Chapter 1: Vector Analysis

1.1 Vector and Scalar Quantities

A scalar quantity: has magnitude only.
A vector quantity**:** has both magnitude and direction.

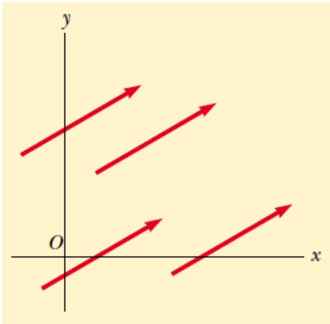
Example: Which of the following are vector quantities and which are scalar quantities?
(a) temperature (b) acceleration (c) velocity (d) speed (e) mass

1.2 Some Properties of Vectors

Vectors are denoted as bold face type like "**A**" or with a small arrow over the symbol like "$\vec{A}$"

(a) Equality of Two Vectors

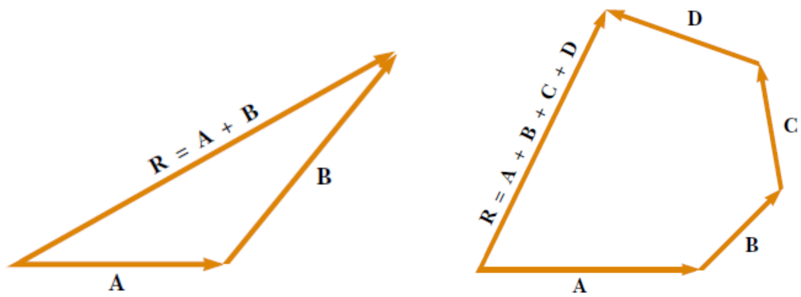
A = **B** only if $A = B$ and if **A** and **B** point in the same direction along parallel lines. For example, the four vectors in this figure are equal.



Thus, we can move a vector to a position parallel to itself in a diagram without affecting the vector.

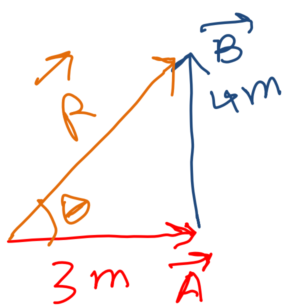
(b) Adding Vectors

R = **A** + **B** + **C** + **D**. where **R** is the vector drawn from the tail of the first vector "**A**" to the tip of the last vector " **D**".



Example 1.1

If you walked 3.0 m toward the east and then 4.0 m toward the north what would be the magnitude and direction of your total displacement (the resultant vector).

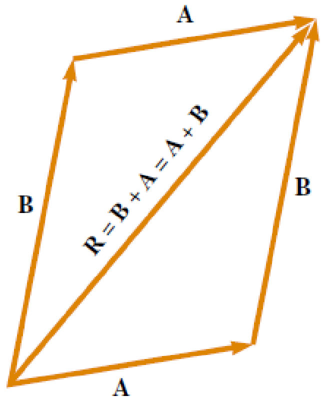


$$|\vec{R}| = \sqrt{3^2 + 4^2} = 5\,m$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53^\circ$$

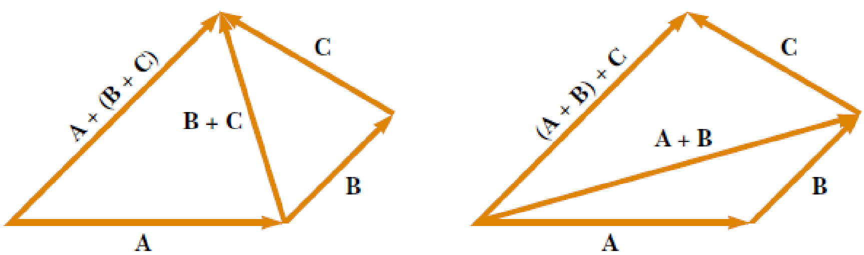
The **commutative** *law of addition*

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$



The associative law of addition:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$



(c) Negative of a Vector

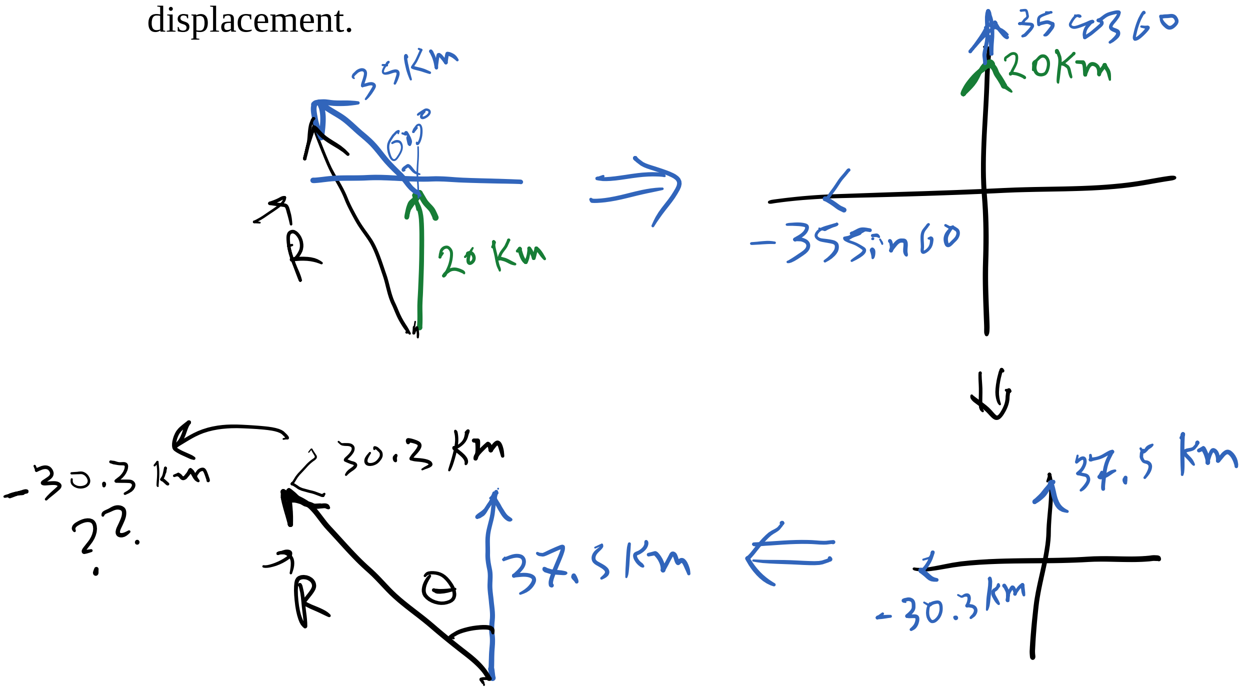
If –**A** is the negative vector of **A**
A + (-**A**) = 0
The vectors **A** and **-A** have the same magnitude, but point in opposite directions.

(d) Subtracting Vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

Example 1.2

A car travels 20.0 km due north and then 35.0 km in a direction 60.0° west of north. Find the magnitude and direction of the car's resultant displacement.



$$|\vec{R}| = R = \sqrt{30.3^2 + 37.5^2} = 48.2 \text{ Km}$$

$$\theta = \tan^{-1}\left(\frac{30.3}{37.5}\right) \simeq 39^\circ \text{ W of N}$$

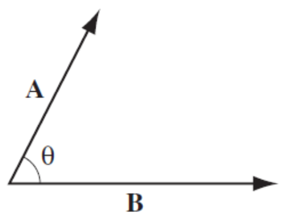
(e) Multiplying a Vector by a Scalar

If m is a scalar quantity

mA is a vector has the same direction as **A** if m is positive or opposite if m is negative. The magnitude of the m**A** vector is mA.

(f) Dot product of two vectors (scalar product)

The dot product of two vectors is defined by
$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$,



The dot product is commutative,
$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$,
and distributive,
$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

(g) Cross product of two vectors (vector product)

$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB\sin\theta\ \hat{n}$
where $\hat{n}$ is a unit vector (vector of magnitude 1) pointing perpendicular to the plane of **A** and **B**.

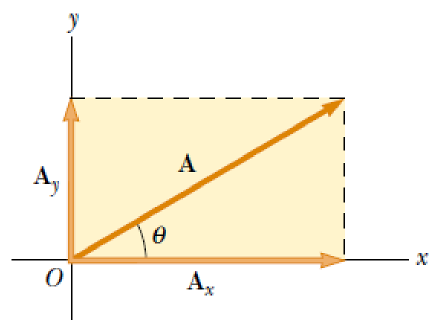
Right-hand rule can be used to determine the direction of the cross product (vector **C**) Let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of vector **C**.

The cross product is distributive,
$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$

<u>but not commutative</u>.
$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$

1.3 Components of a Vector and Unit Vectors

The projections of vectors along coordinate axes are called the components**,** see the following figure *where A_x and A_y are the components of* the vector **A**

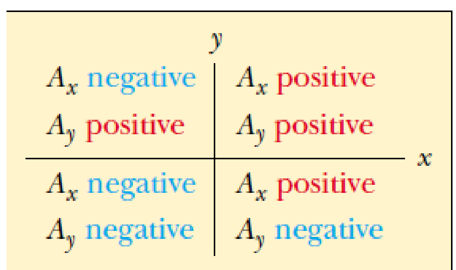


$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$$

$Ax = A \cos \theta$, $Ay = A \sin \theta$

$$A = \sqrt{A_x^2 + A_y^2} \qquad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

The signs of the components A_x and A_y depend on the angle **θ**



we measure the angle θ with respect to the *x* axis

1.3.1 Unit Vectors

Unit vector is a dimensionless vector having a magnitude of exactly 1. It is used to describe a direction in space.

We shall use the symbols $\hat{\boldsymbol{x}}, \hat{y}$ and $\hat{\boldsymbol{z}}$ (some books use $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\widehat{\boldsymbol{k}}$ to represent unit vectors pointing in the positive x, y, and z directions, respectively

The magnitude of each unit vector equals 1

$$|\widehat{\boldsymbol{x}}| = |\widehat{\boldsymbol{y}}| = |\hat{\boldsymbol{z}}| = 1$$

They are used to describe directions in space,

For example,

$$\mathbf{A} = A_x\widehat{\boldsymbol{x}} + A_y\widehat{\boldsymbol{y}}$$

$$\mathbf{B} = B_x\widehat{\boldsymbol{x}} + B_y\widehat{\boldsymbol{y}}$$

The resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$

$$\mathbf{R} = (A_x+B_x)\widehat{\boldsymbol{x}} + (A_y+B_y)\widehat{\boldsymbol{y}}$$

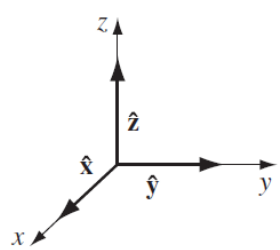
$$R_x = A_x+B_x$$

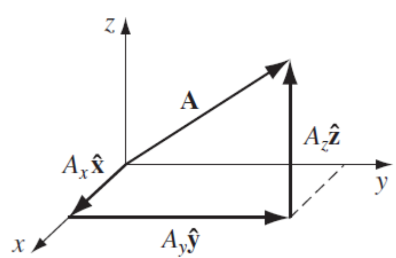
$$R_y = A_y+B_y$$

The magnitude of **R** is $R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$

The angle it makes with the x axis is $\theta = \tan^{-1}(\frac{R_y}{R_x}) = \tan^{-1}(\frac{A_y+B_y}{A_x+B_x})$

What if* R *has x,y and z components ?





Example 1.3

A particle undergoes three consecutive displacements: $\mathbf{d_1} = (15\,\hat{\boldsymbol{x}} + 30\,\hat{\boldsymbol{y}} + 12\,\hat{\boldsymbol{z}})$ cm, $\mathbf{d_2} = (23\,\hat{\boldsymbol{x}} - 14\,\hat{\boldsymbol{y}} - 5\,\hat{\boldsymbol{z}})$ cm and $\mathbf{d_3} = (-13\,\hat{\boldsymbol{x}} + 15\,\hat{\boldsymbol{y}})$ cm. Find the components of the resultant displacement and its magnitude.

$$\vec{d} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3$$
$$= (25\,\hat{x} + 31\,\hat{y} + 7\,\hat{z})\ \text{cm}$$
$$d = \sqrt{25^2 + 31^2 + 7^2} = 40\ \text{cm}$$

1.4 Vector Algebra (component form)

If we have two vectors **A** and **B**

$$\mathbf{A} + \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.$$

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}$$

Dot product:

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1; \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0$$

why?

$$\mathbf{A} \cdot \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= A_xB_x + A_yB_y + A_zB_z.$$

Cross product:

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}$$
$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}},$$
$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}},$$
$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}}.$$

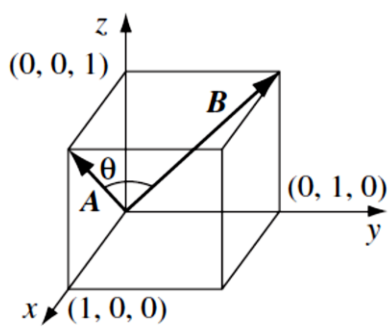
$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}\end{aligned}$$

This expression can be written as a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example 1.4

Find the angle between the face diagonals of a cube



$\vec{A} = 1\hat{x} + 1\hat{z}$ $\qquad$ $\vec{B} = 1\hat{y} + 1\hat{z}$

$\Rightarrow \vec{A} \cdot \vec{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$\because \vec{A} \cdot \vec{B} = AB\cos\theta = \sqrt{2}\,\sqrt{2}\,\cos\theta$

$\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \boxed{\theta = 60^\circ}$

1.5 Differential calculus

Suppose we have a function of one variable: f (x)

$$df = \left(\frac{df}{dx}\right) dx$$

This shows how rapidly the function f (x) changes when we change " x " by a very small amount,   dx

Also, geometrically, the derivative df/dx is the slope of the graph of f(x) versus x

1.5.1 Gradient

What if we have a function of three variables?
For example temperature (scalar quantity) that changes as a function of x, y and z (this function changes with the three directions)

Then

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned}$$

We can call

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

The **<u>gradient</u>** of T and this gradient is a <u>vector quantity</u>

Like any vector, the gradient has magnitude and direction.

Example 1.5

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

Solution:

$$\begin{aligned}\nabla r &= \frac{\partial r}{\partial x}\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\hat{\mathbf{z}} \\ &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{x}} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{y}} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{z}} \\ &= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.\end{aligned}$$

1.5.2 The Divergence

Let us take a vector quantity **v**

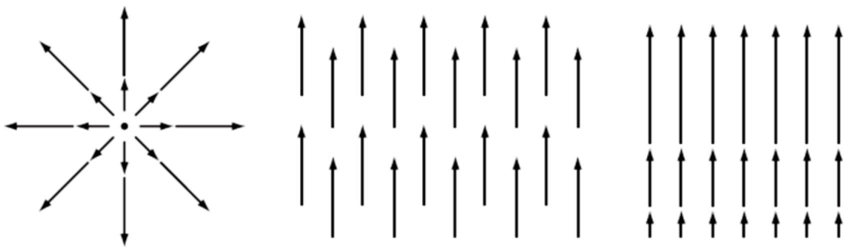
$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}$$

The divergence of the vector **v** ($\nabla \cdot \mathbf{v}$) is a scalar

Geometrically, $\nabla \cdot v$ is a measure of how much the vector v spreads out (diverges) from a given point.

For example, the vector function in the following figure

- In " figure a" has a large (positive) divergence (if the arrows pointed in, it would be a negative divergence)
- The function in "figure b" has zero divergence
- The function in "figure c" has a positive divergence.



(a) (b) (c)

Why?

Example 1.6

Suppose the function in Figure (a) above is:

$\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}},$

Calculate its divergence.

Solution:

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

1.5.3 The Curl

From the definition of ∇ we can construct the curl:

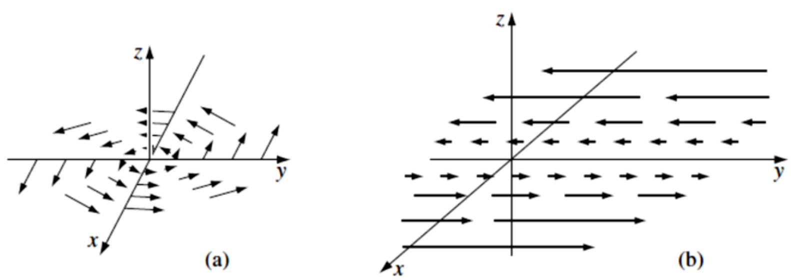
$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

The curl of a vector v is a vector (like any cross product)

$\nabla \times$v (the curl of vector **v**) is a measure of how much the vector v whirls around a selected point.

We can say that the functions in the following figures have a considerable curl pointing in the positive z direction, using the right-hand rule.



Example 1.7

What do you think the curl of the function mentioned in Example 1.6? Prove your answer.

Solution:

It has zero curl (no rotation shown in its sketch)

$$\nabla \times v_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} y\right)\hat{x} + \left(\frac{\partial}{\partial z} x - \frac{\partial}{\partial x} z\right)\hat{y} + \left(\frac{\partial}{\partial x} y - \frac{\partial}{\partial y} x\right)\hat{z}$$

$$= 0$$

In summary, ∇ is called "del" or "nabla symbol"

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

From the previous discussion we can say that and it is an operator acting upon functions. It physically does not mean much until we provide it with a function to act upon

It can act upon:

- a scalar function T : **∇T** (the gradient)
- a vector function like **v**, via the dot product: **∇ · v** (**the divergence**)
- a vector function v, via the cross product: **∇**×**v** (**the curl**).

1.6 Curvilinear coordinate systems

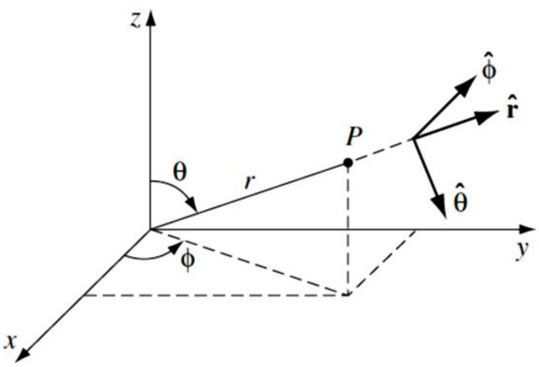
1.6.1 Spherical Coordinates

Let us take point P with Cartesian coordinates (x, y, z), this point can be described using spherical coordinates (r, θ , ϕ);

r is the distance from the origin (the magnitude of the position vector r),
θ (the angle down from the z axis) is called the polar angle,
and ϕ (the angle around from the x axis) is the azimuthal angle.

Their relations to Cartesian coordinates are:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

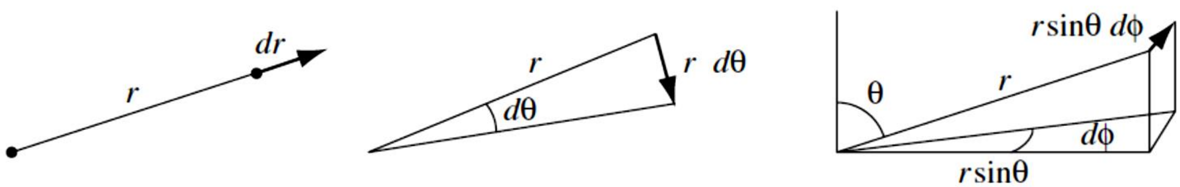


For example, vector **A** can be expressed as follows:

$$\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}$$

Also the general infinitesimal displacement d**l** is:

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$



Example 1.9

Find the volume of a sphere of radius R.

Solution:

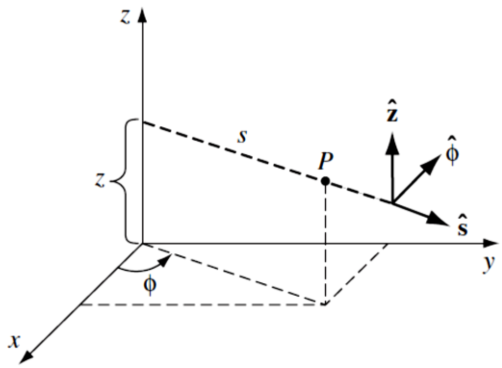
$$V = \int d\tau = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \left(\int_0^R r^2\,dr\right)\left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$= \left(\frac{R^3}{3}\right)(2)(2\pi) = \frac{4}{3}\pi R^3$$

1.6.2 Cylindrical Coordinates

The cylindrical coordinates (s, ϕ , z) of a point P are defined in the following figure.



Notice that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin.

The relations to Cartesian coordinates are

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z.$$

The infinitesimal displacements are

$$dl_s = ds, \qquad dl_\phi = s\,d\phi, \qquad dl_z = dz,$$

so

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}},$$