pb 5.23 p. 78 Textbook

Show that as $\tau \to \infty$ and $\theta\tau^{1/\gamma} \to \xi$, a constant in the transformed beta distribution the result is the inverse transformed gamma distribution.

Ans:

The transformed beta pdf is

Given
$$f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\, x^{\gamma\tau-1}}{\Gamma(\alpha)\,\Gamma(\tau)\,\theta^{\gamma\tau}\,(1+x^{\gamma}\theta^{-\gamma})^{\alpha+\tau}} \qquad (1)$$

At $\theta\tau^{1/\gamma} = \xi = \text{Const.} \Rightarrow \theta = \xi\tau^{-1/\gamma}$ (2)

Substitut (2) in (1), we get
$$f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\, x^{\gamma\tau-1}}{\Gamma(\alpha)\,\Gamma(\tau)\,\xi^{\gamma\tau}\,\tau^{-\tau}\,[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}} \qquad (3)$$

$$x^{\gamma}\theta^{-\gamma} = \left(\frac{x}{\theta}\right)^{\gamma} = \left(\frac{x}{\xi\tau^{-1/\gamma}}\right)^{\gamma} = x^{\gamma}\xi^{-\gamma}\tau$$

* By using Stirling's formula

Hint
$$\lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \Rightarrow \lim_{\tau\to\infty}\frac{e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2}}{\Gamma(\tau)} = 1$$

i.e. $\Gamma(\tau) = e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2}$ as $\tau \to \infty$ (4)

Also $\Gamma(\alpha+\tau) = e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}$ as $\alpha+\tau \to \infty$ (5)

where α is constant and $\tau \to \infty$ in our pb.

Substitut (4), (5) in (3)
$$f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}\cancel{(2\pi)^{1/2}}\,\gamma\, x^{\gamma\tau-1}}{\Gamma(\alpha)\,e^{\tau}\,\tau^{\tau-1/2}\cancel{(2\pi)^{1/2}}\,\xi^{\gamma\tau}\,\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$f(x) = \frac{e^{-\alpha}\left(\frac{\alpha+\tau}{\tau}\right)^{\alpha+\tau-1/2}\gamma\, x^{\gamma\tau-1}\cdot x^{-\gamma\tau-\gamma\alpha}}{\Gamma(\alpha)\,\tau^{-\alpha}\,\tau^{-\tau}\,\xi^{\gamma(\tau+\alpha)}\,\xi^{-\gamma\alpha}\,x^{-\gamma(\tau+\alpha)}(1+x^{\gamma}\xi^{-\gamma}\tau)^{\alpha+\tau}}$$

Taking $\tau \to \infty$, we get
$$f(x) = \frac{e^{-\alpha}e^{\alpha}\,\gamma\, x^{-\gamma\alpha-1}}{\Gamma(\alpha)\,\xi^{-\gamma\alpha}\left(\frac{1}{\tau}\right)^{\alpha+\tau}(\xi/x)^{\gamma(\tau+\alpha)}(1+x^{\gamma}\xi^{-\gamma}\tau)^{\alpha+\tau}}$$

where $\lim_{\tau\to\infty}\left(\frac{\alpha+\tau}{\tau}\right)^{\alpha+\tau-1/2} = \lim_{\tau\to\infty}\left(1+\frac{\alpha}{\tau}\right)^{\tau+\alpha-1/2} = e^{\alpha}$

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$$f(x) = \frac{\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\, \xi^{-\gamma\alpha}\left[\frac{1}{\tau}\left(\frac{\xi}{x}\right)^{\gamma}\right]^{\alpha+\tau}\left[1+x^{\gamma}\xi^{-\gamma}\tau\right]^{\alpha+\tau}}$$

$$f(x) = \frac{\gamma\,(\xi/x)^{\gamma\alpha}}{\Gamma(\alpha)\,x} \cdot \frac{1}{\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}}$$

$$\because \lim_{\tau\to\infty}\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau} = e^{(\xi/x)^{\gamma}}$$

$$\therefore f(x) = \frac{\gamma\,(\xi/x)^{\gamma\alpha}\, e^{-(\xi/x)^{\gamma}}}{\Gamma(\alpha)\,x} \quad \text{as } \tau\to\infty$$

which is the inverse transformed gamma pdf

with parameters α, ξ and γ. See p. 497

For Inverse Transformed gamma (α, θ, τ)

$$f(x) = \frac{\tau\,(\theta/x)^{\tau\alpha}\, e^{-(\theta/x)^{\tau}}}{x\,\Gamma(\alpha)}$$

$\alpha \to \alpha$

$\tau \to \gamma$ الثابت

$\theta \to \xi$

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