.t $n = 2k$

$\rightarrow n^2 = 4k^2$

$\rightarrow 5n^2 = 20k^2$

$\rightarrow 5n^2+3 = 20k^2+3$

$= 20k^2+2+1$

$= 2(10k^2+1)+1$

which is odd.

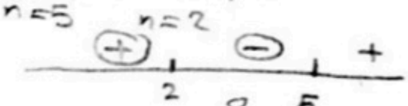
B. Prove that the statement "$n^2 - 7n + 10 \geq 0$ **is false** for every integer number $n \in \mathbb{Z}$.

باعطاء مثال.

$\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

note.

$$\begin{cases} n^2-7n+10 \geq 0 \\ (n-5)(n-2) \geq 0 \\ n=5 \quad n=2 \end{cases}$$



if $n=3$ $\rightarrow$ $n^2-7n+10$

$= 3^2 - 7(3) + 10$

$= 9 - 21 + 10 = -2 < 0$

$\therefore n^2-7n+10 \geq 0 \;\; \forall n \in Z$ is false.

Question IV:

A. Prove **by cases** that "$n^2 - n$" is even for every integer number n.

there are two cases

Case 1) n is even $\xrightarrow{?}$ $n^2 - n$ is even

suppose n is even

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k$

$\rightarrow n^2 = 4k^2$

$\rightarrow n^2 - n = 4k^2 - 2k$

$= 2(2k^2 - k)$

$= 2t \qquad t = 2k^2 - k \in \mathbb{Z}$

which is even.

Case 2) n is odd $\rightarrow$ $n^2 - n$ is even

suppose n is odd

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k + 1$

$\rightarrow n^2 = 4k^2 + 4k + 1$

$\rightarrow n^2 - n = 4k^2 - 4k + \cancel{1} - 2k - \cancel{1}$

$= 4k^2 - 6k$

$= 2(2k^2 - 3k)$

which is even.

from two cases $n^2 - n$ is even for every integer n.

Question Number	1	2	3	4	5	6	Total
Answer	b	a	a	c	b	b	

Question I:

Choose the correct answer, then fill in the table above:

F $\leftrightarrow$ T F

(1) The truth value of "$5 + 1 = 4$ *if and only if* $5 + 1 = 6$" is

(a) True (b) False (c) None of the previous

---

(2) Let $Q(x, y)$ be the statement "$x = y + 5$" then

(a) $Q(5,0)$ is true (b) $Q(0,5)$ is true (c) $Q(3,2)$ is true (d) None of the previous

$5 = 0+5$ ✓ $0 = 5+5$ ✗ $3 = 2+5$ ✗

---

(3) If $P(x)$ is the statement "$2x = x,\ x \in \mathbb{Z}$", then the truth value of $\exists x\, P(x)$ is

$\exists\ x = 0$

(a)True (b) False (c) None of the previous

$2 \cdot 0 = 0$ ✓

True.

---

(4) The inverse of the statement "*if* $x^2 < x$ *then* $0 < x < 1$" is

$x \le 0$ 0 1 $x \ge 1$

(a) *if* $x^2 > x$ *then* $x \le 0 \wedge x \ge 1$

(b) *if* $0 < x < 1$ *then* $x^2 < x$

(c) *if* $x^2 \ge x$ *then* $x \le 0 \vee x \ge 1$

(d) None of the previous

---

(5) $\neg[\forall x\ (x < y \wedge x \text{ is rational})]$ is equivalent to

(a) $\forall x\ (x \ge y \wedge x \text{ is irrational})$

(b) $\exists x\ (x \ge y \vee x \text{ is irrational})$

(c) $\exists x\ (x \ge y \wedge x \text{ is irrational})$

(d) None of the previous

Question II:

A. **Without using truth tables** prove the following:

$$\neg p \to (\neg p \wedge q) \equiv \neg p \to q.$$

$$\boxed{p \to q \equiv \neg p \vee q}$$

$$\begin{aligned}
\text{L.H.S} &= \neg p \to (\neg p \wedge q) \\
&= \neg(\neg p) \vee (\neg p \wedge q) \\
&= p \vee (\neg p \wedge q) \\
&= (p \vee \neg p) \wedge (p \vee q) \\
&\equiv T \wedge (p \vee q) \\
&\equiv p \vee q \\
&\equiv \neg p \to q \equiv \text{R.H.S}
\end{aligned}$$

Question III:

A. If n is an integer, prove that "$7n + 4$ *is even if and only if n is even*".

$7n+4$ is even $\leftrightarrow$ n is even

we must prove that

① ②

$7n+4$ is even $\rightarrow$ n is even

by Contrapostive

$\equiv$ n is odd $\rightarrow$ $7n+4$ is odd

let n is odd

$\rightarrow n = 2t+1 \quad t \in \mathbb{Z}$

$\rightarrow 7n = 14t + 7$

$\rightarrow 7n+4 = 14t+7+4$

$= 14t + 10 + 1$

$= 2(7t+5)+1$

which is odd

n is even $\rightarrow$ $7n+4$ is even

direct proof

let n is even

$\rightarrow n = 2k \quad k \in \mathbb{Z}$

$\rightarrow 7n = 14k$

$\rightarrow 7n+4 = 14k+4$

$= 2(7k+2)$

which is even

from ① and ②,

The statment is true.

B. Prove that the statement "$for\ every\ x, y \in \mathbb{R},\ (x+y)^2 = x^2 + y^2$" is **false**.

if $x=3 \quad y=4$

$L.H.S = (x+y)^2 = (3+4)^2 = 7^2 = 49$

$R.H.S = x^2+y^2 = 3^2+4^2 = 9+16 = 25$

$L.H.S \neq R.H.S$

by counter example.

$\forall x, y \ (x+y)^2 = x^2+y^2$ is false.

Question IV:

A. Prove **by cases** that "$2n^2 + 3$ *is odd*", for every integer number n.

There are two cases

Case 1: if n is odd $\overset{?}{\rightarrow}$ $2n^2+3$ is odd.

let n is odd $\rightarrow$ $n = 2k+1$ $\quad k \in \mathbb{Z}$

$\rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$\rightarrow 2n^2 = 8k^2 + 8k + 2$

$\rightarrow 2n^2 + 3 = 8k^2 + 8k + 4 + 1 = 2(4k^2 + 4k + 2) + 1$ which is odd.

Case 2: if n is even $\rightarrow$ $2n^2+3$ is odd.

let n is even $\rightarrow$ $n = 2k$ $\quad k \in \mathbb{Z}$

$\rightarrow n^2 = 4k^2$

$\rightarrow 2n^2 = 8k^2$

$\rightarrow 2n^2 + 3 = 8k^2 + 3 = 8k^2 + 2 + 1 = 2(4k^2 + 1) + 1$ which is odd

Question Number	1	2	3	4	5	6	Total
Answer	a	b	c	a	b	c	

Question I:

$(p \wedge q) \vee (\neg p \vee \neg q)$
$\equiv (p \wedge q) \vee \neg (p \wedge q)$
$\equiv T$

أو بالجدول

A. Choose the correct answer, then fill in the table above:

(1) The statement "$(p \wedge q) \vee (\neg p \vee \neg q)$ " is a

✓ (a) Tautology (b) Contradiction (c) Contingency (d) None of the previous

---

(2) Let $P(x, y)$ be the statement" $y - x$ *is odd* " then

(a) $P(1,5)$ is true ✓ (b) $P(2,5)$ is true (c) $P(3,5)$ is true (d) None of the previous

$5-1=4$ (F) $5-2=3$ (T) $5-3=2$ (F)

---

(3) The contrapositive of the statement "$if\ |x| = x\ then\ x \geq 0$"is

Contrapostive
$\neg q \to \neg p$.

(a) $if\ |x| \neq x\ then\ x < 0$ (b) $if\ \ x \geq 0\ then\ |x| = x$

✓ (c) "$if\ x < 0\ then\ |x| \neq x$ (d) None of the previous

---

(4) $\neg(\exists x: (x^2 > 2 \vee y \neq 2))$ is equivalent to $\rightsquigarrow \forall x (x^2 \leq 2 \wedge y = 2)$

✓ (a) $\forall x: (x^2 \leq 2 \wedge y = 2)$ (b) $\exists x: (x^2 \leq 2 \wedge y = 2)$

(c) $\forall x: (x^2 \leq 2 \vee y = 2)$(d) None of the previous

---

(5) $\exists! x\ (|x| = 1)$ is

$|1| = 1$ and $|-1| = 1$

(a) True ✓ (b) False (c) None of the previous

Question II:

A. **Without using truth tables** prove the following:

$$\neg p \vee (r \to \neg q) \equiv (p \to \neg r) \vee \neg q$$

$$L.H.S \equiv \neg p \vee (r \to \neg q)$$

$$\equiv \neg p \,\underline{\vee}\, (\neg r \,\underline{\vee}\, \neg q)$$

$$\equiv (\neg p \vee \neg r) \vee \neg q$$

$$\equiv (p \to \neg r) \vee \neg q \equiv R.H.S$$

Question III:

A. Prove that the product of two rational numbers is rational.

The statment: x, y are rational numbers $\xrightarrow{?} x \cdot y$ is rational

use direct proof

Suppose that x and y are rational numbers and show that $x \cdot y$ is also rational

$x = \frac{a}{b}$ $a, b \in \mathbb{Z}, b \neq 0$

$y = \frac{c}{d}$ $c, d \in \mathbb{Z}, d \neq c$

$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ $\rightarrow ac \in \mathbb{Z}$, $\rightarrow bd \in \mathbb{Z}$

$bd \neq 0$ sine $b \neq 0$ and $d \neq 0$

$\therefore$ The product of two rational numbers is rational.

B. Prove **by cases** that "$n^2 + 3n$ *is even*", for every integer number$n$.

there are two cases

Case 1

n is even

n is even $\xrightarrow{?} n^2 + 3n$ is even

Suppose n is even

$n = 2k$ $\exists k \in \mathbb{Z}$

$n^2 = 4k^2$

$n^2 + 3n = 4k^2 + 3(2k)$

$= 4k^2 + 6k$

$= 2(2k^2 + 3k)$

$= 2S$ $S = 2k^2 + 3k$

$\rightarrow$ even

$\therefore n^2 + 3n$ is even

Case 2

n is odd.

n is odd $\xrightarrow{?} n^2 + 3n$ is even.

Suppose n is odd.

$\rightarrow n = 2k + 1$ $\exists k \in \mathbb{Z}$

$\rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$\rightarrow n^2 + 3n = 4k^2 + 4k + 1 + 3(2k+1)$

$= 4k^2 + 4k + 1 + 6k + 3$

$= 4k^2 + 10k + 4$

$= 2(\underbrace{2k^2 + 5k + 2}_{m})$

$= 2m$ $m \in \mathbb{Z}$

$\therefore n^2 + 3n$ is even.

Question IV:

A. Prove that the statement "$n^2 - 4n \leq 0$ *for every integer number* n" is **false**.

Let $n = 10$

$n^2 - 4n = (10)^2 - 4(10)$

$= 100 - 40$

$= 60 \leq 0$ False.

$\therefore n^2 - 4n \leq 0$ for every integer number n is false.

Question	1	2	3	4	5	6
Answer	A	B	A	C	A	C

I) Choose the correct answer (write it on the table above):

1) The truth value of the proposition "$10 - 1 = 8$ *if and only if* $3 + 2 = 1$" is

F $\Leftrightarrow$ F : T

(A) True	(B) False	(C) Undetermined	(D) None

2) Let $P(x, y)$ be the statement "*$y + x$ is a perfect square*". Then

$P(x, y)$: $y + x$ perfect square

$P(3,2) = 3+2 = 5$ ✗

$P(2,2) = 2+2 = 4$

$4 = 2^2$

(A) $P(3, 2)$ is true	(B) $P(2, 2)$ is true	(C) $P(1, 1)$ is true	(D) None

3) The converse of the statement "*If $x > 2$, then $x + 2 > x^2$* is

if $x+2>x^2$, then $x>2$

$\exists \int \frac{1}{a} = \alpha$ C

(A) If $x + 2 > x^2$, then $x > 2$	(B) If $x \leq 2$, then $x + 2 \leq x^2$	(C) If $x + 2 \leq x^2$, then $x \leq 2$	(D) None

6) The negation of $\exists x \in \mathbb{R}(x > 2 \wedge x \leq 3)$ is

(A) $\exists x \in \mathbb{R}$ $(x \leq 2 \vee x > 3)$	(B) $\forall x \in \mathbb{R}$ $(x \leq 2 \wedge x > 3)$	(C) $\forall x \in \mathbb{R}$ $(x \leq 2 \vee x > 3)$	(D) None

II) A) Without using truth tables, prove that $p \to (q \to r) \vee q$ is a tautology.

$$p \to (q \to r) \vee q \overset{?}{\equiv} T$$

$$
\begin{aligned}
L.H.S &= p \to (q \to r) \vee q \\
&= \neg p \vee (q \to r) \vee q \\
&\equiv \neg p \vee (\neg q \vee r) \vee q \\
&\equiv (\neg p \vee r) \vee (\neg q \vee q) \\
&\equiv (\neg p \vee r) \vee T \\
&\equiv T = R.H.S
\end{aligned}
$$

III) Prove the theorem:

"If n is an integer number, then n is odd if and only if $5n+3$ is even".

Solution: we must prove

① n is odd $\xrightarrow{?}$ $5n+3$ is even

Suppose n is odd and prove $5n+3$ is even

suppose n is odd.

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k+1$

$\rightarrow 5n = 10k+5$

$\rightarrow 5n+3 = 10k+8$

$= 2(5k+4)$

$= 2m \qquad m = 5k+4 \in \mathbb{Z}$

$\therefore$ $5n+3$ is even

---

② $5n+3$ is even $\xrightarrow{?}$ n is odd.

by Contrapostive.

n is even $\xrightarrow{?}$ $5n+3$ odd.

suppose n is even

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k$

$\rightarrow 5n = 10k$

$\rightarrow 5n+3 = 10k+3$

$= 10k+2+1$

$= 2(5k+1)+1$

$= 2t+1 \qquad t = 5k+1 \in \mathbb{Z}$

is odd.

IV) A) Prove, by cases, that, for every real number x, the inequality $x \geq -|x|$ holds.

we have three cases

1) if $x = 0$ $\quad 0 \geqslant -|0|$ is true.

موجبة

2) if $x > 0$

$-|x| = -x < 0$

$\therefore x > -|x|$ is true.

سالبة

3) if $x < 0$

$-|x| = -x < 0 \qquad \therefore x = -|x|$

$\therefore x \geqslant -|x|$

from three case. $\quad x \geqslant -|x| \quad \forall x \in R$

B) Prove that the statement "$n^2 - 7n + 10 \geq 0$, *for all integer numbers* n" is false.

if $n = 3$

$9 - 7 \cdot 3 + 10 = 19 - 21 = -2 \ngeq 0$

$\therefore$ The statment is false

by Counter example.

Question	1	2	3	4	5	6	7	8	9	10
Answer	A	A	B	B	B	B	B	C	A	C

I) Choose the correct answer (write it on the table above):

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$[\neg q \wedge (p \to q)] \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

tautology

1) The statement $[\neg q \wedge (p \to q)] \to \neg p$ is a

(A) tautology	(B) contradiction	(C) None

2) The argument $\neg[p \vee (\neg p \wedge q)]$ is logically equivalent to

(A) $\neg p \wedge \neg q$	(B) $\neg(p \wedge q)$	(C) $p \vee \neg q$	(D) None

$\neg[p \vee (\neg p \wedge q)]$
$\equiv \neg p \wedge (p \vee \neg q)$
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$
$\equiv F \vee (\neg p \wedge \neg q)$
$\equiv \neg p \wedge \neg q$

$A \times B$

4) The Cartesian product of $A = \{a, b\}$ and $B = \{x, y, z\}$ is

هنا هوا أزواج مرتبة

(A) $\{a, b, x, y, z\}$	(B) $\{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$	(C) $\{ax, ay, az, bx, by, bz\}$	(D) None

5) If the universal set is the set $\mathbf{Z}$ of integers, then the statement $\exists x(x^2 = 2)$ is

$\{\sqrt{2}, -\sqrt{2}\} \notin \mathbb{Z}$

(A) true	(B) false

6) Given that propositions p and q are true and propositions r and s are false, the truth value of the expression $(p \leftrightarrow r) \wedge (\neg q \to s)$ is

$(\overset{T}{p} \leftrightarrow \overset{F}{r}) \wedge (\overset{F}{\neg \overset{T}{q}} \to \overset{F}{s})$

$F \wedge T$

F

(A) true	(B) false

II) A) Write the **contrapositive** of the statement:

"*Your guarantee is valid only if you bought your CD player less than 90 days ago*".

$$\underbrace{\text{Your guarantee is valid}}_{p}\ \underbrace{\text{only if}}_{\rightarrow}\ \underbrace{\text{you bought your CD player less than 90 days ago}}_{q}$$

statment: if your guarantee is valid, then you bought your CD player less than 90 days ago.

Contra postive: $\neg q \rightarrow \neg p$

if you didn't by your CD player less than 90 dayes ago, then your guarantee is not valid.

B) Find the negation of the statement:

"*All students enrolled in Math* 151 *are older than* 20 *years and taller than* 130 *cm*".

$$\underbrace{\text{All}}_{\forall}\ \text{students enrolled in Math 151 are}\ \underbrace{\text{older than 20 years}}_{p}\ \underbrace{\text{and}}_{\wedge}\ \underbrace{\text{taller than 130 cm}}_{q}$$

$\forall x\,(p \wedge q)$

negation: There exist a student enrolled in Math 151 who is not oldor than 20 or not taller than 130 cm.

C) **Without** using a truth table, prove the equivalence

$$(p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p.$$

$$\begin{aligned}
L.H.S &= (p \rightarrow q) \wedge (p \rightarrow \neg q) \\
&\equiv (\neg p \vee q) \wedge (\neg p \vee \neg q) && \text{Conditional law} \\
&\equiv \neg p \vee (q \wedge \neg q) && \text{distributive.} \\
&\equiv \neg p \vee F && \text{domination} \\
&\equiv \neg p = R.H.S && \text{identity.}
\end{aligned}$$

III) A) Prove the theorem:

"If n is an integer number, then n is odd if and only if 5n + 2 is odd".

n is odd $\Leftrightarrow$ 5n+2 is odd.

we must proof that

① n is odd $\overset{?}{\rightarrow}$ 5n+2 is odd ~ direct proof ~

suppose that n is odd and show that 5n+2 is odd

let n is odd

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k+1$

$$5n = 10k + 5$$

$$5n + 2 = 10k + 7$$

$$5n + 2 = 10k + 6 + 1$$

$$= 2(5k+3) + 1$$

$$= 2m + 1 \qquad m = 5k+3 \in \mathbb{Z}$$

$\therefore$ 5n+2 is odd.

② 5n+2 is odd $\rightarrow$ n is odd ~ by Contrapostive ~

Contrapostive:

n is even $\overset{?}{\rightarrow}$ 5n+2 is even

suppose n is even, show that 5n+2 is even

let n is even

$\rightarrow \exists k \in \mathbb{Z}$ s.t $n = 2k$

$$5n = 10k$$

$$5n + 2 = 10k + 2$$

$$= 2(5k+1)$$

$$= 2m \qquad m = 5k+1 \in \mathbb{Z}$$

$\therefore$ 5n+2 is even.

$\therefore$ n is odd $\Leftrightarrow$ 5n+2 is odd is true $\forall n \in \mathbb{Z}$

B) Prove that $2^n \leq n^2$, for all integers n, with $1 < n < 5$.

Domain = $\{2, 3, 4\}$ prove by exhaustive proof

if $n=2$ $2^2 \leq 2^2 \rightarrow 4 \leq 4$ True.

if $n=3$ $2^3 \leq 3^2 \rightarrow 8 \leq 9$ True

if $n=4$ $2^4 \leq 4^2 \rightarrow 16 \leq 16$ True.

$\therefore$ The statment is true for all integer n

with $1 < n < 5$

Question IV:

Let $\{a_n\}$ be a sequence defined inductively as:

$$a_0 = 0, \qquad a_1 = 2, \qquad a_{n+1} = 4a_n - 3a_{n-1}, \forall n \geq 1.$$

Use **strong induction** to prove that: $a_n = 3^n - 1, \forall n \geq 0.$ $P(n)$: $\boxed{a_n = 3^n - 1}$

$\boxed{1}$ Basic step: $n=0$

$a_0 = 0$ (given) , $3^0 - 1 = 1 - 1 = 0$

$\therefore a_0 = 3^0 - 1$. True for $n=0$

$\boxed{2}$ Inductive step:

Suppose $P(j)$ is true $\forall j \leq k$

i.e. $a_j = 3^j - 1 \quad \forall j \leq k$ is true (*)

we want to prove that $P(k+1)$ is true.

$$a_{k+1} \overset{??}{=} 3^{k+1} - 1$$

Now as $a_{k+1} = 4a_k - 3a_{k-1}$. From (*)

we have

$$\begin{aligned} &= 4(3^k - 1) - 3(3^{k-1} - 1) \\ &= 4 \cdot 3^k - 4 - 3 \cdot 3^{k-1} + 3 \\ &= 4 \cdot 3^k - 3^k - 1 \\ &= 3^k(4-1) - 1 \\ &= 3^k(3) - 1 \\ &= 3^{k+1} - 1 \end{aligned}$$

$\therefore a_{k+1} = 3^{k+1} - 1 \quad \therefore P(k+1)$ is true.

From principle of strong induction, we proved $P(n)$ is true $\forall n \geq 0$, $a_n = 3^n - 1$.

II) Show that $4 + 9 + 14 + \cdots + (5n - 1) = \frac{n}{2}(3 + 5n)$, for all positive integers n.

by principle of mathematical induction

let $p(n)$: $4+9+14+\cdots+(5n-1) = \frac{n}{2}(3+5n), n \geq 1$

① basis step: show that $p(1)$ is true.

$L.H.S = 4 \qquad R.H.S = \frac{1}{2}(3+5) = \frac{8}{2} = 4$

$\because L.H.S = R.H.S$

$\therefore p(1)$ is true.

② inductive step: $\forall k \geq 1 \, [P(k) \longrightarrow p(k+1)]$

suppose $p(k)$ is true and show that $p(k+1)$ is true.

suppose

$p(k)$: $4+9+14+\cdots+(5k-1) = \frac{k}{2}(3+5k)$

Show

$p(k+1)$: $4+9+14+\cdots+(5k-1)+(5(k+1)-1) \overset{?}{=} \frac{k+1}{2}(3+5(k+1))$

$$L.H.S = \underbrace{4+9+14+\cdots+(5k-1)} + \underbrace{(5(k+1)-1)}$$

$$= \frac{k}{2}(3+5k) + (5(k+1)-1)$$

$$= \frac{k}{2}(3+5k) + (5k+5-1)$$

$$= \frac{3}{2}k + \frac{5}{2}k^2 + 5k + 4$$

$$= \boxed{\frac{5}{2}k^2 + \frac{13}{2}k + 4}$$

$$R.H.S = \frac{k+1}{2}(3+5(k+1))$$

$$= \frac{k+1}{2}(3+5k+5) = \frac{k+1}{2}(5k+8) = \frac{5}{2}k^2 + \frac{5}{2}k + \frac{8}{2}k + \frac{8}{2}$$

$$= \boxed{\frac{5}{2}k^2 + \frac{13}{2}k + 4}$$

$$= \frac{k+1}{2}(5k+8).$$

$\because L.H.S = R.H.S \rightarrow p(k+1)$ is true.

③ Conclusion

$p(n)$ is true $\forall n \geq 1$

Question III:

Use mathematical induction to prove that $n! \geq 2^{n-1}$, for all $n = 1, 2, 3, \ldots\ldots$

by using principal of mathematical induction

let $p(n)$: " $n! \geqslant 2^{n-1}$ $n \geqslant 1$ "

1) Basis step: show that $p(1)$ is true.

$L.H.S = 1! = 1$
$R.H.S = 2^{1-1} = 2^0 = 1$
$1 \geqslant 1$
$\therefore p(1)$ is true.

2) Inductive step: $\forall k \geqslant 1\ [p(k) \xrightarrow{?} p(k+1)]$

suppose $p(k)$ is true.

$p(k)$: $k! \geqslant 2^{k-1}$

show that $p(k+1)$ is true.

$p(k+1)$: $\underbrace{(k+1)!}_{(k+1)k!} \geqslant 2^k$??

sine $p(k)$ is true.

$k! \geqslant 2^{k-1}$

$(k+1)k! \geqslant (k+1) \cdot 2^{k-1} \geqslant 2 \cdot 2^{k-1}$ since $k+1 \geqslant 2$

$1+1 \geqslant 2$
$2+1 \geqslant 2$
$\vdots$

Good Luck☺

$\rightarrow (k+1)k! \geqslant 2 \cdot 2^{k-1}$

$\rightarrow (k+1)! \geqslant 2^k$

$\therefore p(k+1)$ is true.

3) Conclusion: $p(n)$ is true $\forall n \geqslant 1$

Question II:

Let $\{a_n\}$ be a sequence defined inductively as

$$a_0 = 0, \qquad a_1 = 2, \qquad a_{n+1} = 4a_n - 3a_{n-1}, \ \forall n \geq 1.$$

Using **Strong Induction** prove that

$$a_n = 3^n - 1, \ \forall n \geq 0.$$

$a_0 = 0$

$a_1 = 2$

$a_{n+1} = 4a_n - 3a_{n-1} \quad n \geq 1$

$\boxed{a_{k+1} = 4a_k - 3a_{k-1}}$

$P(n): \quad a_n = 3^n - 1 \quad n \geq 0$

By strong induction.

① Basic step show that $p(0)$, $p(1)$ true.

$P(0): \quad a_0 \stackrel{?}{=} 3^0 - 1$

$0 \stackrel{?}{=} 1 - 1$

$0 = 0$ ✓

$\therefore p(0)$ is true

$p(1): \quad a_1 \stackrel{?}{=} 3^1 - 1$

$2 \stackrel{?}{=} 3 - 1$

$2 = 2$ ✓

$\therefore p(1)$ is true.

② inductive step $\forall k \geq 0 \, [p(0) \wedge \cdots \wedge p(k) \stackrel{?}{\longrightarrow} p(k+1)]$

suppose that $p(k), p(k-1)$ are true

$p(k): \quad a_k = 3^k - 1$ ✓

$p(k-1): \quad a_{k-1} = 3^{k-1} - 1$ ✓

show that $p(k+1)$ is true.

$p(k+1): \quad a_{k+1} = 3^{k+1} - 1$??

$L.H.S = a_{k+1} = 4a_k - 3a_{k-1} = 4(3^k - 1) - 3(3^{k-1} - 1)$

فرض   فرض

$= 4 \cdot 3^k - 4 - 3 \cdot 3^{k-1} + 3 = 4 \cdot 3^k - 3^k - 1$

جمع أسس

$= 3 \cdot 3^k - 1 = 3^{k+1} - 1 = R.H.S$

$\therefore p(k+1)$ is true.

③ Conclusion: $p(n)$ is true $\forall n \geq 0$

IV) A) Using mathematical induction, prove that, for all positive integers n,

by Principle M.I $1\cdot2+2\cdot3+3\cdot4+\cdots+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}$.

let $P(n)$: $1\cdot2+2\cdot3+3\cdot4+\dots+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}$, $n\geq1$

1) basis step : show that $P(1)$ is true

L.H.S $= 1\cdot(2) = 2$

R.H.S $=\frac{1(1+1)(1+2)}{3}=\frac{6}{3}=2$

L.H.S = R.H.S $\rightarrow P(1)$ is true

2) inductive step $\forall k\geq1$ $(P(k)\longrightarrow P(k+1)]$

assume $P(k)$ is true

$P(k)$: $1\cdot2+2\cdot3+3\cdot4+\dots+k\cdot(k+1)=\frac{k(k+1)(k+2)}{3}$, $k\geq1$

show that $P(k+1)$ is true

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$P(k+1)$: $\underbrace{1\cdot2+2\cdot3+3\cdot4+\dots+k\cdot(k+1)}+(k+1)\cdot(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$

L.H.S $=\frac{k(k+1)(k+2)}{3}+(k+1)\cdot(k+2)=\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$

$=\frac{(k+1)(k+2)(k+3)}{3}$ = R.H.S $\rightarrow$ $P(k+1)$ is true

3) conclution $P(n)$ is true $\forall$ postive integers