Vector Calculus

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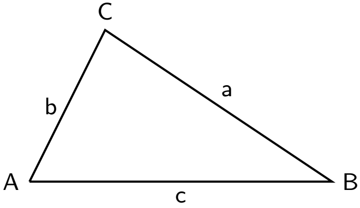
Definition

In $\mathbb{R}^2$, if $\overrightarrow{u} = (u_1, u_2)$ and $\overrightarrow{v} = (v_1, v_2)$, the dot product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is the number $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = u_1 v_1 + u_2 v_2$. The dot product is also denoted by: $\overrightarrow{u}.\overrightarrow{v}$.
Recall that the angle θ between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as follows,

$$\overrightarrow{u}.\overrightarrow{v} = \langle \overrightarrow{u}, \overrightarrow{v} \rangle = \| \overrightarrow{u} \| \, \| \overrightarrow{v} \| \cos\theta.$$

The Cosine Law

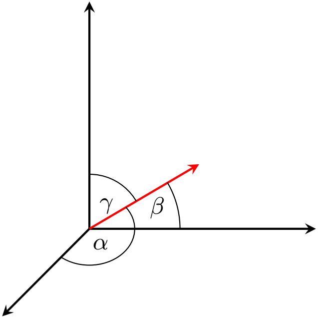


Recall from trigonometry:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where $\theta = m\angle ACB$.

In $\mathbb{R}^3$, if $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$, the dot product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is the number $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$.
The norm of a vector $\overrightarrow{u}$ is $\|\overrightarrow{u}\| = \sqrt{\langle \overrightarrow{u}, \overrightarrow{u} \rangle}$.



The direction angles associated to a vector $\overrightarrow{u}$ are given by: $\cos\alpha = \frac{\langle \overrightarrow{u}, \overrightarrow{\mathbf{i}} \rangle}{\| \overrightarrow{u} \|}$, $\cos\beta = \frac{\langle \overrightarrow{u}, \overrightarrow{\mathbf{j}} \rangle}{\| \overrightarrow{u} \|}$, $\cos\gamma = \frac{\langle \overrightarrow{u}, \overrightarrow{\mathbf{k}} \rangle}{\| \overrightarrow{u} \|}$.

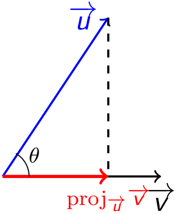
Definition

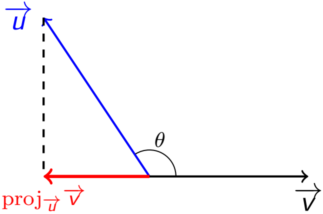
Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, the component of $\overrightarrow{u}$ along $\overrightarrow{v}$ is

$$\mathrm{comp}_{\overrightarrow{v}}\overrightarrow{u} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{v}\|} = \|\overrightarrow{u}\| \cos\theta,$$

where θ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.
The projection of the vector $\overrightarrow{u}$ on the vector $\overrightarrow{v}$ is

$$\mathrm{comp}_{\overrightarrow{v}}\overrightarrow{u} . \frac{1}{\|\overrightarrow{v}\|}\overrightarrow{v}.$$


$\overrightarrow{u}$
θ
$\text{proj}_{\overrightarrow{u}}\overrightarrow{v}$
$\overrightarrow{v}$


$\overrightarrow{u}$
θ
$\text{proj}_{\overrightarrow{u}}\overrightarrow{v}$
$\overrightarrow{v}$

The Cross Product

Definition

If $\overrightarrow{u}_1 = (x_1, y_1, z_1)$ and $\overrightarrow{u}_2 = (x_2, y_2, z_2)$, then the cross product of $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ is the vector

$$\overrightarrow{u}_1 \wedge \overrightarrow{u}_2 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{i}} + \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \overrightarrow{\mathbf{k}}.$$

Remark

1. The vector $\overrightarrow{u}_1 \wedge \overrightarrow{u}_2$ is orthogonal to the vectors $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ and its direction is given by the right-hand rule i.e. the determinant $|\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_1 \wedge \overrightarrow{u}_2|$ is non negative.
2. $\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\|$ is the area of the parallelogram spanned by $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$, i.e.,

$$\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\| = \|\overrightarrow{u}_1\| \, \|\overrightarrow{u}_2\| \sin\theta.$$

3. Two vectors $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ are parallel if and only if $\overrightarrow{u}_1 \wedge \overrightarrow{u}_2 = 0$.
4. $\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\|^2 = \|\overrightarrow{u}_1\|^2\|\overrightarrow{u}_1\|^2 - \langle u_1, v_2\rangle^2$. Indeed $\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\|^2 = \|\overrightarrow{u}_1\|^2\|\overrightarrow{u}_2\|^2 \sin^2\theta = \|\overrightarrow{u}_1\|^2\|\overrightarrow{u}_2\|^2(1 - \cos^2\theta)$.

Example

Compute the area of the triangle with vertices $(2,3,-1)$, $(1,3,2)$, $(3,0,-2)$.
Two sides are: $\overrightarrow{u} = (-1,0,3)$, $\overrightarrow{v} = (1,-3,-1)$,
$\overrightarrow{u} \wedge \overrightarrow{v} = (9,2,3)$, $\|\overrightarrow{u} \wedge \overrightarrow{v}\|^2 = 81+4+9 = 104 = 8.13$. The area of the triangle is $\sqrt{26}$.

Theorem (Cross Product Properties)

Let $\overrightarrow{u}_1$, $\overrightarrow{u}_2$, and $\overrightarrow{u}_3$ be vectors and let c be a constant:

1. $\overrightarrow{u}_1 \wedge \overrightarrow{u}_2 = -\overrightarrow{u}_2 \wedge \overrightarrow{u}_1$;
2. $(c\overrightarrow{u}_1) \wedge \overrightarrow{u}_2 = c(\overrightarrow{u}_1 \wedge \overrightarrow{u}_2) = \overrightarrow{u}_1 \wedge (c\overrightarrow{u}_2)$;
3. $\overrightarrow{u}_1 \wedge (\overrightarrow{u}_2 + \overrightarrow{u}_3) = \overrightarrow{u}_1 \wedge \overrightarrow{u}_2 + \overrightarrow{u}_1 \wedge \overrightarrow{u}_3$;
4. $(\overrightarrow{u}_1 + \overrightarrow{u}_2) \wedge \overrightarrow{u}_3 = \overrightarrow{u}_1 \wedge \overrightarrow{u}_3 + \overrightarrow{u}_2 \wedge \overrightarrow{u}_3$;
5. $\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle = \langle (\overrightarrow{u}_1 \wedge \overrightarrow{u}_2), \overrightarrow{u}_3\rangle$;
6. $\overrightarrow{u}_1 \wedge (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3) = (\langle \overrightarrow{u}_1, \overrightarrow{u}_3\rangle)\overrightarrow{u}_2 - (\langle \overrightarrow{u}_1, \overrightarrow{u}_2\rangle)\overrightarrow{u}_3$.

Scalar Triple Product

The scalar triple product of three vectors $\overrightarrow{u}_1$, $\overrightarrow{u}_2$, and $\overrightarrow{u}_3$ is the determinant

$$\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3) \rangle = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The volume of the parallelepiped formed by the vectors $\overrightarrow{u}_1$, $\overrightarrow{u}_2$, and $\overrightarrow{u}_3$ is

$$|\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle|.$$

The vectors $\overrightarrow{u}_1$, $\overrightarrow{u}_2$ and $\overrightarrow{u}_1$ are in the same plane if the scalar triple product $\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle$ is 0.

Example

Compute the volume of the parallelepiped spanned by the 3 vectors $\overrightarrow{u}_1 = (2, 3, -1)$, $\overrightarrow{u}_2 = (1, 3, 2)$ and $\overrightarrow{u}_3 = (3, 0, -2)$.
$\overrightarrow{u}_2 \wedge \overrightarrow{u}_3 = (-6, 8, -9)$, $\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3) \rangle = 21$.

Remark

$$\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle = \langle (\overrightarrow{u}_1 \wedge \overrightarrow{u}_2), \overrightarrow{u}_3\rangle.$$

Lines

A line L in three-dimensional space is determined by

- A point $M_0 = (x_0, y_0, z_0)$ on the line
- A vector $\overrightarrow{v} = (a, b, c)$ that gives the direction of the line.

Any point M on the line can be expressed as $M_0 + t\overrightarrow{v}$ for some real number t called the parameter.

Line - Vector Equation

The parametrization $t \longmapsto M_0 + t\overrightarrow{v}$ is called the vector equation of a line L, where M_0 is a point on the line and $\overrightarrow{v}$ is the direction of the line.

Line - Parametric Equation

If $M_0 = (x_0, y_0, z_0)$ and $\overrightarrow{v} = (a, b, c)$, the equations

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct \end{cases}$$

give the parametric equations for the line passing through M_0 and in direction of the vector $\overrightarrow{v}$.

Line Symmetric Equation

If we begin with the parametric equations of a line:

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct \end{cases}$$

we can eliminate the parameter to get the symmetric equation of a line;

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Let $M_1 = (x_1, y_1, z_1), M_2 = (x_2, y_2, z_2)$ be two points on the space. The parametric equation of the line passing through M_1 to M_2 is the parametric equation of the line with M_1 on the line and the direction $\overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
The vector equation of the line is: $M(t) = M_1 + t\overrightarrow{M_1M_2}$. If $t \in [0, 1]$, this equation in the is segment which goes from M_1 to M_2.

Theorem (Distance between a point to a line)

The distance between a point and a line is

$$\frac{\|\overrightarrow{M_0M} \wedge \overrightarrow{u}\rangle\|}{\|\overrightarrow{u}\|},$$

where M_0 on the line.

Example

Find the distance from $M = (2, -3, 1)$ to the line containing $M_1 = (1, 3, -1)$ and $M_2 = (2, -1, 1)$.
$\overrightarrow{M_1M_2} = (1, -4, 2)$, $\overrightarrow{MM_1} = (1, -6, 2)$, $\overrightarrow{M_1M_2} \wedge \overrightarrow{MM_1} = (4, 0, -2)$.
The distance is $\dfrac{\sqrt{20}}{\sqrt{21}}$.

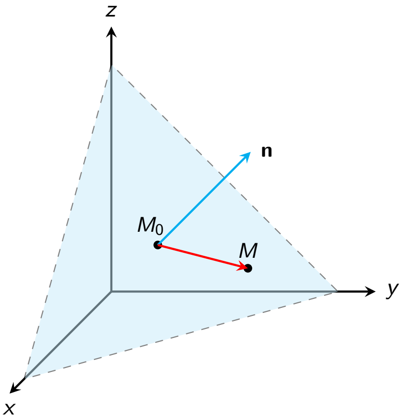
Planes

In order to find the equation of a plane, we need:

- a point on the plane $M_0 = (x_0, y_0, z_0)$
- a vector that is orthogonal to the plane $\overrightarrow{n} = (a, b, c)$. This vector is called the normal vector the to plane.

Plane - Vector Equation

Any point M of the plane verifies $\langle \overrightarrow{M_0M}, \overrightarrow{u} \rangle = 0$. This is the vector equation of the plane.


z
n
M_0
M
y
x

Plane - Scalar Equation
The scalar (or component) equation of the plane is
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$

Example

Determine the equation of the plane that passes through the points $M_1 = (1, 2, 3)$, $M_2 = (3, 2, 1)$ and $M_3 = (-1, -2, 2)$.
$\overrightarrow{M_1M_2} = (2, 0, -2)$, $\overrightarrow{M_1M_3} = (-2, -4, -1)$,
$\overrightarrow{M_1M_2} \wedge \overrightarrow{M_1M_3} = (-8, 6, -8)$. The scalar equation of the plane is
$-8(x-1) + 6(y-2) - 8(z-3) = 0 \iff 4x - 3y + 4z = 10$.

Remark

Two planes are parallel if and only if the normal vectors are parallel i.e. $\overrightarrow{n_1} \wedge \overrightarrow{n_2} = 0$.

Theorem [Distance between a point and a plane]

The distance between a point and a plane is

$$\frac{\left|\langle \overrightarrow{M_0 M}, \overrightarrow{n} \rangle\right|}{\|\overrightarrow{n}\|},$$

where M_0 on the plane.
If $M_0 = (x_0, y_0, z_0)$, $M = (x, y, z)$ and $\overrightarrow{n} = (a, b, c)$, then

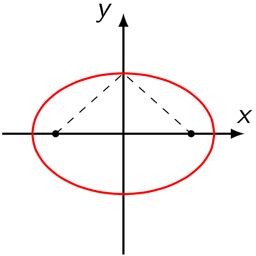
$$d(M, P) = \frac{|a(x - x_0) + b(y - y_0) + c(z - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example

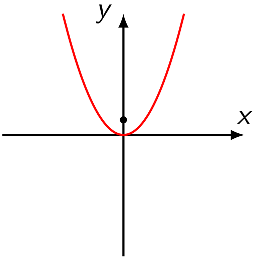
Find the distance from $M = (1, 2, 0)$ to the plane $2x - 3y + 2z = 1$.
$M_1 = (-1, -1, 0)$, $\overrightarrow{n} = (2, -3, 2)$, $\overrightarrow{M_1 M} = (2, 3, 0)$.

$$d(M, P) = \frac{5}{\sqrt{17}}.$$

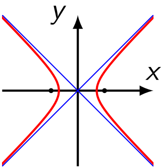
A quadratic curve is the graph of a second-degree equation in two variables taking one of the forms



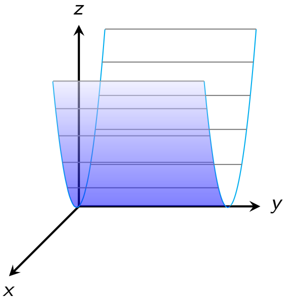
The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci $(\pm c, 0)$, where $a^2 = b^2 + c^2$.



The parabola $x^2 = 4py$ with focus at $(0, p)$ and directrix at $y = -p$

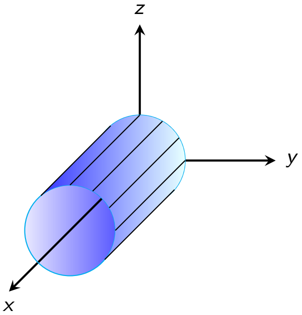


The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci at $(\pm c, 0)$ where $c^2 = a^2 + b^2$.



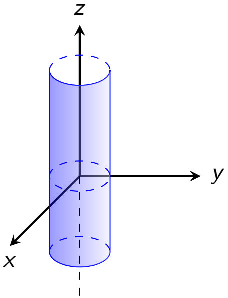
Cylinders which consist of all lines (called rulings) that are parallel to a given line and pass through a given plane curve

Example

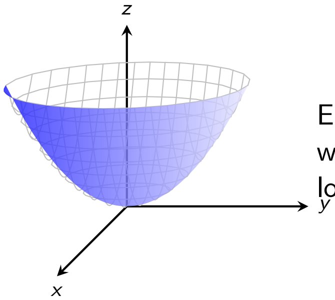


$$y^2 + z^2 = 1$$

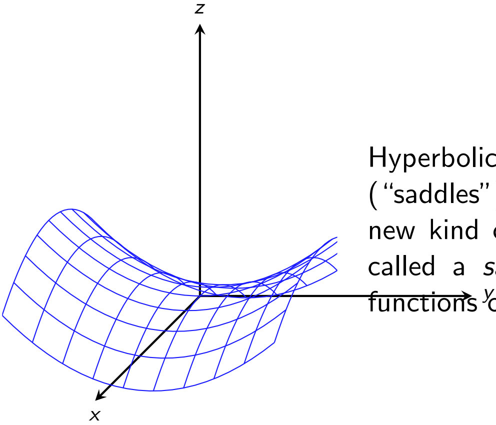
Example



The set of points (x, y, z) that satisfy the equation $x^2 + y^2 = 1$ is the cylinder of radius 1 centered at $(0, 0, 0)$ whose axis of symmetry is the z-axis.



Elliptic Paraboloids which will model functions with local maxima or minima



Hyperbolic Paraboloids ("saddles") which model a new kind of critical point, called a *saddle point*, for functions of two variables

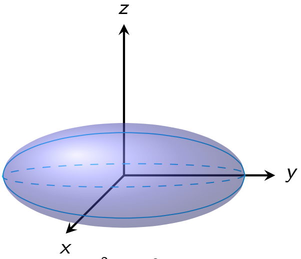
Quadric Surfaces in $\mathbb{R}^3$

A quadric surface is the graph of a second-degree equation in x, y, and z taking one of the standard forms

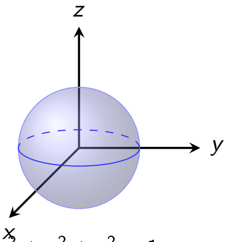
$$Ax^2 + By^2 + Cz^2 + D = 0, \quad Ax^2 + By^2 + Cz = 0.$$

We can graph a quadric surface by studying its traces in planes parallel to the x, y, and z axes. The traces are always quadratic curves.

The ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$



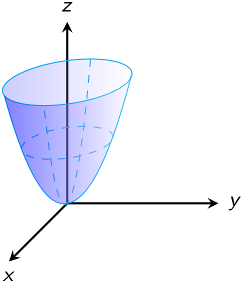
$x^2 + \frac{y^2}{4} + \frac{z^2}{3} = 1$



$x^2 + y^2 + z^2 = 1$

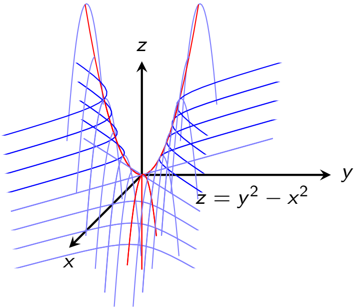
The set of points (x, y, z) that satisfy the equation $x^2 + y^2 + z^2 = 1$ is the sphere of radius 1 centered at $(0, 0, 0)$.

Elliptic Paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

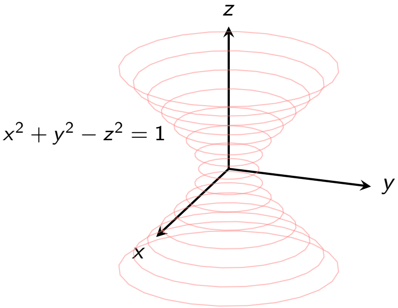


$z = x^2 + y^2$

Hyperbolic Paraboloid (Saddle) $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



Hyperboloid of one sheet $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



Cone: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z^2$
You can have similar equations with x, y, z permuted or with the origin shifted.

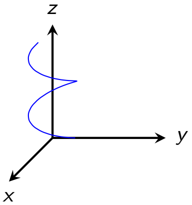
Definition

A vector-valued function is a function $\mathbf{r}(t)$ whose domain is a set of real numbers and whose range is a set of vectors in two- or three-dimensional space. We can specify $\mathbf{r}(t)$ through its component functions:

$$\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

Example

$$\mathbf{r}(t) = (\cos t, t, \sin t).$$



Definition

The limit of a vector-valued function is the limit of the component functions:

$$\lim_{t\to a}(x(t), y(t), z(t)) = \left(\lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\right).$$

A vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ is continuous if each of the component functions $x(t)$, $y(t)$, $z(t)$ is continuous.
A vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ is differentiable if each of the component functions $x(t)$, $y(t)$, $z(t)$ is differentiable and we have $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$.

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

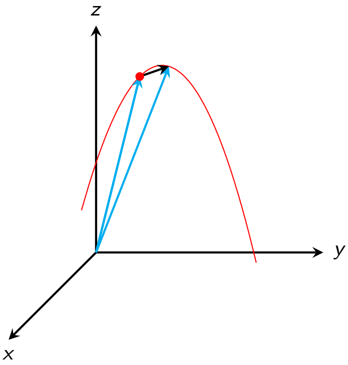
Definition

The integral of a vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ on an interval $[a, b]$ is defined by:

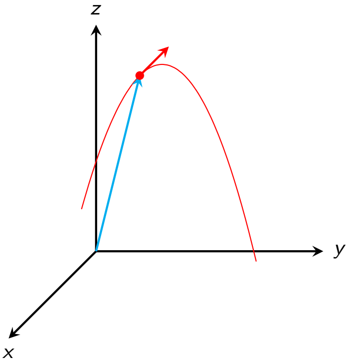
$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\right).$$

Remark

The vector $\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ measures the displacement from t to $t+h$.



The vector $\mathbf{r}'(t)$ gives the instantaneous change in displacement The magnitude $|\mathbf{r}'(t)|$ gives instantaneous speed.



Tangent Lines

Consider the curve $\mathbf{r}(t) = (2t, e^{-t}, \cos t - t^2)$.
$\mathbf{r}'(t) = (2, -e^{-t}, -\sin t - 2t)$ and $\mathbf{r}'(0) = (2, -1, 0)$. The parametric equations for the tangent line to the curve at $(0, 1, 1)$ is

$$\begin{cases} x = 2t, \\ y = 1 - t, \\ z = 1 \end{cases}$$

Definition

The unit tangent to $\mathbf{r}(t)$ is the vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Definition

The arc length of a plane curve $\mathbf{r}(t) = (x(t), y(t))$, $t \in [a, b]$ is

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_a^b \|\mathbf{r}'(t)\| dt.$$

The arc length of a plane curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ $t \in [a, b]$ is

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt = \int_a^b \|\mathbf{r}'(t)\| dt.$$

If $\mathbf{r}(t)$ is the space curve of a moving body and if t is time:

1. $\mathbf{r}'(t)$ is the velocity of the moving body
2. $\|\mathbf{r}'(t)'\|$ is the speed of the moving body
3. $\mathbf{r}''(t)$ is the acceleration of the moving body

Definition-(The Arc Length Function)

Let $\mathscr{C}$ be a space curve given by a vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

for $t \in [a, b]$.
the arc length function for $\mathscr{C}$ is defined by:

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du.$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$

The Partial Derivatives

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$, the partial derivatives of f with respect to x and y if they exist are defined by:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, , y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Consider a smooth function $f\colon D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$, the partial derivatives of f with respect to x, y and z if they exist are defined by:

$$\frac{\partial f}{\partial x}(x,y,z) = f_x(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h};$$

$$\frac{\partial f}{\partial y}(x,y,z) = f_y(x,y,z) = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h};$$

$$\frac{\partial f}{\partial z}(x,y,z) = f_z(x,y,z) = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}.$$

Theorem (Schwarz's Theorem)

Let f be a function defined on a domain D that contains the point (a, b). If the functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

The Directional Derivative

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$ and $\overrightarrow{u} = (a, b)$ a unit vector in $\mathbb{R}^2$. The directional derivative of f in the direction of $\overrightarrow{u}$ at (x_0, y_0) if it exists is

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h \to 0} \frac{f((x_0, y_0) + hu) - f(x_0, y_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}. \end{aligned}$$

If f is a function defined on a domain $D \subset \mathbb{R}^3$. For $(x_0, y_0, z_0) \in D$ and $\overrightarrow{u} = (a, b, c)$ a unit vector in $\mathbb{R}^3$. The directional derivative of f in the direction of $\overrightarrow{u}$ at (x_0, y_0, z_0) if it exists is

Example

1. If $\overrightarrow{u} = (a, b)$, $D_u f(x_0, y_0)$ is the same as the derivative of $f(x_0 + at, y_0 + bt)$ at $t = 0$. We can compute this by the chain rule and get

$$D_u f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0).$$

2. Find the directional derivative of $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in the direction $\overrightarrow{u} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
3. Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of $\overrightarrow{u} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Tangent Plane

The derivatives $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ define a tangent plane to the graph of f at $(a, b, f(a, b))$.
The differential of $z = f(x, y)$ is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Definition

If f has continuous partial derivatives, the tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$ is

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Chain Rule for Functions of One Variable

If $y = f(u)$ and $u = u(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f'(u).\frac{du}{dx}.$$

The Chain Rule, 2 Variables

If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

If

$$z = f(x, y), x = g(s, t), y = h(s, t),$$

then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t},$$

Definition

A two-dimensional transformation is a function f that maps each point (x,y) in a domain $\Omega \subset \mathbb{R}^2$ to a point $f(x, y) = (u(x, y), v(x, y))$ in $\mathbb{R}^2$.
A two-dimensional vector field is a function f that maps each point (x,y) in a domain $\Omega \subset \mathbb{R}^2$ to a two-dimensional vector $f(x, y) = u(x, y)\overrightarrow{\mathbf{i}} + v(x, y)\overrightarrow{\mathbf{j}}$, where $\overrightarrow{\mathbf{i}} = (1, 0)$ and $\overrightarrow{\mathbf{j}} = (0, 1)$.

Definition

A three-dimensional transformation is a function f that maps each point (x,y,z) in a domain $\Omega \subset \mathbb{R}^3$ to a point $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ in $\mathbb{R}^3$.
A three-dimensional vector field maps (x,y,z) in a domain $\Omega \subset \mathbb{R}^3$ to a three-dimensional vector $f(x,y,z) = u(x,y,z)\overrightarrow{\mathbf{i}} + v(x,y,z)\overrightarrow{\mathbf{j}} + w(x,y,z)\overrightarrow{\mathbf{k}}$, where $\overrightarrow{\mathbf{i}} = (1,0,0)$, $\overrightarrow{\mathbf{j}} = (0,1,0)$ and $\overrightarrow{\mathbf{k}} = (0,0,1)$.

Vector fields have many important applications, as they can be used to represent many physical quantities:

- Mechanics: the classical example is a gravitational field.
- Electricity and Magnetism: electric and magnetic fields.
- Fluid Mechanics: wind speed or the velocity of some other fluid.

If $\mathbf{r}(t) = x(t)\overrightarrow{\mathbf{i}} + y(t)\overrightarrow{\mathbf{j}} + z(t)\overrightarrow{\mathbf{k}}$ is the position vector field of an object. We can define various physical quantities associated with the object as follows:
velocity: $v(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = x'(t)\overrightarrow{\mathbf{i}} + y'(t)\overrightarrow{\mathbf{j}} + z'(t)\overrightarrow{\mathbf{k}}$,
acceleration:
$a(t) = v'(t) = \frac{dv}{dt} = \mathbf{r}''(t) = \frac{d^2\mathbf{r}}{dt^2} = x''(t)\overrightarrow{\mathbf{i}} + y''(t)\overrightarrow{\mathbf{j}} + z''(t)\overrightarrow{\mathbf{k}}$,
The norm $\|v(t)\|$ of the velocity vector is called the speed of the object.

Example

1. The gravitational force field between the Earth with mass M and a point particle with mass m is given by:

$$F(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}),$$

   where G is the gravitational constant, and the (x,y,z) coordinates are chosen so that $(0,0,0)$ is the center of the Earth.

2. **The Electrostatic fields**:
   In $3D$, $E = -\dfrac{q}{4\pi\varepsilon_0\|\mathbf{r}\|^3}\mathbf{r}$.
   In $2D$, $E = \dfrac{\rho}{2\pi\varepsilon_0\|\mathbf{r}\|^2}\mathbf{r}$.

Gradient Fields

Let f be a scalar function of two variables, the gradient of f is defined by

$$\nabla f(x,y) = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)).$$

If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x,y,z) = (\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)).$$

The operator ∇ will be denoted by:
$\nabla = \frac{\partial}{\partial x}\overrightarrow{\mathbf{i}} + \frac{\partial}{\partial y}\overrightarrow{\mathbf{j}} + \frac{\partial}{\partial z}\overrightarrow{\mathbf{k}}$ or $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ as a vector.

Remark

Let f be a function. The vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface of f $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = C\}$ that contains (x_0, y_0, z_0).

Theorem

Consider f and g two smooth scalar functions $\mathbf{F} = (f_1, f_2, f_3)$ and $\mathbf{G} = (g_1, g_2, g_3)$ two smooth vector fields defined on a domain $\Omega \subset \mathbb{R}^3$. We have:

$$\begin{aligned}\nabla(fg) &= \left(\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z}\right) \\ &= f\nabla(g) + g\nabla(f).\end{aligned}$$

$$\begin{aligned}\nabla(\mathbf{F}.\mathbf{G}) &= \nabla(f_1 g_1 + f_2 g_2 + f_3 g_3) \\ &= \nabla(f_1 g_1) + \nabla(f_2 g_2) + \nabla(f_3 g_3) \\ &= f_1\nabla(g_1) + g_1\nabla(f_1) + f_2\nabla(g_2) + g_2\nabla(f_2) \\ &\quad + f_3\nabla(g_3) + g_3\nabla(f_3).\end{aligned}$$

Example

$$\begin{aligned}\mathbf{F} &= \left(\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\\ &= \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}.\end{aligned}$$

Definition (Inverse square field)

Let $\mathbf{r}(x, y, z) = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}$ be the position vector of the point $M(x, y, z)$. The vector field $\mathbf{F}(x, y, z) = \dfrac{c}{\|\mathbf{r}\|^3}\mathbf{r}(x, y, z)$ is called an inverse square field, where $c \in \mathbb{R}$.