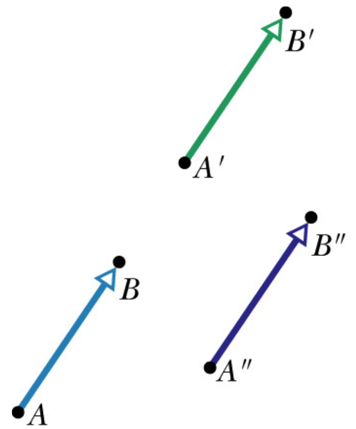
VECTORS

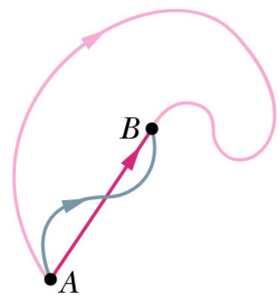
- Physical quantities are classified in two big classes: **vectors & scalars**.
- A **vector** is a physical quantity which is completely defined once we know precisely its **direction** and **magnitude** (for example: force, velocity, displacement)
- A **scalar** is a physical quantity which is completely defined once we know precisely **only** its **magnitude** (for example: speed, mass, density)

DISPLACEMENT

- Vectors are displayed by **arrows** having a "tip" and a "tail". In textbooks a vector quantity is normally denoted by bold letters (ex. **AB** or **A)**
- The simplest vector in nature is displacement. In figure (b) we see vector **AB** representing displacement from position A to B.
- A vector can be shifted without changing its value if its magnitude and direction are not changed.



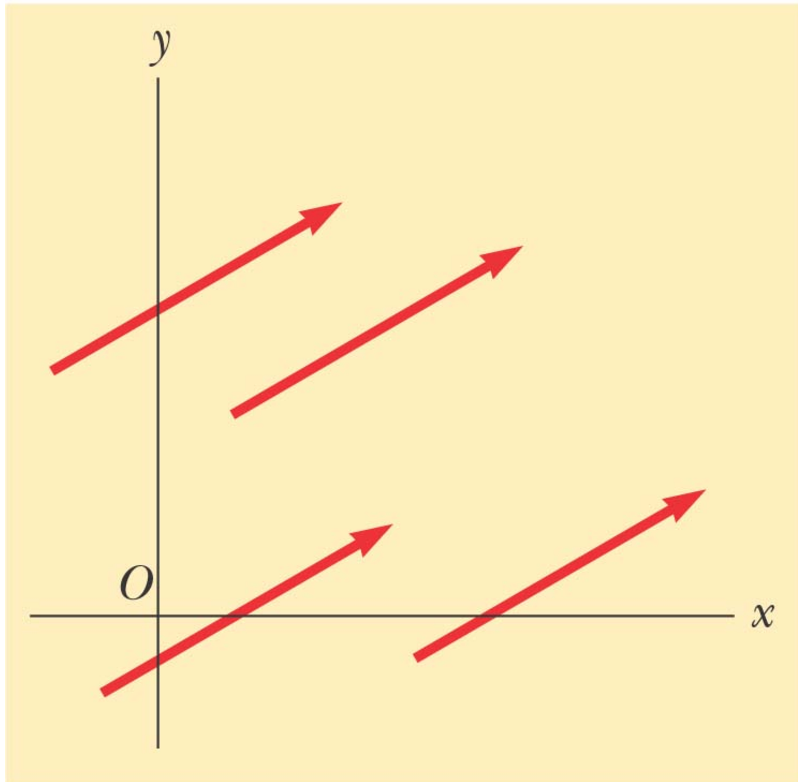
(a)



(b)

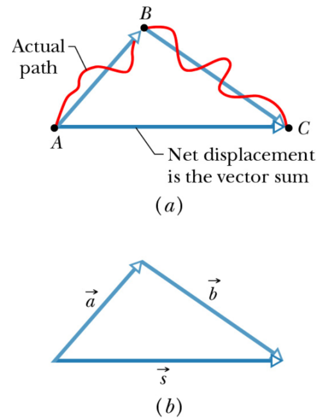
EQUALITY OF TWO VECTORS

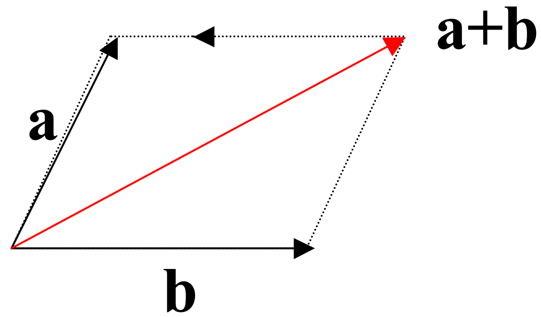
- Two vectors are ***equal*** if they have the same magnitude and the same direction
- **A** = **B** if $A = B$ and they point along parallel lines
- All of the vectors shown are equal



ADDING VECTORS GEOMETRICALLY

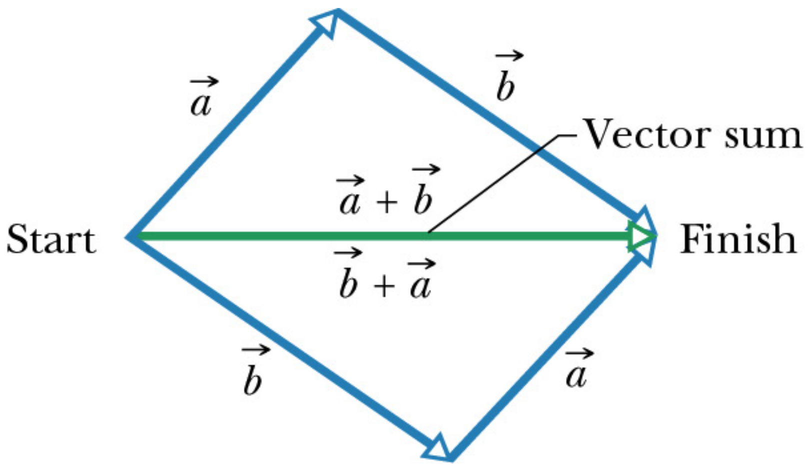
- The concept of displacement helps us very much to understand vector addition. There are **two** ways of vector addition:
- A) By putting the two vectors successively as in fig. (a), (b). In this case the resultant is the vector that runs from the tail of **a** to the tip of **b**
- B) By putting them with common tail, as in the lower figure.
- Both ways are equivalent.





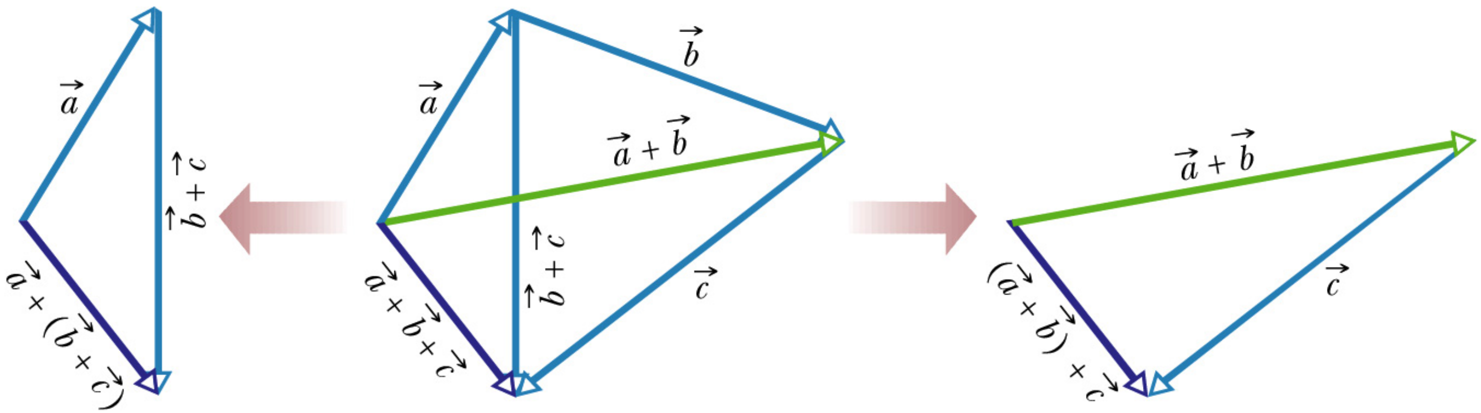
PROPERTIES OF ADDITION OF VECTORS-I

- The order of addition does not matter. Addition of vectors is **commutative**
- **a+b=b+a**



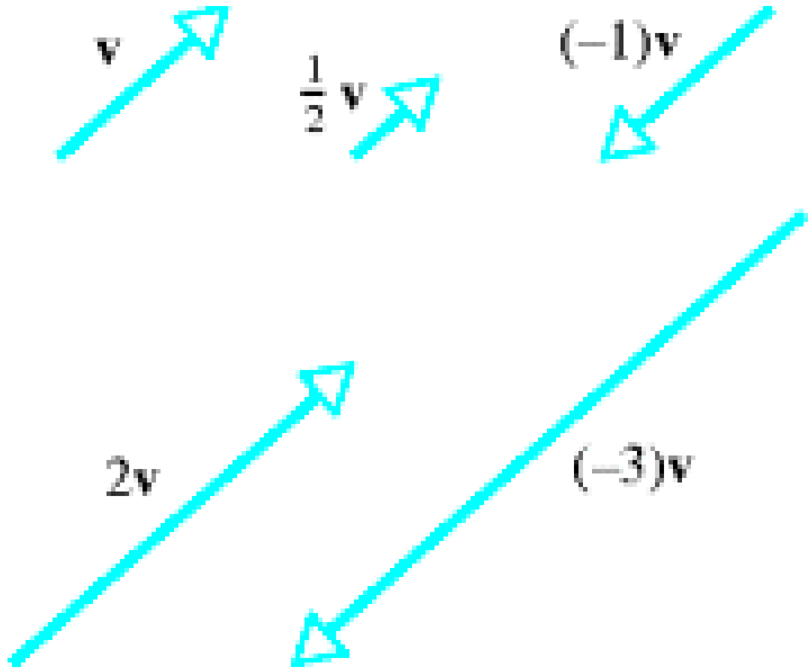
PROPERTIES OF ADDITION OF VECTORS-II

- When there are more than two vectors, we can group them in any order as we add them. The addition of vectors is **associative**:
  - **(a+b)+c = a+(b+c)**



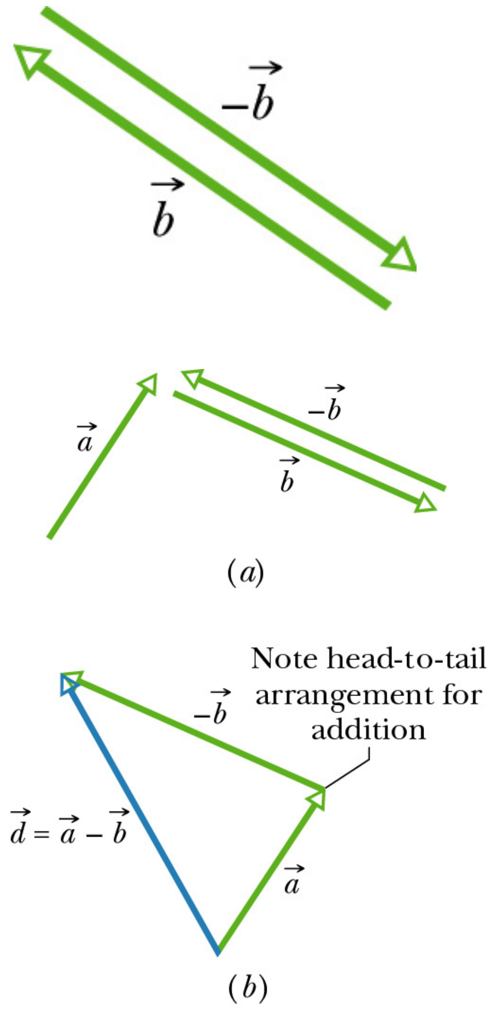
MULTIPLICATION OR DIVISION OF A SCALAR WITH A VECTOR

- The result of the multiplication or division is a vector
- The magnitude of the vector is multiplied or divided by the scalar
- If the scalar is positive, the direction of the result is the same as of the original vector
- If the scalar is negative, the direction of the result is opposite that of the original vector



SUBTRACTION OF VECTORS

- The vector –**b** is the vector with the same magnitude as **b** but opposite direction. Adding these two vectors would yield:
    - **b+(-b)=0**
- Vector **0** is a vector of zero magnitude called **null vector**.
- Thus subtraction is actually the addition of –**b**:
    - **a-b=a+(-b)**

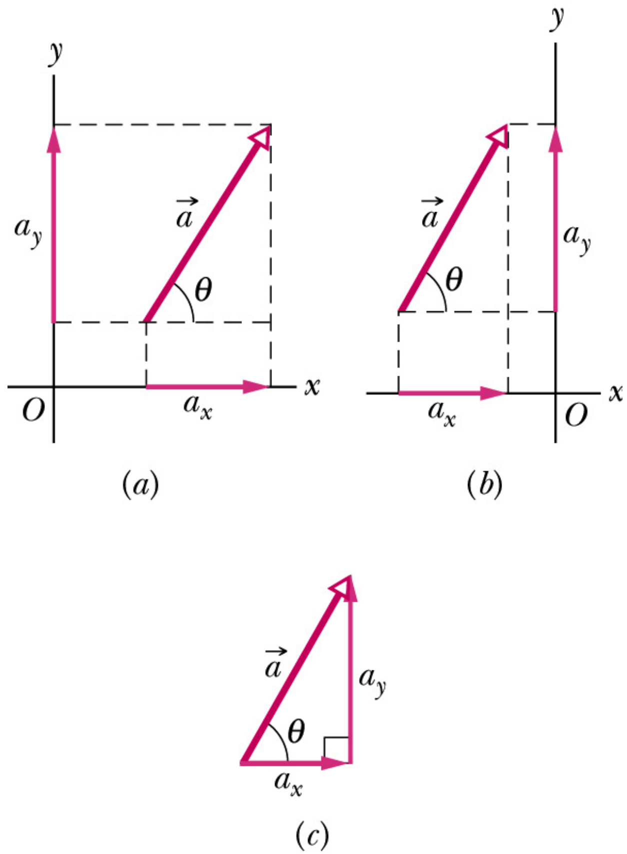


COMPONENTS OF A VECTOR

- A **component** of a vector is the projection of the vector on an axis. To find the projection we draw perpendicular lines from the two ends of the vector to the axis. The process of finding the components of a vector is called **resolving the vector**. Using geometry it is easy to see that:

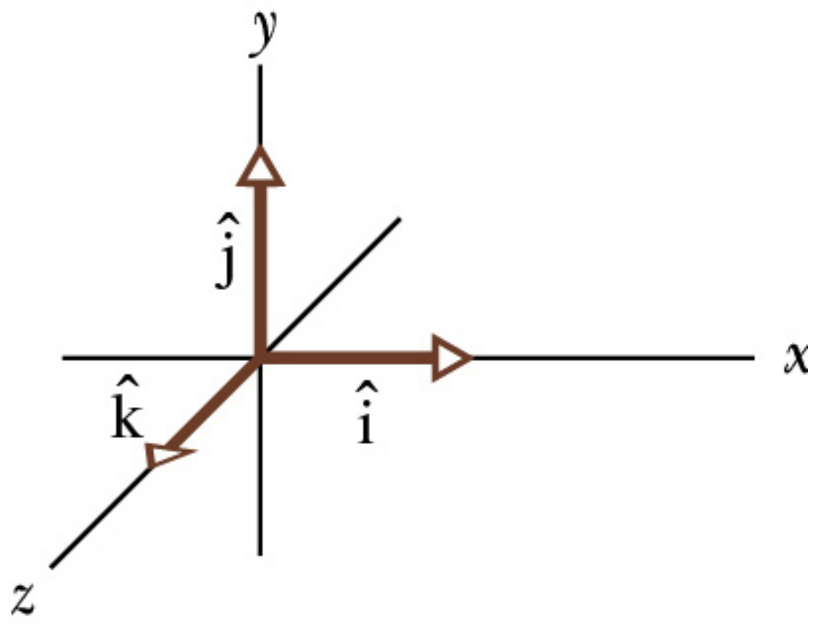
$$a_x = a\cos\vartheta \qquad a_y = a\sin\theta$$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \tan\theta = \frac{a_y}{a_x}$$



UNIT VECTORS

- A **unit** vector is a vector that has magnitude 1 and points in a particular direction. It lacks both dimension and unit.
- The arrangement shown in figure is called **right handed coordinate system**.

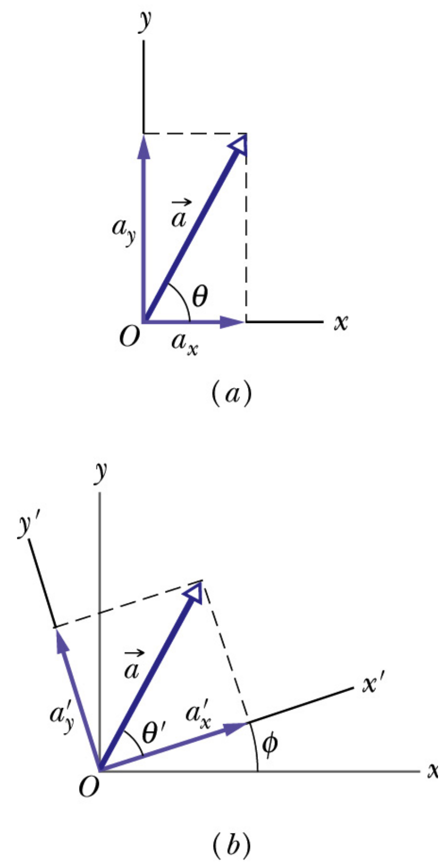


VECTORS AND THE LAWS OF PHYSICS

- The figure shows that you have a freedom in choosing a system of axes. This is so because the relations among vectors do not depend on the location of the origin of the coordinate system or on the orientation of axes.
- In the figure we have:

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$

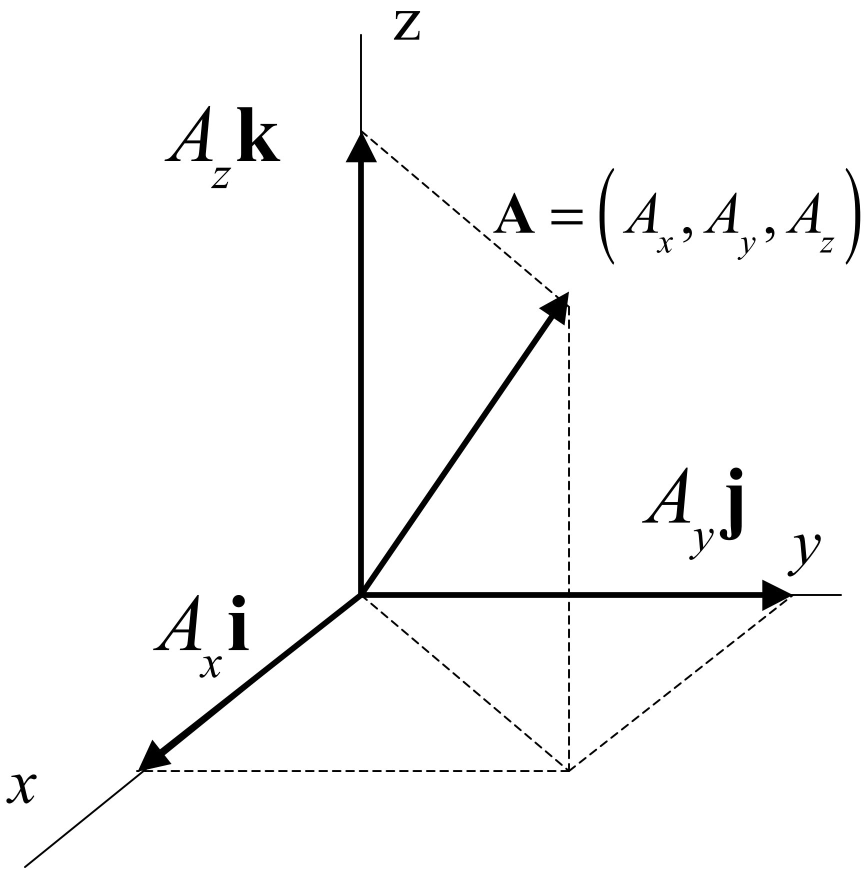
$$\vartheta = \vartheta' + \phi$$



UNIT VECTORS IN VECTOR NOTATION

- $\mathbf{A_x}$ is the same as A_x and $\mathbf{A_y}$ is the same as A_y etc.
- The complete vector can be expressed as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$



VECTOR ALGEBRA IN 3-DIRECTIONS

$$\mathbf{A} = \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right), \quad \mathbf{B} = \left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right)$$

$$\mathbf{R} = \mathbf{A} \pm \mathbf{B}$$

$$\mathbf{R} = \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right) + \left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right)$$

$$\mathbf{R} = \left(A_x \pm B_x\right)\mathbf{i} + \left(A_y \pm B_y\right)\mathbf{j} + \left(A_z \pm B_z\right)\mathbf{k}$$

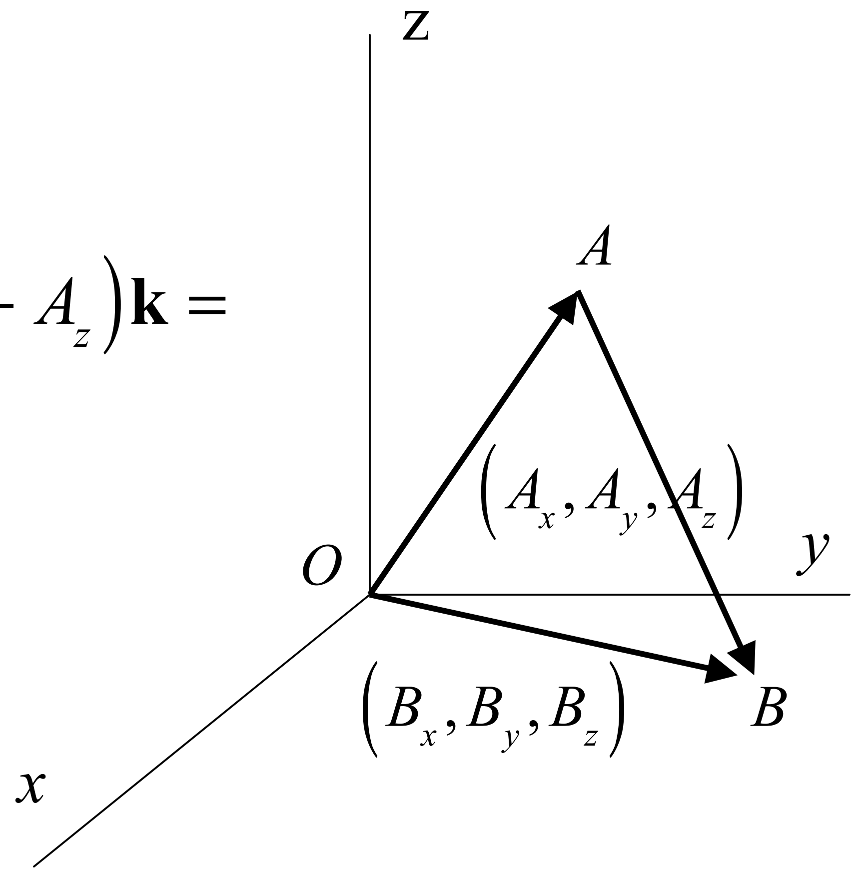
$$\lambda\mathbf{A} = \lambda\left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right) = \lambda A_x\mathbf{i} + \lambda A_y\mathbf{j} + \lambda A_z\mathbf{k}$$

VECTOR WHOSE INITIAL POINT IS NOT AT THE ORIGIN

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} =$$

$$= \left(B_x - A_x\right)\mathbf{i} + \left(B_y - A_y\right)\mathbf{j} + \left(B_z - A_z\right)\mathbf{k} =$$

$$\left(B_x - A_x,\ B_y - A_y,\ B_z - A_z\right)$$



n - DIMENSIONAL SPACES (a)

- The set of all real numbers can be viewed geometrically as a line. It is called the real line and is denoted by R or R^1. The superscript reinforces the intuitive idea that a line is one-dimensional.
- The set of all ordered pairs of real numbers (called 2-tuples) and the set of al ordered triples of real numbers (called 3-tuples) are denoted by R^2 and R^3, respectively. The superscript reinforces the idea that the ordered pairs correspond to points in the plane (two-dimensional) and ordered triples to points in space (three-dimensional).

n - DIMENSIONAL SPACES (b)

- If *n* is a positive integer, then an ordered *n*-tuple is a sequence of *n* real numbers .The set of all ordered *n*-tuples is called *n*-space and is denoted by R^n.

OPERATIONS OF VECTORS IN *n* - DIMENSIONAL SPACES (a)

- The operations of vectors in a n-dimensional space is a natural extension of the operations in a 2- or 3-dimensional space.
- We will denote a vector in n-dimensional space by

$$\mathbf{v} = \left(v_1,\ v_2,\ \ldots, v_N\right)$$

OPERATIONS OF VECTORS IN *n* - DIMENSIONAL SPACES (b)

- We denote with **0** the zeroth vector which is given by:

$$\mathbf{v} = \left(0,\ 0,\ \ldots,0\right)$$

- Two vectors **v** and **w** given by

$$\mathbf{v} = \left(v_1,\ v_2,\ \ldots,v_N\right) \quad \text{and} \quad \mathbf{w} = \left(w_1,\ w_2,\ \ldots,w_N\right)$$

Are said to be equivalent or equal if

$$v_1 = w_1,\ v_2 = w_2,\ldots,v_N = w_N$$

In this case we write **v**=**w**.

OPERATIONS OF VECTORS IN *n* - DIMENSIONAL SPACES (c)

- For the vectors in the *n*-dimensional space we have the following properties:

$$\mathbf{v} + \mathbf{w} = \left(v_1 + w_1,\ v_2 + w_2,\ \ldots, v_N + w_N\right)$$

$$\lambda \mathbf{v} = \left(\lambda v_1,\ \lambda v_2,\ \ldots, \lambda v_N\right)$$

$$\left(-1\right)\mathbf{v} = \left(-v_1,\ -v_2,\ \ldots, -v_N\right)$$

$$\mathbf{v} - \mathbf{w} = \left(v_1 - w_1,\ v_2 - w_2,\ \ldots, v_N - w_N\right)$$

OPERATIONS OF VECTORS IN *n* - DIMENSIONAL SPACES (d)

- For the vectors **w**, **v** and **u** in the *n*-dimensional space and *k* and *m* scalars we can prove the following properties:

a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

c) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

e) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$ f) $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

g) $k(m)\mathbf{u}=(km)\mathbf{u}$ h) $1\mathbf{u}=\mathbf{u}$

i) $0\mathbf{u}=\mathbf{0}$ j) $k\mathbf{0}=\mathbf{0}$ k) $(-1)\mathbf{v}=-\mathbf{v}$

LINEAR COMBINATION OF VECTORS IN n - DIMENSIONAL SPACES

- A vectors **w**, in the n-dimensional space, is said to be in a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \ \ldots + k_r\mathbf{v}_r$$

- Where $k_1, k_2 \ \ldots k_r$ are scalars called the **coefficients** of the linear combinatinon.

NORM OF A VECTORLINEAR IN *n* - DIMENSIONAL SPACES

- If $\mathbf{v} = \left(v_1,\ v_2,\ \ldots, v_N\right)$ is a vector in the n-dimensional space we call ***norm*** (or **magnitude** or **length**) the quantity

$$\left\|\mathbf{v}\right\| = \sqrt{v_1^2 + v_2^2 + \ \ldots + v_N^2}$$

NORM OF A VECTORLINEAR IN *n* - DIMENSIONAL SPACES

- For the norm we can show the following properties:

$$a)\quad \|\mathbf{v}\| \geq 0$$

$$b)\quad \|\mathbf{v}\| = 0 \text{ if and only if } \mathbf{v} = \mathbf{0}$$

$$c)\quad \|k\mathbf{v}\| = |k|\|\mathbf{v}\|$$

THE UNIT VECTOR

- A vector with norm equal to 1 is called a **unit vector**.
- If you are given any vector $\mathbf{v}$ you can construct a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ by using the following relation:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

THE STANDARD UNIT VECTORS IN *n* - DIMENSIONAL SPACES

- In the n-th dimensional space R^{n} the standard unit vectors are:

$$\mathbf{e}_1 = (1,\ 0,\ \ldots,0),\ \mathbf{e}_2 = (0,\ 1,\ \ldots,0),\ \ldots\ ,\ \mathbf{e}_n = (0,\ 0,\ \ldots,1)$$

- In this case the vector $\mathbf{v} = (v_1,\ v_2,\ \ldots, v_N)$ is written as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ \ldots + v_n\mathbf{e}_{\mathrm{n}}$$

DISTANCE BETWEEN POINTS IN *n* - DIMENSIONAL SPACES

- If two points in the n-dimensional space have position vectors

$$\mathbf{v} = \left(v_1,\ v_2,\ \ldots, v_N\right), \mathbf{w} = \left(w_1,\ w_2,\ \ldots, w_N\right)$$

- Then the distance between these points is given by:

$$d = \left\|\mathbf{w} - \mathbf{v}\right\| = \sqrt{\left(w_1 - v_1\right)^2 + \left(w_2 - v_2\right)^2 + \ldots\left(w_n - v_n\right)^2}$$

DOT PRODUCT IN *n* - DIMENSIONAL SPACES

- The dot product of two vectors **v** and **w** in the n-dimensional space is defined as:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \ \ldots + v_N w_N$$

- Two vectors are said to be orthogonal if their dot vector is zero:

$$\mathbf{v} \perp \mathbf{w} \Rightarrow v_1 w_1 + v_2 w_2 + \ \ldots + v_N w_N = 0$$

PYTHAGORAS THEOREM IN *n* - DIMENSIONAL SPACES

- If **u** and **v** are orthogonal vectors in a *n*-dimensional space then:

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

RELATIONS BETWEEN CROSS AND DOT PRODUCT

- If **u**, **v** and **w** are three vectors of the space R^3 then:

$$\mathbf{u}\cdot\left(\mathbf{u}\times\mathbf{v}\right)=0$$

$$\left\|\mathbf{u}\times\mathbf{v}\right\|^2=\left\|\mathbf{u}\right\|^2\left\|\mathbf{v}\right\|^2-\left(\mathbf{u}\cdot\mathbf{v}\right)^2 \quad \text{(Lagrange's Identity)}$$

$$\mathbf{u}\times\left(\mathbf{v}\times\mathbf{w}\right)=\left(\mathbf{u}\cdot\mathbf{w}\right)\mathbf{v}-\left(\mathbf{u}\cdot\mathbf{v}\right)\mathbf{w}$$

PROPERTIES OF CROSS PRODUCT

$$\mathbf{u}\times\mathbf{v}=-\left(\mathbf{v}\times\mathbf{u}\right)$$

$$\mathbf{u}\times\left(\mathbf{v}+\mathbf{w}\right)=\left(\mathbf{u}\times\mathbf{v}\right)+\left(\mathbf{u}\times\mathbf{w}\right)$$

$$\left(\mathbf{v}+\mathbf{w}\right)\times\mathbf{u}=\left(\mathbf{v}\times\mathbf{u}\right)+\left(\mathbf{w}\times\mathbf{u}\right)$$

$$k\left(\mathbf{v}\times\mathbf{u}\right)=\left(k\mathbf{v}\right)\times\mathbf{u}=\mathbf{v}\times\left(k\mathbf{u}\right)$$

$$\mathbf{u}\times\mathbf{u}=\mathbf{0}$$