Vector Fields

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1. Vector Calculus
   - The Dot (or the inner) Product
   - The Cross Product
   - Scalar Triple Product
   - The Directional Derivative
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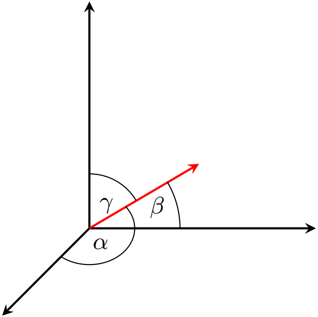
The Dot Product

Definition

In $\mathbb{R}^2$, if $u = (u_1, u_2)$ and $v = (v_1, v_2)$, the dot product of u and v is the number $\langle u, v\rangle = u_1 v_1 + u_2 v_2$.
In $\mathbb{R}^3$, if $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, the dot product of u and v is the number $\langle u, v\rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$.
The norm of a vector u is $\|u\| = \sqrt{\langle u, u\rangle}$.

Recall that if θ is the angle between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, then

$$\langle u, v\rangle = \|u\| \; \|v\| \cos\theta.$$



The direction angles associated to a vector u are given by: $\cos\alpha = \frac{\langle u, i\rangle}{\|u\|}$, $\cos\beta = \frac{\langle u, j\rangle}{\|u\|}$, $\cos\gamma = \frac{\langle u, k\rangle}{\|u\|}$.

The Cross Product

Definition

If $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$, then the cross product of u_1 and u_2 is the vector

$$u_1 \wedge u_2 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{i}} + \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \overrightarrow{\mathbf{k}}.$$

Remark

1. The vector $u_1 \wedge u_2$ is orthogonal to the vectors u_1 and u_2 and its direction is given by the right-hand rule i.e. the determinant $|u_1, u_2, u_1 \wedge u_2|$ is non negative.
2. $|u_1 \wedge u_2|$ is the area of the parallelogram spanned by u_1 and u_2, i.e.,
$$|u_1 \wedge u_2| = |u_1|\,|u_2|\,\sin\theta$$
3. Two vectors u_1 and u_2 are parallel if and only if $u_1 \wedge u_2 = 0$.

Theorem (Cross Product Properties)

Let u_1, u_2, and u_3 be vectors and let c be a constant:

1. $u_1 \wedge u_2 = -u_2 \wedge u_1$;
2. $(cu_1) \wedge u_2 = c(u_1 \wedge u_2) = u_1 \wedge (cu_2)$;
3. $u_1 \wedge (u_2 + u_3) = u_1 \wedge u_2 + u_1 \wedge u_3$;
4. $(u_1 + u_2) \wedge u_3 = u_1 \wedge u_3 + u_2 \wedge u_3$;
5. $u_1 \cdot (u_2 \wedge u_3) = (u_1 \wedge u_2) \cdot u_3$;
6. $u_1 \wedge (u_2 \wedge u_3) = (u_1 \cdot u_3)u_2 - (u_1 \cdot u_2)u_3$.

Scalar Triple Product

The scalar triple product of three vectors u_1, u_2, and u_3 is the determinant

$$\langle u_1, (u_2 \wedge u_3) \rangle = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The volume of the parallelepiped formed by the vectors u_1, u_2, and u_3 is given by

$$|\langle u_1, (u_2 \wedge u_3) \rangle|.$$

The Directional Derivative

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$, the partial derivatives of f with respect to x and y it they exist are defined by:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

$$f_y(x_0, , y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Consider a smooth scalar field $f : D \longrightarrow \mathbb{R}$. The partial derivatives of f in the point $\mathbf{r} = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}} \in D$ when these limits exist:

$$\frac{\partial f}{\partial x}(\mathbf{r}) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h};$$

$$\frac{\partial f}{\partial y}(\mathbf{r}) = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h};$$

$$\frac{\partial f}{\partial z}(\mathbf{r}) = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}.$$

The Directional Derivative

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$ and $u = (a, b)$ a unit vector in $\mathbb{R}^2$. The directional derivative of f in the direction of u at (x_0, y_0) if it exists is

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h \to 0} \frac{f((x_0, y_0) + hu) - f(x_0, y_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}. \end{aligned}$$

Example

1. If $u = (a, b)$, $D_u f(x_0, y_0)$ is the same as the derivative of $f(x_0 + at, y_0 + bt)$ at $t = 0$. We can compute this by the chain rule and get

$$D_u f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0).$$

2. Find the directional derivative of $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in the direction $u = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
3. Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of $u = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Definition

A two-dimensional vector field is a function f that maps each point (x,y) in $\mathbb{R}^2$ to a two-dimensional vector $f(x,y) = (u(x,y), v(x,y))$.
We denote $f(x,y) = u(x,y)\overrightarrow{\mathbf{i}} + v(x,y)\overrightarrow{\mathbf{j}}$, where $\overrightarrow{\mathbf{i}} = (1,0)$ and $\overrightarrow{\mathbf{j}} = (0,1)$.
Similarly a three-dimensional vector field maps (x,y,z) to $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$.
We denote $f(x,y,z) = u(x,y,z)\overrightarrow{\mathbf{i}} + v(x,y,z)\overrightarrow{\mathbf{j}} + w(x,y,z)\overrightarrow{\mathbf{k}}$, where $\overrightarrow{\mathbf{i}} = (1,0,0)$, $\overrightarrow{\mathbf{j}} = (0,1,0)$ and $\overrightarrow{\mathbf{k}} = (0,0,1)$.

Example

The vector fields have many important significations, as they can be used to represent many physical quantities: gravity, electricity, magnetism or a velocity of fluid.
Let $r(t) = x(t)\overrightarrow{\mathbf{i}} + y(t)\overrightarrow{\mathbf{j}} + z(t)\overrightarrow{\mathbf{k}}$ be the position vector of an object. We can define various physical quantities associated with the object as follows:
velocity: $v(t) = r'(t) = \frac{dr}{dt} = x'(t)\overrightarrow{\mathbf{i}} + y'(t)\overrightarrow{\mathbf{j}} + z'(t)\overrightarrow{\mathbf{k}}$,
acceleration:
$a(t) = v'(t) = \frac{dv}{dt} = r''(t) = \frac{d^2r}{dt^2} = x''(t)\overrightarrow{\mathbf{i}} + y''(t)\overrightarrow{\mathbf{j}} + z''(t)\overrightarrow{\mathbf{k}}$,
The norm $\|v(t)\|$ of the velocity vector is called the speed of the object.

Example

The gravitational force field between the Earth with mass M and a point particle with mass m is given by:

$$F(x,y,z) = -GmM\frac{x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

where G is the gravitational constant, and the (x,y,z) coordinates are chosen so that $(0,0,0)$ is the center of the Earth.

Gradient Fields

Let f be a scalar function of two variables, the gradient of f is defined by

$$\nabla f(x,y) = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)).$$

If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x,y,z) = (\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)).$$

The operator ∇ will be denoted by:
$\nabla = \frac{\partial}{\partial x}\overrightarrow{\mathbf{i}} + \frac{\partial}{\partial y}\overrightarrow{\mathbf{j}} + \frac{\partial}{\partial z}\overrightarrow{\mathbf{k}}$ or $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ as a vector.

Remark

Let f be a function. The vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface of f $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = C\}$ that contains (x_0, y_0, z_0).

Consider f and g two smooth scalar functions defined on a domain $D \subset \mathbb{R}^3$ and consider $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ two smooth vector fields.

$$\begin{aligned}\nabla(fg) &= (\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z}) \\ &= f\nabla(g) + g\nabla(f).\end{aligned}$$

$$\begin{aligned}\nabla(\langle F, G\rangle) &= \nabla(f_1g_1 + f_2g_2 + f_3g_3) \\ &= \nabla(f_1g_1) + \nabla(f_2g_2) + \nabla(f_3g_3) \\ &= f_1\nabla(g_1) + +f_2\nabla(g_2) + f_3\nabla(g_3) \\ & \quad g_1\nabla(f_1) + g_2\nabla(f_2) + g_3\nabla(f_3).\end{aligned}$$

Definition

A vector field F is called conservative, if F is the gradient of a function, $F = \nabla f$. In this case, the function f is called a potential of the vector field F.

For example the vector field

$$\begin{aligned} F &= \left(\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \\ &= \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}. \end{aligned}$$

Example (The inverse square field)

Let $\mathbf{r}(x, y, z) = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}$ be the position vector of the point $M(x, y, z)$. The vector field $F(x, y, z) = \dfrac{c}{\|\mathbf{r}\|^3}\mathbf{r}(x, y, z)$ is called the inverse square field, where $c \in \mathbb{R}$.
The inverse field is conservative.

Test of Conservative

If $F = (P, Q) = \nabla f$. Then $P = \dfrac{\partial f}{\partial x}$ and $Q = \dfrac{\partial f}{\partial y}$, and provided that f is smooth, from Schwarz's Theorem, $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2}{\partial y \partial x} f = \dfrac{\partial Q}{\partial x}$. Hence, if $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, F is not conservative.

For a vector field $F = (P, Q, R)$, suppose that $(P, Q, R) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. If z is constant, then $f(x, y, z)$ is a function of x and y, and by Schwarz's Theorem, $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial y}$. Likewise, if y is constant, then $\dfrac{\partial P}{\partial z} = \dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{\partial R}{\partial x}$, and if x is constant, we get $\dfrac{\partial Q}{\partial z} = \dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{\partial R}{\partial y}$.

Conversely, if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, and $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$ then F is conservative.

Example

1. The vector field $(1+3xy, 2x^2-3y^2)$ is not conservative because, $\dfrac{\partial(1+3xy)}{\partial y} = 3x$ and $\dfrac{\partial(2x^2-3y^2)}{\partial x} = 4x$.
2. The vector field $F = (y^2z + y\cos x, 2xyz + \sin x - \sin y, xy^2)$ is conservative because, $F = \nabla(xy^2z + y\sin x + \cos y)$.

The Divergence

Definition

The divergence of a vector field $F = (P, Q, R)$ is

$$\langle \nabla, F \rangle = \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), (P, Q, R) \right\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The curl of a vector field

The curl of $F = (P, Q, R)$ is

$$\nabla \times F = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

If $F = P\overrightarrow{\mathbf{i}} + Q\overrightarrow{\mathbf{j}}$ is a two dimensional vector field, the curl $\nabla \times F$ can also be defined by regarding the $k-$component to be zero, i.e. $F = P\overrightarrow{\mathbf{i}} + Q\overrightarrow{\mathbf{j}} + 0\overrightarrow{\mathbf{k}}$, then $\mathrm{curl} F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \overrightarrow{\mathbf{k}}$.

Theorem (The Curl Test)

Given a vector field $F = (P, Q, R)$ is defined and continuously differentiable everywhere in $\mathbb{R}^3$ (or everywhere in $\mathbb{R}^2$ for vector fields in $\mathbb{R}^2$), then F is conservative if and only if $\mathrm{curl} F = 0$.

Here are two simple but useful facts about divergence and curl.

Theorem

$\langle \nabla, (\nabla \times F) \rangle = 0$. In other words, the divergence of the curl is zero.

Theorem

$\nabla \times (\nabla f) = 0$. That is, the curl of a gradient is the zero vector.

Exercises

Exercise 1 :
A vector field F is said to be incompressible if $\langle \nabla, F \rangle = 0$.
Prove that any vector field of the form
$F(x, y, z) = (f(y, z), g(x, z), h(x, y))$ is incompressible.

Exercise 2 :
Find an f so that $\nabla f = (2x + y^2, 2y + x^2)$, or explain why there is no such f.

Exercise 3 :
Find an f so that $\nabla f = (x^3, -y^4)$, or explain why there is no such f.

Exercise 4 :
Find an f so that $\nabla f = (xe^y, ye^x)$, or explain why there is no such f.

Exercise 5 :
Find an f so that $\nabla f = (y\cos x, y\sin x)$, or explain why there is no such f.

Exercise 6 :
Find an f so that $\nabla f = (y\cos x, \sin x)$, or explain why there is no such f.

Exercise 7 :
Find an f so that $\nabla f = (x^2y^3, xy^4)$, or explain why there is no such f.

Exercise 8 :
Find an f so that $\nabla f = (yz, xz, xy)$, or explain why there is no such f.